The Mathematics Admissions Test (MAT) is a paper based test that has been used by the University of Oxford since 1996. This extract from the 2012 test and associated sample solutions constitute the test undertaken by applicants to the Mathematics, Maths & Philosophy and Maths & Statistics undergraduate degree courses at Oxford.

From 2013, the test will be used as part of the admissions process for applicants to the following courses run by the Department of Mathematics at Imperial College.

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IN 2013 THE ADMISSIONS TESTING SERVICE WILL BE ORGANIZING THE DISTRIBUTION AND RECEIPT OF THE MATHEMATICS TEST. SEE THIS ADMISSIONS TESTING SERVICE PAGE FOR FULL DETAILS.
1. For **ALL APPLICANTS**.

For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part A–J which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

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A. Which of the following lines is a tangent to the circle with equation 
\[ x^2 + y^2 = 4 \]?

(a) \( x + y = 2 \);   (b) \( y = x - 2\sqrt{2} \);   (c) \( x = \sqrt{2} \);   (d) \( y = \sqrt{2} - x \).

B. Let \( N = 2^k \times 4^m \times 8^n \) where \( k, m, n \) are positive whole numbers. Then \( N \) will definitely be a square number whenever

(a) \( k \) is even;
(b) \( k + n \) is odd;
(c) \( k \) is odd but \( m + n \) is even;
(d) \( k + n \) is even.
C. Which is the smallest of the following numbers?

(a) \((\sqrt{3})^3\),  (b) \(\log_3 (9^2)\),  (c) \((3\sin \frac{\pi}{3})^2\),  (d) \(\log_2 (\log_2 (8^5))\).

D. Shown below is a diagram of the square with vertices \((0, 0)\), \((0, 1)\), \((1, 1)\), \((1, 0)\) and the line \(y = x + c\). The shaded region is the region of the square which lies below the line; this shaded region has area \(A(c)\).

Which of the following graphs shows \(A(c)\) as \(c\) varies?
E. Which one of the following equations could possibly have the graph given below?

(a) \( y = (3 - x)^2 (3 + x)^2 (1 - x); \)
(b) \( y = -x^2 (x - 9) (x^2 - 3); \)
(c) \( y = (x - 6) (x - 2)^2 (x + 2)^2; \)
(d) \( y = (x^2 - 1)^2 (3 - x). \)

F. Let

\[
T = \left( \int_{-\pi/2}^{\pi/2} \cos x \, dx \right) \times \left( \int_{\pi}^{2\pi} \sin x \, dx \right) \times \left( \int_{0}^{\pi/8} \frac{dx}{\cos 3x} \right).
\]

Which of the following is true?

(a) \( T = 0; \)  (b) \( T < 0; \)  (c) \( T > 0; \)  (d) \( T \) is not defined.
G. There are positive real numbers $x$ and $y$ which solve the equations

$$2x + ky = 4, \quad x + y = k$$

for

(a) all values of $k$;  \quad (b) no values of $k$;  \quad (c) $k = 2$ only;  \quad (d) only $k > -2$.

H. In the region $0 < x \leq 2\pi$, the equation

$$\int_0^x \sin(\sin t) \, dt = 0$$

has

(a) no solution;  \quad (b) one solution;  \quad (c) two solutions;  \quad (d) three solutions.
I. The vertices of an equilateral triangle are labelled X, Y and Z. The points X, Y and Z lie on a circle of circumference 10 units. Let P and A be the numerical values of the triangle’s perimeter and area, respectively. Which of the following is true?

(a) \( \frac{A}{P} = \frac{5}{4\pi} \);  
(b) \( P < A \);  
(c) \( \frac{P}{A} = \frac{10}{3\pi} \);  
(d) \( P^2 \) is rational.

J. If two chords \(QP\) and \(RP\) on a circle of radius 1 meet in an angle \(\theta\) at \(P\), for example as drawn in the diagram below, then the largest possible area of the shaded region \(RPQ\) is

(a) \( \theta \left( 1 + \cos \left( \frac{\theta}{2} \right) \right) \);  
(b) \( \theta + \sin \theta \);  
(c) \( \frac{\pi}{2} (1 - \cos \theta) \);  
(d) \( \theta \).

Turn over
2. For ALL APPLICANTS.

Let 
\[ f(x) = x + 1 \quad \text{and} \quad g(x) = 2x. \]

We will, for example, write \( fg \) to denote the function "perform \( g \) then perform \( f \)" so that
\[ fg(x) = f(g(x)) = 2x + 1. \]

If \( i \geq 0 \) is an integer we will, for example, write \( f^i \) to denote the function which performs \( f \) \( i \) times, so that
\[ f^i(x) = \underbrace{ff\cdots f}_i(x) = x + i. \]

(i) Show that \( f^2g(x) = gf(x) \).

(ii) Note that \( gf^2g(x) = 4x + 4 \).

Find all the other ways of combining \( f \) and \( g \) that result in the function \( 4x + 4 \).

(iii) Let \( i, j, k \geq 0 \) be integers. Determine the function \( f^i g^j f^k(x) \).

(iv) Let \( m \geq 0 \) be an integer. How many different ways of combining the functions \( f \) and \( g \) are there that result in the function \( 4x + 4m \)?
3.

For APPLICANTS IN {MATHEMATICS
MATHEMATICS & STATISTICS
MATHEMATICS & PHILOSOPHY
MATHEMATICS & COMPUTER SCIENCE} ONLY.

Computer Science and Computer Science & Philosophy applicants should turn to page 14.

Let \( f(x) = x^3 + ax^2 + bx + c \), where the coefficients \( a \), \( b \) and \( c \) are real numbers. The figure below shows a section of the graph of \( y = f(x) \). The curve has two distinct turning points; these are located at \( A \) and \( B \), as shown. (Note that the axes have been omitted deliberately.)

![Graph of f(x)](image)

(i) Find a condition on the coefficients \( a \), \( b \), \( c \) such that the curve has two distinct turning points if, and only if, this condition is satisfied.

It may be assumed from now on that the condition on the coefficients in (i) is satisfied.

(ii) Let \( x_1 \) and \( x_2 \) denote the \( x \) coordinates of \( A \) and \( B \), respectively. Show that

\[
x_2 - x_1 = \frac{2}{3} \sqrt{a^2 - 3b}.
\]

(iii) Suppose now that the graph of \( y = f(x) \) is translated so that the turning point at \( A \) now lies at the origin. Let \( g(x) \) be the cubic function such that \( y = g(x) \) has the translated graph. Show that

\[
g(x) = x^2 \left( x - \sqrt{a^2 - 3b} \right).
\]

(iv) Let \( R \) be the area of the region enclosed by the \( x \)-axis and the graph \( y = g(x) \). Show that if \( a \) and \( b \) are rational then \( R \) is also rational.

(v) Is it possible for \( R \) to be a non-zero rational number when \( a \) and \( b \) are both irrational? Justify your answer.
4.
For APPLICANTS IN \{ \text{MATHEMATICS}
\text{MATHEMATICS & STATISTICS}
\text{MATHEMATICS & PHILOSOPHY} \} \text{ ONLY.} \n
Mathematics & Computer Science, Computer Science and Computer Science & Philosophy applicants should turn to page 14.

The diagram below shows the parabola \( y = x^2 \) and a circle with centre \((0, 2)\) just ‘resting’ on the parabola. By ‘resting’ we mean that the circle and parabola are tangential to each other at the points \(A\) and \(B\).

![Diagram of parabola and circle](image)

(i) Let \((x, y)\) be a point on the parabola such that \(x \neq 0\). Show that the gradient of the line joining this point to the centre of the circle is given by

\[
\frac{x^2 - 2}{x}.
\]

(ii) With the help of the result from part (i), or otherwise, show that the coordinates of \(B\) are given by

\[
\left( \sqrt{\frac{3}{2}}, \frac{3}{2} \right).
\]

(iii) Show that the area of the sector of the circle enclosed by the radius to \(A\), the minor arc \(AB\) and the radius to \(B\) is equal to

\[
\frac{7}{4} \cos^{-1} \left( \frac{1}{\sqrt{7}} \right).
\]

(iv) Suppose now that a circle with centre \((0, a)\) is resting on the parabola, where \(a > 0\). Find the range of values of \(a\) for which the circle and parabola touch at two distinct points.

(v) Let \(r\) be the radius of a circle with centre \((0, a)\) that is resting on the parabola. Express \(a\) as a function of \(r\), distinguishing between the cases in which the circle is, and is not, in contact with the vertex of the parabola.
5. For **ALL APPLICANTS**.

A particular robot has three commands:
- **F**: Move forward a unit distance;
- **L**: Turn left $90^\circ$;
- **R**: Turn right $90^\circ$.

A *program* is a sequence of commands. We consider particular programs $P_n$ (for $n \geq 0$) in this question. The basic program $P_0$ just instructs the robot to move forward:

$$P_0 = \text{F}.$$ 

The program $P_{n+1}$ (for $n \geq 0$) involves performing $P_n$, turning left, performing $P_n$ again, then turning right:

$$P_{n+1} = P_n \text{L} P_n \text{R}.$$ 

So, for example, $P_1 = \text{F} \text{L} \text{F} \text{R}$.

(i) Write down the program $P_2$.

(ii) How far does the robot travel during the program $P_n$? In other words, how many **F** commands does it perform?

(iii) Let $l_n$ be the total number of commands in $P_n$; so, for example, $l_0 = 1$ and $l_1 = 4$.

Write down an equation relating $l_{n+1}$ to $l_n$. Hence write down a formula for $l_n$ in terms of $n$. No proof is required. **Hint**: consider $l_n + 2$.

(iv) The robot starts at the origin, facing along the positive $x$-axis. What direction is the robot facing after performing the program $P_n$?

(v) The left-hand diagram on the opposite page shows the path the robot takes when it performs the program $P_1$. On the right-hand diagram opposite, draw the path it takes when it performs the program $P_4$.

(vi) Let $(x_n, y_n)$ be the position of the robot after performing the program $P_n$, so $(x_0, y_0) = (1, 0)$ and $(x_1, y_1) = (1, 1)$. Give an equation relating $(x_{n+1}, y_{n+1})$ to $(x_n, y_n)$.

What is $(x_8, y_8)$? What is $(x_{8k}, y_{8k})$?
Mark Scheme:

Each part of Question 1 is worth four marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

QUESTION 1:

A. One can proceed by elimination. Point (1, 1) lies on line (a) and is clearly inside the given circle. $(\sqrt{2}, 0)$ lies on both lines (c) and (d) which is again inside the given circle. The answer is (b).

B. We can rewrite $N$ as

$$N = 2^k \times 4^m \times 8^n = 2^{k+2m+3n}.$$ 

Now $2^r$ is a square when $r$ is even and is not a square when $r$ is odd. So we need

$$k + 2m + 3n = k + n + 2(m + n)$$

to be even, which is equivalent to needing $k + n$ to be even. The answer is (d).

C. Note that $\log_3(9^2) = 2 \log_3 9 = 4$. By comparison with this number

$$\left(\sqrt{3}\right)^3 = 3\sqrt{3} > 4 \quad \text{as on squaring } 27 > 16;$$

$$\left(3 \sin \frac{\pi}{3}\right)^2 = \left(\frac{3\sqrt{3}}{2}\right)^2 = \frac{27}{4} > 4;$$

$$\log_2 \left(\log_2(8^5)\right) = \log_2(5 \log_2 8) = \log_2 15 < \log_2 16 = 4.$$ 

The answer is (d).

D. If $-1 < c < 0$ then the horizontal and vertical sides of the shaded triangle have length $1 + c$ and hence

$$A(c) = \frac{1}{2} (1 + c)^2.$$ 

This is a parabolic curve with minimum at $(-1, 0)$ and so the answer is (a).

E. The curve is that of a quintic with a negative leading coefficient which eliminates (c). Two of its roots are repeated which eliminates (b). Finally the smallest and middle of these roots are the repeated ones which eliminates (a) which has its smallest and largest roots repeated. The answer is (d).
F. None of the integrals needs to be calculated. We need only note
\[
\int_{-\pi/2}^{\pi/2} \cos x \, dx > 0 \quad \text{as} \quad \cos x > 0 \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2};
\]
\[
\int_{\pi}^{2\pi} \sin x \, dx < 0 \quad \text{as} \quad \sin x < 0 \quad \text{for} \quad \pi < x < 2\pi;
\]
and
\[
\int_{0}^{\pi/8} \frac{dx}{\cos 3x} \quad \text{is defined and positive as} \quad \cos 3x > \cos \frac{3\pi}{8} > 0 \quad \text{for} \quad 0 < 3x < \frac{3\pi}{8};
\]
The product of two positive numbers and a negative one is negative. **The answer is (b).**

G. Using the second equation to eliminate \( x \) we see that
\[
2(k - y) + ky = 4 \quad \Longrightarrow \quad (k - 2)y = (4 - 2k) = -2(k - 2).
\]
If \( k \neq 2 \) then we can divide by \( k - 2 \) and we see \( y = -2 \). So positive solutions aren’t possible when \( k \neq 2 \). When \( k = 2 \) then the equations are
\[
2x + 2y = 4, \quad x + y = 2
\]
which are both clearly satisfied by \( x = y = 1 \). **The answer is (c).**

H. A sketch of \( y = \sin(\sin t) \) in the range \( 0 < t \leq 2\pi \) looks like

To appreciate this we need to realise how the graph \( y = \sin(\sin t) \) relates to the graph \( y = \sin t \). The value \( \sin t \) is in the range \( -1 \leq \sin t \leq 1 \) and for values in the range \( -1 \leq \theta \leq 1 \) then \( \sin \theta \) is a value which is smaller than but of the same sign as \( \theta \). Importantly \( \sin \) is also odd.

So for \( 0 < x \leq \pi \) then
\[
\int_{0}^{x} \sin(\sin t) \, dt > 0.
\]
Also, by the oddness of \( \sin \) we have that
\[
\int_{\pi}^{2\pi} \sin(\sin t) \, dt = -\int_{0}^{\pi} \sin(\sin t) \, dt.
\]
So
\[
\int_{0}^{x} \sin(\sin t) \, dt
\]
remains positive for \( \pi < x < 2\pi \), the integral only becoming 0 when the signed area between \( \pi \) and \( 2\pi \) cancels out the area between 0 and \( \pi \). The only root of the equation in the given range is \( x = 2\pi \) and **the answer is (b).**
I. If the radius of the circle is $r$ then we have $2\pi r = 10$ and $r = \frac{5}{\pi}$. This distance $r$ is also the distance from the centre of the triangle to any of its vertices. So we have

$$A = 3 \times \frac{1}{2} r^2 \sin \left(\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{4} r^2; \quad P = 6 \times r \sin \left(\frac{\pi}{3}\right) = 3\sqrt{3} r.$$  

Hence

$$\frac{A}{P} = \frac{3\sqrt{3} r^2 / 4}{3\sqrt{3} r} = \frac{r}{4} = \frac{5}{4\pi}$$

and the answer is (a).

J. The area $QPR$ is largest when $Q$ and $R$ are symmetrically placed about $P$, for if (say) $PQ$ were longer than $PR$ then $Q$ could be moved so as to gain more area than would be lost by the corresponding move of $R$. This means that the angles $QPO$ and $RPO$ are both $\theta/2$; the angle $QOR$ is $2\theta$ as the angle subtended by $QR$ at the centre $O$ is twice that subtended at the circumference $P$.

We then see that the area of $PQR$ equals

$$2 \times \frac{1}{2} \times 1^2 \times \sin QOP + \frac{1}{2} \times 1^2 \times 2\theta = \sin (\pi - \theta) + \theta = \sin \theta + \theta$$

and the answer is (b).
2. (i) Note that
\[
f^2g(x) = f(f(g(x))) = f(f(2x)) = f(2x + 1) = 2x + 2;
gf(x) = g(x + 1) = 2(x + 1) = 2x + 2.
\]

(ii) Using the identity \( f^2g = gf \) we see that
\[
gf^2g = g(f^2g) = g(gf) = g^2f
\]
and also that
\[
gf^2g = gf(fg) = f^2g(fg) = f^2gfg
\]
and finally that
\[
f^2gfg = f^2(gf)g = f^2(f^2g)g = f^4g^2.
\]
These four, \( f^4g^2, f^2gfg, gf^2g, g^2f \) are the only sequences that lead to \( 4x + 4 \).

(iii) Note that
\[
f^k(x) = x + k;
gf^k(x) = 2(x + k) = 2x + 2k;
f^jg^k(x) = (2x + 2k) + j = 2x + 2k + j;
gf^igf^k(x) = 2(2x + 2k + j) = 4x + 4k + 2j;
f^igf^igf^k(x) = (4x + 4k + 2j) + i = 4x + 4k + 2j + i.
\]

(iv) We need to consider the ways we can have \( 4k + 2j + i = 4m \) where \( i, j, k \geq 0 \). Clearly \( k \) can take the values \( 0 \leq k \leq m \) and then we need
\[
2j + i = 4(m - k).
\]
Then \( j \) can take values \( 0 \leq j \leq 2(m - k) \) – that is \( 2m - 2k + 1 \) choices for \( j \) (given \( k \)). The choice of \( j \) then determines \( i \). So the number of possible combinations is
\[
\sum_{k=0}^{m} (2m - 2k + 1) = (2m + 1) \left( \sum_{k=0}^{m} 1 \right) - 2 \left( \sum_{k=0}^{m} k \right)
\]
\[
= (2m + 1)(m + 1) - 2 \times \frac{1}{2}m(m + 1)
\]
\[
= (m + 1)(m + 1)
\]
\[
= (m + 1)^2.
\]
3. (i) For there to be two distinct turning points the derivative \( f'(x) = 3x^2 + 2ax + b \) must have two distinct real roots. This is determined by the discriminant
\[
(2a)^2 - 4 \times 3 \times b = 4 \left( a^2 - 3b \right).
\]
Thus \( y = f(x) \) has two distinct turning points if, and only if, \( a^2 - 3b > 0 \).

(ii) The \( x \)-coordinates of the turning points are given by
\[
x = \frac{-2a \pm \sqrt{4a^2 - 12b}}{2 \times 3} = -\frac{a \pm \sqrt{a^2 - 3b}}{3}.
\]
If we call these \( x_1 \) and \( x_2 \) with \( x_1 < x_2 \) then
\[
x_2 - x_1 = \frac{1}{3} \left( -a + \sqrt{a^2 - 3b} + a + \sqrt{a^2 - 3b} \right) = \frac{2}{3} \sqrt{a^2 - 3b}.
\]

(iii) The equation of the translated graph has a repeated root at \( x = 0 \) and another root at \( t \) (say), where \( t \) is a positive real number. It is thus the case that
\[
g(x) = x^2(x - t) = x^3 - tx^2, \quad \text{and} \quad g'(x) = 3x^2 - 2tx = x(3x - 2t).
\]
There are then turning points at \( x = 0 \) and \( x = \frac{2}{3}t \). Thus, using the result from (ii), we see that
\[
\frac{2}{3}t = \frac{2}{3} \sqrt{a^2 - 3b}
\]
and hence \( t = \sqrt{a^2 - 3b} \), as required.

(iv) The area of the region \( R \) is given by
\[
-\int_0^t g(x) \, dx = -\int_0^t x^2(x - t) \, dx = \left[ \frac{x^4}{4} - \frac{x^3}{3} t \right]_0^t = \frac{t^4}{12} = \frac{(a^2 - 3b)^2}{12},
\]
which is rational when the coefficients \( a, b \) are rational.

(v) Yes, it is possible for \( R \) to be rational when \( a \) and \( b \) are both irrational. For example, let \( a = 2\sqrt{2} \) and \( b = \sqrt{2} \).
4. (i) We need the gradient of the line segment joining \((x, x^2)\) to \(C = (0, 2)\). This is, for \(x \neq 0\), given by
\[
\frac{x^2 - 2}{x - 0} = \frac{x^2 - 2}{x}.
\]
(ii) Let the coordinates of \(B\) be \((x_1, x_1^2)\). Then the gradient of the parabola (and also of the circle) at \(B\) is equal to \(2x_1\). The tangent to these curves at \(B\) is perpendicular to the line segment joining \((x_1, x_1^2)\) to \((0, 2)\), so that
\[
\frac{x_1^2 - 2}{x_1} \times 2x_1 = -1.
\]
Solving this, and taking the appropriate solution, gives \(x_1 = \sqrt{\frac{3}{2}}\) and \(x_1^2 = \frac{3}{2}\).
(iii) The square of the radius \(CB\) is
\[
\left(\sqrt{\frac{3}{2}}\right)^2 + \left(2 - \frac{3}{2}\right)^2 = \frac{7}{4},
\]
so the area of the circle is \(\frac{7\pi}{4}\). The angle at the centre subtended by the minor arc \(AB\) is
\[
2\cos^{-1}\left(\frac{\frac{3}{4}}{\sqrt{\frac{7}{4}}}\right) = 2\cos^{-1}\left(\frac{1}{\sqrt{7}}\right),
\]
from which we see the sector’s area equals
\[
\frac{1}{2}r^2\theta = \frac{1}{2} \times \frac{7}{4} \times 2\cos^{-1}\left(\frac{1}{\sqrt{7}}\right) = \frac{7}{4}\cos^{-1}\left(\frac{1}{\sqrt{7}}\right).
\]
(iv) Consider the equation
\[
\frac{x^2 - a}{x} \times 2x = -1.
\]
This can be rearranged to give \(x^2 = a - \frac{1}{2}\). For two distinct real solutions we require \(a > \frac{1}{2}\).
(v) If \(0 < a \leq \frac{1}{2}\) then, from (iv), the circle is resting on the vertex of the parabola. In this case \(a = r\).
On the other hand, if \(a > \frac{1}{2}\) then there are two distinct points points of contact, and we have
\[
r^2 = x^2 + (x^2 - a)^2.
\]
Now, using the fact that \(x^2 = a - \frac{1}{2}\), we obtain
\[
r^2 = \left(a - \frac{1}{2}\right) + \frac{1}{4} \implies a = r^2 + \frac{1}{4}.
\]
5. (i) From the recursive definition
\[ P_0 = F, \]
\[ P_1 = P_0 L P_1 R = FLFR, \]
\[ P_2 = P_1 L P_2 R = FLFRLFLFRR. \]

(ii) Say there are \( f_n \) commands \( F \) in \( P_n \). As \( P_{n+1} = P_n L P_R \) then \( f_{n+1} = 2f_n \). As \( f_0 = 1 \) then \( f_n = 2^n \).

(iii) Let \( l_n \) denote the total number of commands in \( P_n \). As \( P_{n+1} = P_n L P_R \) then
\[ l_{n+1} = l_n + 1 + l_n + 1 = 2l_n + 2. \]

If we set \( m_n = l_n + 2 \) (following the hint) then we see
\[ m_{n+1} = l_{n+1} + 2 = 2l_n + 4 = 2(l_n + 2) = 2m_n. \]

As \( m_0 = l_0 + 2 = 3 \) then \( m_n = 3 \times 2^n \) and
\[ l_n = m_n - 2 = 3 \times 2^n - 2. \]

(iv) The robot again faces along the positive \( x \)-axis after each \( P_n \) because each \( P_n \) contains as many \( L \)s as it does \( R \)s.

(v) \( P_4 \) is sketched below:

![Diagram of P_4]

(vi) After performing \( P_n \) the robot sits at \( (x_n, y_n) \) facing "east". Turning left it now faces "north" so what otherwise would have led to a movement of \( (x_n, y_n) \) instead achieves \( (-y_n, x_n) \). So from the recursion \( P_{n+1} = P_n L P_R \) we can see that
\[ (x_{n+1}, y_{n+1}) = (x_n, y_n) + (-y_n, x_n) = (x_n - y_n, x_n + y_n). \]

Note then that
\[ (x_{n+2}, y_{n+2}) = (x_{n+1} - y_{n+1}, x_{n+1} + y_{n+1}) = (-2y_n, 2x_n); \]
\[ (x_{n+4}, y_{n+4}) = (-2y_n, 2x_{n+2}) = (-4x_n, -4y_n); \]
\[ (x_{n+8}, y_{n+8}) = (-4x_{n+4}, -4y_{n+4}) = (16x_n, 16y_n). \]

Hence
\[ (x_8, y_8) = (16, 0) \quad \text{and} \quad (x_{8k}, y_{8k}) = (16^k, 0). \]