Mathematical Methods (Spring term 2016)

Aims
The aim of the module is to study mathematical methods useful in Physics.

Examinable Content
The module is split into five parts:
1. CALCULUS OF VARIATIONS
Shortest curve joining two points and Brachistochrone problems, Euler-Lagrange equation as a stationarity condition on

$$S = \int_{a}^{b} L(y(x), y'(x), x) \, dx,$$

with $y(a)$ and $y(b)$ fixed. If $L$ does not depend explicitly on $x$ then

$$H = y' \frac{\partial L}{\partial y'} - L$$

is constant.

Hamilton’s Principle (Lagrangian mechanics), $L = T - V$ for conservative forces, generalised coordinates and momenta, cyclic coordinates, Lagrangian for a charged particle using scalar and vector potentials.

Hanging rope problem, Lagrange multipliers, isoperimetric problems.

Geodesics and the metric tensor $g_{ij}$; transformation properties of $g_{ij}$. Einstein’s summation convention, inverse metric $g^{ij}$, Kronecker delta $\delta^{i}_{j}$.

2. COMPLEX VARIABLES
Complex differentiation, analytic functions, Cauchy-Riemann equations, entire functions, harmonic property of analytic functions.

Complex integration, Fundamental Theorem of Calculus, Cauchy’s theorem (anti derivative and standard form), Deformation theorem, Cauchy’s integral formula and applications (proof of Liouville’s theorem and proof that analytic functions are infinitely differentiable), Taylor’s theorem, isolated singularities and Laurent series, poles and essential singularities, meromorphic functions, Laurent’s theorem.

Residue theorem and application to computing real integrals, Principal value of real and complex integrals, Residue theorem with simple poles on the contour.

3. FOURIER TRANSFORMS
Review of Fourier transforms and Fourier integrals, computation of Fourier transforms using contour integration.
Heaviside and sign function, delta function as a derivative, delta function as a limit of smooth functions, definition through the integral formula

\[ \int_{-\infty}^{\infty} f(x) \delta(x - a) \, dx = f(a). \]

Properties of the delta function, Fourier transform (and Fourier integral representation) of the delta function. Higher dimensional delta functions and Green’s functions. Application of Fourier transforms to solving linear ODEs and PDEs (eg. driven oscillator ODEs and Laplace’s equation in an infinite strip and half-plane).

4. TENSORS

5 NUMERICAL METHODS
Numerical integration (trapezium rule and Simpson’s rule), Newton-Raphson method, Runge-Kutta algorithm.

Examination
2 hour paper with 5 questions (answer question 1 and two other questions. Question 1 comprises 8 short answer questions covering the whole module and is worth 40%. Questions 2 to 5 are worth 30% each.

Mathematical Proofs
The examination will predominantly be a test of technical skill rather than the ability to reproduce proofs. However, students are expected to be able to give short proofs (of results given in the lectures and problem sheets or simple results not seen previously). Students will not be asked for any long proofs, e.g. Taylor’s theorem.