

# Distributed Hypothesis Testing Over Discrete Memoryless Channels

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## Abstract

A distributed binary hypothesis testing problem is studied in which multiple observers transmit their observations to a remote detector over orthogonal discrete memoryless channels. The detector uses the received information from the observers along with its own observations to test for the joint distribution of the data. Single-letter inner and outer bounds for the type 2 error exponent (T2EE) is established for the special case of testing against conditional independence (TACI), in the regime of vanishing type 1 error probability constraint. Specializing this result for the one-encoder problem, a single-letter characterization of the optimal T2EE is obtained. For the general hypothesis testing problem, a lower bound on the T2EE is established by using a separation based scheme that performs independent channel coding and hypothesis testing. It is shown that this separation based scheme recovers the optimal T2EE when specialized for the case of TACI. A joint hypothesis testing and channel coding scheme based on hybrid coding is also proposed and the T2EE achievable under this scheme is analyzed. In general, this scheme is expected to achieve a better performance compared to the separation based scheme.

## I. INTRODUCTION

Given data samples, statistical hypothesis testing (HT) deals with the problem of ascertaining the true assumption, that is, the true hypothesis, about the data from among a set of hypotheses. In modern communication networks (like in sensor networks, cloud computing and Internet of things (IoT)), data is gathered at multiple remote nodes, referred to as *observers*, and transmitted over noisy links to another node for further processing. Often, there is some prior statistical

knowledge available about the data, for example, that the joint probability distribution of the data belongs to a certain prescribed set. In such scenarios, it is of interest to identify the true underlying probability distribution, and this naturally leads to the problem of distributed HT over noisy channels, which is depicted in Fig. 1. Each encoder  $l$ ,  $l = 1, \dots, L$ , observes  $k$  samples independent and identically distributed (i.i.d) according to  $P_{U_l}$ , and communicates its observation to the detector by  $n$  uses of the discrete memoryless channel (DMC), characterized by the conditional distribution  $P_{Y_l|X_l}$ . The detector decides between the two hypotheses,  $H_0$  and  $H_1$ , based on the channel outputs  $Y_1^n, \dots, Y_L^n$  as well as its own observations  $V^k$  and  $Z^k$ , where  $H_0$  (resp.  $H_1$ ) is the hypothesis that the data  $(U_1, \dots, U_L, V, Z)$  is distributed according to the joint distribution  $P_{U_1 \dots U_L V Z}$  (resp.  $Q_{U_1 \dots U_L V Z}$ ). Our goal is to characterize the set of achievable type 2 error exponents (T2EE's) for a prescribed constraint on the type 1 error probability. We will refer to this problem as the *general hypothesis testing with side-information* (GHTS) problem. The instance of the GHTS problem in which  $H_0 : P_{U_1 \dots U_L V Z}$  and  $H_1 : P_{U_1 \dots U_L | Z} \times P_{V|Z} \times P_Z$  will be referred to as the *testing against conditional independence* (TACI) problem. The special cases of the GHTS and the TACI problem when the side information  $Z$  is absent is referred to as the *testing against independence* (TAI) and the *general hypothesis testing* (GHT) problem, respectively.

In the non-distributed scenario in which, the detector performs a binary HT on the probability distribution of the data observed directly, the optimal T2EE is characterized by the well-known lemma of Stein [1] (see also [2]). The study of distributed statistical inference under communication constraints was conceived by Berger in [3]. In [3], and in the follow up literature summarized below, communication from the observers to the detector are assumed to be over rate-limited error-free channel. Ahlswede and Csiszár studied the GHT problem for the case of a single observer ( $L = 1$ ) [4]. They proved a tight single-letter characterization of the optimal T2EE for the TAI problem and also established a lower bound for the GHT problem. Furthermore, they also proved a *strong converse* which states that the optimal achievable T2EE is independent of the constraint on the type 1 error probability. A more general lower bound for the TAI problem with a single observer is established by Han [5], which recovers the corresponding lower bound in [4]. Han also considered complete data compression in a related setting where either  $U_1$ , or  $V$ , or both (also referred to as two-sided compression setting) are compressed and communicated to the detector using a message set of cardinality two. It is shown that, asymptotically, the optimal T2EE achieved in these three settings are equal. In contrast, even the TAI problem with

two-sided compression and general rate constraints remains open till date. Shalaby et. al [6] extended the complete data compression result of Han to show that the optimal T2EE is not improved even if the rate constraint is relaxed to that of zero-rate compression (sub-exponential message set with respect to block-length). Shimokawa et. al [7] obtained a better lower bound for the GHT problem by considering quantization and binning at the encoder along with a minimum empirical-entropy decoder. Rahman and Wagner [8] established inner bound for the TACI problem with  $L$  observers, by performing quantization and binning at the encoders. This quantize-bin-test bound is then shown to be tight, and also to coincide with the one achieved by the Shimokawa-Han-Amari scheme in [7] for the case of a single observer, thereby implying the optimality of both these schemes. The optimal T2EE for the TAI problem with two decision centers is obtained in [9], where the encoder communicates to both detectors via a common bit-pipe in addition to individual private bit-pipes to each. The TACI problem with multiple observers is still open, although a special case has been solved in [10] when the observed data follows a certain Markovian condition. The T2EE for more complex settings involving interaction between two observers, where one of the observer also acts as the detector has also been studied in [11], [12]. The observers exchange messages over a noiseless link for  $K$  rounds of interaction under a constraint on the total exchange rate. On completion of  $K$  rounds, the HT decision is taken at the observer which receives the last message. The optimal T2EE for TAI in this model with  $K = 1$  and  $K > 1$  is obtained in [11] and [12], respectively. A lower bound on the optimal T2EE for the general HT case in this setting is established in [13]. The authors also prove a single-letter expression for the optimal T2EE in the zero-rate compression regime, analogous to that of [6]. When the detector also performs lossy source reconstruction in addition to hypothesis testing, the set of all simultaneously achievable T2EE-distortion pairs for the GHT problem is studied in [14]. Therein, the authors also prove a single-letter characterization of the T2EE-distortion region for the special case of TAI. Recently, the GHT problem has been studied in a multi-hop network scenario where the communication from the observer to the detector happens over a relay network [16].

While the works mentioned above have studied the unsymmetric case of focusing on the T2EE for a constraint on the type 1 error probability, other works have analyzed the trade-off between the type 1 and type 2 error probabilities in the exponential sense. In this direction, the optimal trade-off between the type 1 and type 2 error exponents in the non-distributed scenario is obtained in [17]. The distributed version of this problem is first studied in [18], where inner bounds on the

above trade-off are established for the GHT problem. This problem has also been explored from an information geometric perspective for the zero-rate compression scenario in [19] and [20], which provides further insight into the geometric properties of the optimal trade-off between the two exponents. A Neyman-Pearson like test for the GHT problem in the zero-rate compression scenario is proposed in [21], which in addition to achieving the optimal trade-off between the two exponents, also achieves the optimal second order asymptotic performance among all symmetric (type-based) encoding schemes. However, the optimal trade-off between the type 1 and type 2 error exponents for the general distributed HT problem remains open till date. Recently, an inner bound for this trade-off is obtained in [22], by using the reliability function of the optimal channel detection codes.

In contrast, HT in distributed settings that involve communication over noisy channels has received relatively less attention in the past. In noiseless rate-limited settings, the encoder can reliably communicate its observation subject to a rate constraint. However, this is no longer the case in noisy settings, which complicates the study of error exponents in HT. A measure of the noisiness of the channel is the so-called reliability function  $E(R)$  (function of the communication rate  $R$ ) of the channel [23].  $E(R)$  denotes the exponent (first order) of best asymptotic decay of the probability of error achievable in channel coding when the rate of messages is  $R$ . It is reasonable to expect that  $E(R)$  plays a role in the characterization of the achievable T2EE. The problem of designing a channel codebook that achieves a reliability of  $E(R)$  is an open problem in general. However, it is well known that  $E(R) \geq E_r(R)$ , where  $E_r(R)$  is the random coding exponent [23]. As the name suggests, the existence of a channel codebook achieving  $E_r(R)$  can be shown by the standard random coding method.

The goal of this paper is to study the best attainable T2EE for the GHTS problem, and obtain a computable characterization of the same. Although a complete solution is not to be expected (since even the corresponding noiseless case is still open), the aim is to provide an achievable scheme for the general problem, and to discuss special cases in which a tight characterization can be obtained. The main contributions can be summarized as follows. We establish single-letter lower and upper bounds on the achievable T2EE for the TACI problem with multiple observers. This is done by first mapping the problem to an equivalent joint source channel coding (JSCC) problem with helpers. The Berger-Tung bounds [24] and the source-channel separation theorem in [25] are then used to obtain the desired bounds. Subsequently, these bounds are shown to be tight for the special case of a single observer. This tight single-letter characterization for TACI

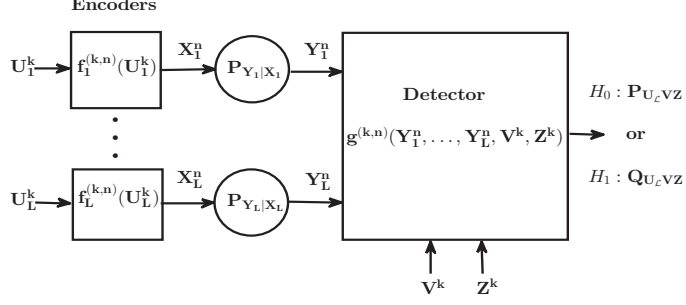


Fig. 1: Illustration of the system model.

reveals that the optimal T2EE depends only on the marginal distributions of the observed data and the channel rather than on their joint distribution. Motivated by this, we obtain a lower bound on the T2EE for the GHTS problem for the case of a single observer, by using a separation based scheme that performs independent hypothesis testing and channel coding. This scheme recovers the optimal T2EE for TACI derived previously. We then analyze the T2EE achieved by a generalization of the separation based scheme, in which, hybrid coding is used to communicate the observations from the encoder to the detector.

The rest of the paper is organized as follows. In Section II, we introduce the system model, definitions and supporting lemmas. Following this, we introduce the  $L$ -helper JSCC problem and obtain lower and upper bounds for the achievable T2EE for the TACI problem in Section III. In Section IV, we describe the separation based scheme followed by the hybrid coding based scheme for the GHTS problem, and state the T2EE attained by these schemes. The proofs of these results are presented in the Appendices. Finally, we conclude the paper in Section V.

#### A. Notations

Random variables (r.v.'s) are denoted by capital letters (e.g.,  $X$ ), their realizations by the corresponding lower case letters (e.g.,  $x$ ), and their support by calligraphic letters (e.g.,  $\mathcal{X}$ ). The cardinality of  $\mathcal{X}$  is denoted by  $|\mathcal{X}|$ . The joint distribution of r.v.'s  $X$  and  $Y$  is denoted by  $P_{XY}$  and its marginals by  $P_X$  and  $P_Y$ .  $X - Y - Z$  denotes that  $X$ ,  $Y$  and  $Z$  form a Markov chain. For  $m, l \in \mathbb{Z}^+$ ,  $X^m$  denotes the sequence  $X_1, \dots, X_m$ , while  $X_l^m$  denotes the sequence  $X_{l,1}, \dots, X_{l,m}$ .  $D(P_X || Q_X)$ ,  $H_{P_X}(X)$ ,  $H_{P_{XY}}(X|Y)$  and  $I_{P_{XY}}(X; Y)$  represent the standard quantities of Kullback-Leibler (KL) divergence between distributions  $P_X$  and  $Q_X$ , the entropy of  $X$  with distribution  $P_X$ , the conditional entropy of  $X$  given  $Y$  and the mutual

information between  $X$  and  $Y$  with joint distribution  $P_{XY}$ , respectively. When the distribution of the r.v.'s involved are clear from the context, the entropic and mutual information quantities are denoted simply by  $I(X;Y)$ ,  $H(X)$  and  $H(X|Y)$ , respectively. Given realizations  $X^n = x^n$  and  $Y^n = y^n$ ,  $H_e(x^n|y^n)$  denote the conditional empirical entropy defined as

$$H_e(x^n|y^n) = H_{P_{\tilde{X}\tilde{Y}}}(\tilde{X}|\tilde{Y}), \quad (1)$$

where  $P_{\tilde{X}\tilde{Y}}$  denote the joint type of  $(x^n, y^n)$ . For  $a \in \mathbb{R}^+$ ,  $[a]$  denotes the set of integers  $\{1, 2, \dots, [a]\}$ . All logarithms considered in this paper are with respect to the base 2. For any set  $\mathcal{G}$ ,  $\mathcal{G}^c$  denotes the set complement.  $a_k \xrightarrow{(k)} 0$  indicates that  $\lim_{k \rightarrow \infty} a_k = 0$ . For functions  $f_1 : \mathcal{A} \rightarrow \mathcal{B}$  and  $f_2 : \mathcal{B} \rightarrow \mathcal{C}$ ,  $f_2 \circ f_1$  denotes function composition. Finally,  $\mathbb{1}(\cdot)$ ,  $O(\cdot)$  and  $o(\cdot)$  denote the indicator function, the Big-o and the Little-o notation of Landau, respectively.

## II. SYSTEM MODEL AND DEFINITIONS

All the r.v.'s considered henceforth are discrete with finite support. Let  $k, n \in \mathbb{Z}^+$  be arbitrary. Let  $\mathcal{L} = \{1, \dots, L\}$  denote the set of observers which communicate to the detector over orthogonal noisy channels, as shown in Fig. 1. For  $l \in \mathcal{L}$ , encoder  $l$  observes  $U_l^k$ , and transmits codeword  $X_l^n = f_l^{(k,n)}(U_l^k)$ , where  $f_l^{(k,n)} : \mathcal{U}_l^k \rightarrow \mathcal{X}_l^n$  is a stochastic mapping. Let  $\tau := \frac{n}{k}$  denote the *bandwidth ratio*. The channel output  $Y_{\mathcal{L}}^n$  is given by the probability law  $P_{Y_{\mathcal{L}}^n|X_{\mathcal{L}}^n}(y_{\mathcal{L}}^n|x_{\mathcal{L}}^n) = \prod_{l=1}^L \prod_{j=1}^n P_{Y_l|X_l}(y_{l,j}|x_{l,j})$ , i.e., the channels between the observers and the detector are independent of each other and memoryless. Depending on the received symbols  $Y_{\mathcal{L}}^n$  and its own observations  $(V^k, Z^k)$ , the detector makes a decision between the two hypotheses  $H_0 : P_{U_{\mathcal{L}}VZ}$  or  $H_1 : Q_{U_{\mathcal{L}}VZ}$  according to the map  $g^{(k,n)} : \mathcal{Y}_{\mathcal{L}}^n \times \mathcal{V}^k \times \mathcal{Z}^k \rightarrow \{H_0, H_1\}$ . Let  $\mathcal{A}_{(k,n)} \subseteq \mathcal{Y}_{\mathcal{L}}^n \times \mathcal{V}^k \times \mathcal{Z}^k$  denote the acceptance region for  $H_0$ , i.e.,  $g^{(k,n)}(y_{\mathcal{L}}^n, v^k, z^k) = H_0$  if  $(y_{\mathcal{L}}^n, v^k, z^k) \in \mathcal{A}_{(k,n)}$  and  $g^{(k,n)}(y_{\mathcal{L}}^n, v^k, z^k) = H_1$  otherwise. It is assumed that  $P_{U_{\mathcal{L}}VZ} \ll Q_{U_{\mathcal{L}}VZ}$ , i.e., the joint distribution of the data under  $H_0$  is absolutely continuous<sup>1</sup> with respect to that under  $H_1$ .

Let  $\bar{\alpha} \left( k, n, f_1^{(k,n)}, \dots, f_L^{(k,n)}, g^{(k,n)} \right) := P_{Y_{\mathcal{L}}^n V^k Z^k}(\mathcal{A}_{(k,n)}^c)$  and  $\bar{\beta} \left( k, n, f_1^{(k,n)}, \dots, f_L^{(k,n)}, g^{(k,n)} \right) := Q_{Y_{\mathcal{L}}^n V^k Z^k}(\mathcal{A}_{(k,n)})$  denote, respectively, the type 1 and type 2 error probabilities for the encoding

<sup>1</sup>This assumption is necessary for the Kullback-Leibler divergence quantities like  $D(P_{Y^n V^k Z^k} || Q_{Y^n V^k Z^k})$  that characterize the T2EE to be finite.

functions  $f_1^{(k,n)}, \dots, f_L^{(k,n)}$  and decision rule  $g^{(k,n)}$ . Define

$$\beta' \left( k, n, f_1^{(k,n)}, \dots, f_L^{(k,n)}, \epsilon \right) := \inf_{g^{(k,n)}} \bar{\beta} \left( k, n, f_1^{(k,n)}, \dots, f_L^{(k,n)}, g^{(k,n)} \right), \quad (2)$$

such that

$$\bar{\alpha} \left( k, n, f_1^{(k,n)}, \dots, f_L^{(k,n)}, g^{(k,n)} \right) \leq \epsilon,$$

and

$$(Z^k, V^k, U_{l^c}^k) - U_l^k - X_l^n - Y_l^n, \quad l \in \mathcal{L},$$

where  $X_l^n = f_l^{(k,n)}(U_l^k)$ ,  $l^c := \mathcal{L} \setminus l$  and

$$\beta(k, \tau, \epsilon) := \inf_{\substack{f_1^{(k,n)}, \dots, f_L^{(k,n)} \\ n \leq \tau k}} \beta' \left( k, n, f_1^{(k,n)}, \dots, f_L^{(k,n)}, \epsilon \right).$$

**Definition 1.** A T2EE  $\kappa$  is said to be  $(\tau, \epsilon)$  achievable for the GHT problem if there exists a sequence of integers  $k$ , corresponding sequences of encoding functions  $f_1^{(k, n_k)}, \dots, f_L^{(k, n_k)}$  and decoding functions  $g^{(k, n_k)}$  such that  $n_k \leq \tau k$ ,  $\forall k$ , and for any  $\delta > 0$ ,

$$\limsup_{k \rightarrow \infty} \frac{\log(\beta(k, \tau, \epsilon))}{k} \leq -(\kappa - \delta). \quad (4)$$

Let

$$\kappa(\tau, \epsilon) := \sup \{ \kappa' : \kappa' \text{ is } (\tau, \epsilon) \text{ achievable} \}, \text{ and} \quad (5)$$

$$\theta(\tau) := \sup_{k \in \mathbb{Z}^+} \theta(k, \tau), \quad (6)$$

where

$$\theta(k, \tau) := \sup_{\substack{f_1^{(k,n)}, \dots, f_L^{(k,n)} \\ n \leq \tau k}} \frac{D(P_{Y_{\mathcal{L}}^n V^k Z^k} || Q_{Y_{\mathcal{L}}^n V^k Z^k})}{k}. \quad (7)$$

Next, we obtain single-letter inner and outer bounds on  $\kappa(\tau, \epsilon)$  for the problem of TACI over noisy channels. Our approach is similar to that in [4], in that we first obtain bounds on  $\kappa(\tau, \epsilon)$  in terms of  $\theta(\tau)$ , and subsequently show that  $\theta(\tau)$  has a single-letter characterization in terms of information theoretic quantities. We establish this characterization by considering an equivalent JSCC problem with noisy helpers.

**Lemma 2.** *For the GHT problem with any bandwidth ratio  $\tau \geq 0$ , we have*

- (i)  $\limsup_{k \rightarrow \infty} \frac{\log(\beta(k, \tau, \epsilon))}{k} \leq -\theta(\tau), \forall \epsilon \in (0, 1).$
- (ii)  $\lim_{\epsilon \rightarrow 0} \liminf_{k \rightarrow \infty} \frac{\log(\beta(k, \tau, \epsilon))}{k} \geq -\theta(\tau).$

*Proof:* Note that for  $\tau = 0, n = 0$ , which implies that the observer does not transmit anything. Then, from Stein's lemma [4] for ordinary hypothesis testing, (i) and (ii) follows, where  $\theta(0) := D(P_{VZ} || Q_{VZ})$ . When  $\tau > 0$ , the proof is similar to that of Theorem 1 in [4]. Here, we prove (i), which states that a T2EE of  $\theta(\tau)$  is achievable. The proof of (ii) follows in a straightforward manner from the proof given in [4] and is omitted here. Let  $k \in \mathbb{Z}^+$  and  $\tilde{\epsilon} > 0$  be arbitrary, and  $\tilde{n}_k, \tilde{f}_l^{(k, \tilde{n}_k)}, l \in \mathcal{L}$ , and  $\tilde{Y}_{\mathcal{L}}^{\tilde{n}_k}$  be the channel block length, encoding functions and channel outputs respectively, such that  $k\theta(k, \tau) - D(P_{Y_{\mathcal{L}}^{\tilde{n}_k} V^k Z^k} || Q_{Y_{\mathcal{L}}^{\tilde{n}_k} V^k Z^k}) < k\tilde{\epsilon}$ . For each  $l \in \mathcal{L}$ ,  $\{\tilde{Y}_l^{\tilde{n}_k}(j)\}_{j \in \mathbb{Z}^+}$  form an infinite sequence of i.i.d. r.v.'s indexed by  $j$ . Hence, by the application of Stein's Lemma [4] to the sequences  $\{\tilde{Y}_{\mathcal{L}}^{\tilde{n}_k}(j), V^k(j), Z^k(j)\}_{j \in \mathbb{Z}^+}$ , we have

$$\begin{aligned} \limsup_{j \rightarrow \infty} \frac{\log(\beta(kj, \tau, \epsilon))}{kj} &\leq \frac{-D(P_{Y_{\mathcal{L}}^{\tilde{n}_k} V^k Z^k} || Q_{Y_{\mathcal{L}}^{\tilde{n}_k} V^k Z^k})}{k}, \\ &\leq -(\theta(k, \tau) - \tilde{\epsilon}). \end{aligned} \quad (8)$$

For  $m \geq kj$ ,  $\beta(m, \tau, \epsilon) \leq \beta(kj, \tau, \epsilon)$ . Hence,

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{\log(\beta(m, \tau, \epsilon))}{m} &\leq \limsup_{j \rightarrow \infty} \frac{\log(\beta(kj, \tau, \epsilon))}{kj} \\ &\leq -(\theta(k, \tau) - \tilde{\epsilon}). \end{aligned}$$

Note that the left hand side (L.H.S) of the above equation does not depend on  $k$ . Taking infimum with respect to  $k$  on both sides of the equation and noting that  $\tilde{\epsilon}$  is arbitrary, proves (i). ■

**Remark 3.** *Part (ii) of Lemma 2 is known as the weak converse for the HT problem in the literature and (i) and (ii) together imply that  $\theta(\tau)$  is the optimal T2EE as  $\epsilon \rightarrow 0$ , i.e.,  $\lim_{\epsilon \rightarrow 0} \kappa(\tau, \epsilon) = \theta(\tau)$ . For  $\epsilon = 1$ , note that the optimal T2EE is  $\infty$  since it is possible to choose  $\mathcal{A}_{(k, n)}^c = \mathcal{Y}^n \times \mathcal{V}^k \times \mathcal{Z}^k$  while satisfying the type 1 error probability constraint.*

Part (i) of Lemma 2 proves the achievability of the T2EE  $\theta(\tau)$  using Stein's Lemma. In



Appendix A, we show an explicit proof of the achievability by computing the type 1 and type 2 errors for a block-memoryless stochastic encoding function at the observer and a joint typicality detector. Note that for the TACI problem, the KL-divergence in (7) becomes mutual information, and we have

$$\theta(\tau) = \sup_{\substack{f_1^{(k,n)}, \dots, f_L^{(k,n)} \\ k, n \leq \tau k}} \frac{I(V^k; Y_{\mathcal{L}}^n | Z^k)}{k} \text{ s.t.} \\ (Z^k, V^k) - U_l^k - X_l^n - Y_l^n, \quad l \in \mathcal{L}.$$

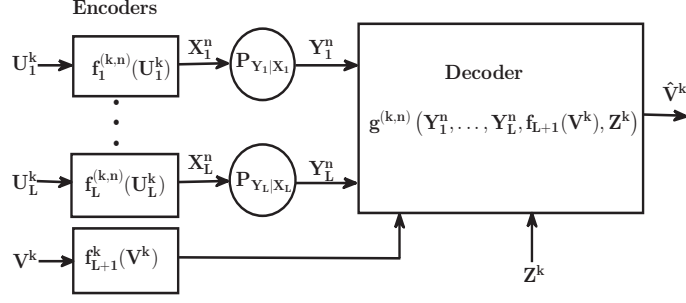
Although Lemma 2 implies that  $\theta(\tau)$  is an achievable T2EE, it is in general not computable as it is characterized in terms of a multi-letter expression. However, as we will show below, for the TACI problem, single-letter bounds for  $\theta(\tau)$  can be obtained. By the memoryless property of the sequences  $V^k$  and  $Z^k$ , we can write

$$\theta(\tau) = H(V|Z) - \inf_{\substack{f_1^{(k,n)}, \dots, f_L^{(k,n)} \\ k, n \leq \tau k}} \frac{H(V^k | Y_{\mathcal{L}}^n, Z^k)}{k} \text{ s.t.} \quad (9) \\ (Z^k, V^k) - U_l^k - X_l^n - Y_l^n, \quad l \in \mathcal{L}.$$

In the next section, we introduce the  $L$ -helper JSCC problem and show that the multi-letter characterization of this problem coincides with obtaining the infimum in (9). The computable characterization of the lower and upper bounds for (9) then follows from the single-letter outer and inner bounds available for the  $L$ -helper JSCC problem.

### III. $L$ -HELPER JSCC PROBLEM

Consider the model shown in Fig. 2 where there are  $L + 2$  correlated discrete memoryless sources  $(U_{\mathcal{L}}, V, Z)$  i.i.d. with joint distribution  $P_{U_{\mathcal{L}}VZ}$ . For  $1 \leq l \leq L$ , encoder  $f_l^{(k,n)} : \mathcal{U}_l^k \rightarrow \mathcal{X}_l^n$  observes the sequence  $U_l^k$  and transmits  $X_l^n = f_l^{(k,n)}(U_l^k)$  over the corresponding DMC  $P_{Y_l|X_l}$ , while encoder  $f^k : \mathcal{V}^k \rightarrow \mathcal{M} = \{1, \dots, 2^{kR}\}$  observes  $V^k$ , and outputs  $M = f^k(V^k)$ . Decoder  $g^{(k,n)}$  has access to side-information  $Z^k$ , receives  $f_{L+1}^k(V^k)$  error-free, observes  $Y_{\mathcal{L}}^n$  and outputs  $\hat{V}^k$  according to the map  $g^{(k,n)} : \mathcal{Y}_{\mathcal{L}}^n \times \mathcal{M} \times \mathcal{Z}^k \rightarrow \hat{\mathcal{V}}^k$ . The goal of  $g^{(k,n)}$  is to reconstruct  $V^k$  losslessly. We will first establish the multi-letter characterization of the rate region of the  $L$ -helper JSCC. Prior to stating the result, we require some new definitions.

Fig. 2:  $L$ -helper JSCC problem.

**Definition 4.** For a given bandwidth ratio  $\tau$ , a rate  $R$  is said to be achievable for the  $L$ -helper JSCC problem if for every  $\lambda \in (0, 1]$ , there exist sequences of positive numbers  $\delta_k$  tending to 0 as  $k \rightarrow \infty$ , encoders  $f^k(\cdot)$ ,  $f_l^{(k,n_k)}(\cdot)$ , and decoder  $g^{(k,n_k)}(\cdot, \cdot, \cdot)$  such that  $n_k \leq \tau k$  and

$$\mathbb{P} \left( g_h^{(k,n_k)}(Y^{n_k}, M, Z^k) = V^k \right) \geq 1 - \lambda \text{ and}$$

$$\frac{\log(|\mathcal{M}|)}{k} \leq R + \delta_k.$$

Let

$$R(\tau) := \inf \{ R : R \text{ is achievable.} \} \quad (10)$$

We next show that the problem of obtaining the infimum in (9) coincides with the multi-letter characterization of  $R(\tau)$  in (10). Define

$$R_k := \inf_{\substack{f_1^{(k,n)}, \dots, f_L^{(k,n)} \\ n \leq \tau k}} \frac{H(V^k | Y_{\mathcal{L}}^n, Z^k)}{k} \text{ s.t.} \quad (11)$$

$$(Z^k, V^k) - U_l^k - X_l^n - Y_l^n, \quad l \in \mathcal{L}.$$

**Theorem 5.** For the  $L$ -helper JSCC problem,

$$R(\tau) = \inf_k R_k.$$

*Proof:* The proof is given in Appendix B. ■

Having shown the equivalence between the multi-letter characterizations of  $\theta(\tau)$  for the TACI problem over noisy channels and  $R(\tau)$  for the  $L$ -helper JSCC problem, our next step is to obtain computable single-letter lower and upper bounds on  $R(\tau)$ , which can then be used to

obtain bounds on  $\theta(\tau)$ . For this purpose, we use the *source-channel separation theorem* [25, Th. 2.4] for orthogonal multiple access channels. The theorem states that all achievable average distortion-cost tuples in a multi-terminal JSCC (MT-JSCC) problem over an orthogonal multiple access channel (MAC) can be obtained by the intersection of the rate-distortion region and the MAC region. We need a slight generalization of this result when there is side information  $Z$  at the decoder, which can be proved similarly to [25]. Note that the  $L$ -helper JSCC problem is a special case of the MT-JSCC problem with  $L + 1$  correlated sources  $P_{U_{\mathcal{L}}V}$  and side information  $Z$  available at the decoder, where the objective is to reconstruct  $V$  losslessly. Although the above theorem proves that separation holds, a single-letter expression is not available in general for the multi-terminal rate distortion problem [24]. However, single-letter inner and outer bounds have been given in [24], which enable us to obtain single-letter upper and lower bounds on  $R(\tau)$  as follows.

**Theorem 6.** *For  $\mathcal{G} \subseteq \mathcal{L}$ , let  $C_{\mathcal{G}} := \sum_{l \in \mathcal{G}} C_l$ , where  $C_l := \max_{P_{X_l}} I(X_l; Y_l)$  denotes the capacity of the channel  $P_{Y_l|X_l}$ . For the  $L$ -helper JSCC problem with bandwidth ratio  $\tau$ , define*

$$R^i(\tau) := \inf_{W_{\mathcal{L}}} \max_{\mathcal{G} \subseteq \mathcal{L}} F_{\mathcal{G}}, \quad (12)$$

where

$$F_{\mathcal{G}} = H(V|W_{\mathcal{G}^c}, Z) + I(U_{\mathcal{G}}; W_{\mathcal{G}}|W_{\mathcal{G}^c}, V, Z) - \tau \sum_{l \in \mathcal{G}} C_l$$

for some auxiliary r.v.'s  $W_l$ ,  $l \in \mathcal{L}$ , such that

$$(Z, V, U_{l^c}, W_{l^c}) - U_l - W_l, \quad (13)$$

$$|\mathcal{W}_l| \leq |\mathcal{U}_l| + 4, \text{ and } \forall \mathcal{G} \subseteq \mathcal{L},$$

$$I(U_{\mathcal{L}}; W_{\mathcal{G}}|V, W_{\mathcal{G}^c}, Z) \leq \tau C_{\mathcal{G}}. \quad (14)$$

Similarly, let  $R^o(\tau)$  denote the right hand side (R.H.S) of (12), when the auxiliary r.v.'s  $W_l$ ,  $l \in \mathcal{L}$  satisfy  $|\mathcal{W}_l| \leq |\mathcal{U}_l| + 4$ , Eqn.(14) and

$$(V, U_{l^c}, Z) - U_l - W_l. \quad (15)$$

Then,

$$R^o(\tau) \leq R(\tau) \leq R^i(\tau), \text{ and} \quad (16)$$

$$H(V|Z) - R^i(\tau) \leq \theta(\tau) \leq H(V|Z) - R^o(\tau). \quad (17)$$

*Proof:* From the source-channel separation theorem, an upper bound on  $R(\tau)$  can be obtained by the intersection of the Berger-Tung (BT) inner bound [24, Th. 12.1] with the capacity region  $(C_1, \dots, C_L, C_{L+1})$ , where  $C_{L+1}$  is the rate available over the noiseless link from the encoder of source  $V$  to the decoder. Writing the BT inner bound <sup>2</sup> explicitly, we obtain that for all  $\mathcal{G} \subseteq \mathcal{L}$  (including the null-set),

$$\begin{aligned} I(U_{\mathcal{G}}; W_{\mathcal{G}}|V, W_{\mathcal{G}^c}, Z) &\leq \sum_{l \in \mathcal{G}} \tau C_l, \\ I(U_{\mathcal{G}}; W_{\mathcal{G}}|V, W_{\mathcal{G}^c}, Z) + H(V|W_{\mathcal{G}^c}, Z) &\leq \sum_{l \in \mathcal{G}} \tau C_l + C_{L+1}, \end{aligned}$$

where the auxiliary r.v.'s  $W_{\mathcal{L}}$  satisfy (13) and  $|\mathcal{W}_l| \leq |\mathcal{U}_l| + 4$ . Taking the infimum of  $C_{L+1}$  over all such  $W_{\mathcal{L}}$  and denoting it by  $R^i(\tau)$ , we obtain the second inequality in (16). The other direction in (16) is obtained similarly by using the BT outer bound [24, Th. 12.2]. Since  $R(\tau)$  is equal to the infimum in (9), substituting (16) in (9) proves (17). ■

The BT inner bound is tight for the two terminal case, when one of the distortion requirements is zero (lossless) [24, Ch.12]. Thus, we have the following result (for convenience, we drop the index 1 from the associated variables).

**Proposition 7.** *For the TACI problem with  $L = 1$  and bandwidth ratio  $\tau$ ,*

$$\theta(\tau) = \sup_W I(V; W|Z), \quad (18)$$

$$\text{such that } I(U; W|Z) \leq \tau C, \quad (19)$$

$$(Z, V) - U - W, \quad |\mathcal{W}| \leq |\mathcal{U}| + 4. \quad (20)$$

*Proof:* Note that the Markov chain conditions in (13) and (15) are identical for  $L = 1$ .

<sup>2</sup> $R^i(\tau)$  can be improved by introducing a time sharing r.v.  $T$  (independent of all the other r.v.'s) in the BT inner bound, but it is omitted here for simplicity.

Hence,

$$R^i(\tau) = R^o(\tau) = R(\tau). \quad (21)$$

Using the BT inner bound in [24, Ch.12], we obtain  $R(\tau)$  as the infimum of  $R'$  such that

$$H(V|Z, W) \leq R', \quad (22)$$

$$I(U; W|V, Z) \leq \tau C, \quad (23)$$

$$H(V|Z, W) + I(U; W|Z) \leq \tau C + R', \quad (24)$$

for some auxiliary r.v.  $W$  satisfying (20). Hence,

$$\begin{aligned} R(\tau) = \inf_W \max & \left( H(V|W, Z), H(V|W, Z) \right. \\ & \left. + I(U; W|Z) - \tau C \right), \end{aligned} \quad (25)$$

such that (20) and (23) hold. We next prove that (25) can be simplified as

$$R(\tau) = \inf_W H(V|Z, W), \quad (26)$$

such that (19) and (20) are satisfied. This is done by showing that, for every r.v.  $W$  for which  $I(U; W|Z) > \tau C$ , there exists a r.v.  $\bar{W}$  such that

$$I(U; \bar{W}|Z) = \tau C, \quad (27)$$

$$H(V|\bar{W}, Z) \leq H(V|W, Z) + I(U; W|Z) - \tau C, \quad (28)$$

and (20) and (23) are satisfied with  $W$  replaced by  $\bar{W}$ . Setting

$$\bar{W} = \begin{cases} W, & \text{with probability } 1-p, \\ \text{constant}, & \text{with probability } p, \end{cases} \quad (29)$$

suffices, where  $p$  is chosen such that  $I(U; \bar{W}|Z) = \tau C$ . To see this, first note that  $H(V|\bar{W}, Z)$  is an increasing function of  $p$ , while  $I(U; \bar{W}|Z)$  and  $I(U; \bar{W}|V, Z)$  are decreasing functions of  $p$ . Hence, it is possible to choose  $p$  such that (27) and (23) are satisfied with  $\bar{W}$  in place of  $W$ . It is clear that such a choice of  $\bar{W}$  also satisfies (20). To complete the proof of (26), it remains

to be shown that for such a  $\bar{W}$ , (28) holds. We can write,

$$H(V|\bar{W}, Z) = (1 - p)H(V|W, Z) + pH(V|Z). \quad (30)$$

Taking derivative with respect to  $p$ , we obtain

$$\frac{d}{dp}H(V|\bar{W}, Z) = I(V; W|Z). \quad (31)$$

Similarly,

$$\frac{d}{dp}H(U|\bar{W}, Z) = I(U; W|Z). \quad (32)$$

For the Markov chain in (20), we have  $I(V; W|Z) \leq I(U; W|Z)$  by the data processing inequality [23]. Hence, we have that

$$\frac{d}{dp}H(V|\bar{W}, Z) \leq \frac{d}{dp}H(U|\bar{W}, Z). \quad (33)$$

From (33), it follows that

$$F(p) := H(V|\bar{W}, Z) + I(U; \bar{W}|Z) - \tau C \quad (34)$$

is a decreasing function of  $p$ . Together with the fact that  $H(V|\bar{W}, Z)$  is increasing with  $p$ , it follows that (28) is satisfied for  $\bar{W}$  chosen in (29). Having shown (26), (18) now follows from (17) and (21). This completes the proof.  $\blacksquare$

#### IV. T2EE FOR THE GHTS PROBLEM

As shown in Appendix A, the optimal T2EE for the GHT problem is achieved when the input to the channel  $X^n$  is generated correlated with the observed source sequence  $U^k$ , according to some conditional distribution  $P_{X^n|U^k}$  and the optimal test achieving the Stein's exponent, i.e., the Neyman-Pearson test, is performed on blocks of data (received and observed) at the detector. The encoder and the detector for such a scheme would be computationally complex to implement from a practical viewpoint. In this section, we propose and analyze the T2EE achieved by two simple schemes for the GHTS problem with  $L = 1$ . The first one is a separation based scheme that performs independent hypothesis testing and channel coding while the second one is a joint hypothesis testing and channel coding scheme that utilizes hybrid coding.

### A. Separate Hypothesis Testing and Channel Coding Scheme

In [7], Shimokawa et.al. obtained an achievable T2EE for the GHT problem ( $L = 1$ ) in the rate-limited noiseless channel setting using a quantization and binning scheme. In this scheme, the type of the observed sequence is sent by the encoder to the detector, which aids in the HT<sup>3</sup>. Since the number of types is polynomial in the block-length, these can be communicated error-free at asymptotically zero-rate. Intuitively, it is desirable to do the same in the noisy channel setting as well; however, this is not possible in general.

In this subsection, we propose a *separate hypothesis testing and channel coding* (SHTCC) scheme for the GHTS problem and analyze its T2EE. In the SHTCC scheme, the encoding and decoding functions are restricted to be of the form  $f^{(k,n)} = f_C^{(k,n)} \circ f_S^{(k)}$  and  $g^{(k,n)} = g_S^{(k)} \circ g_C^{(k,n)}$ , respectively. The source encoder  $f_S^{(k)} : \mathcal{U}^k \rightarrow \mathcal{M} = \{0, 1, \dots, \lceil 2^{kR} \rceil\}$  generates an index  $M \in \mathcal{M}$  based on the observed sequence  $U^k$  and the channel encoder  $f_C^{(k,n)} : \mathcal{M} \rightarrow \tilde{\mathcal{C}} = \{X^n(j), j \in [0 : \lceil 2^{kR} \rceil]\}$  maps  $M$  into the codeword  $X^n(M)$  from the channel codebook  $\tilde{\mathcal{C}}$ . Note that the rate of this coding scheme is  $\frac{kR}{n} = \frac{R}{\tau}$  bits per channel transmission. The channel decoder  $g_C^{(k,n)} : \mathcal{Y}^n \rightarrow \mathcal{M}$  maps the received sequence into an index  $M$ , and  $g_S^{(k)} : \mathcal{M} \times \mathcal{V}^k \rightarrow \{H_0, H_1\}$  outputs the result of the HT. The codewords  $X^n(j)$ ,  $j \in [1 : \lceil 2^{kR} \rceil]$ , in  $\tilde{\mathcal{C}}$  are generated i.i.d. according to the distribution  $P_X$  that achieves the exponent  $E_r(\frac{R}{\tau})$  for channel decoding error. We let  $M = 0$  denote a special *error message* indicating that the observed sequence  $U^k$  is not typical. Since maximum possible reliability is desirable for  $M = 0$ , it is clear that  $X^n(0)$  has to be chosen such that it achieves the best exponent for the error probability at the channel decoder. By generating all the  $\lceil 2^{kR} \rceil + 1$  codewords in  $\tilde{\mathcal{C}}$  i.i.d according to  $P_X$ , it is clear that this exponent is at least  $E_r(\frac{R}{\tau})$ , but it can be higher in some cases.

We next state as a result the T2EE achieved by the SHTC scheme. The subscript denoting the observer index from all the relevant variables is omitted as  $L = 1$ . Also, since the side-information  $Z$  and  $V$  play the same role for the GHT problem, it is enough to consider only one side-information, which we denote by  $V$ .

**Theorem 8.** *Consider the GHT problem with  $L = 1$ , bandwidth ratio  $\tau$ , and noisy channel  $P_{Y|X}$*

<sup>3</sup>In order to achieve the T2EE proposed in [7], it is in fact sufficient to send a message indicating whether the observed sequence is typical or not, instead of sending the type of the observed sequence.

with capacity  $C$ . Then,  $\kappa(\tau, \epsilon) \geq \kappa_s$  for  $0 < \epsilon \leq 1$ , where

$$\kappa_s := \sup_{(P_{W|U}, R) \in \mathcal{B}} \min (E_1(P_{W|U}), E_2(R, P_{W|U}), E_3(R, P_{W|U}, \tau), E_4(R, P_{W|U}, \tau)), \quad (35)$$

where

$$\mathcal{B} := \{(P_{W|U}, R) : I_P(U; W|V) \leq R < \tau C, V - U - W\},$$

$$E_1(P_{W|U}) := \min_{P_{\tilde{U}\tilde{V}\tilde{W}} \in \mathcal{T}_1(P_{UW}, P_{VW})} D(P_{\tilde{U}\tilde{V}\tilde{W}} \| Q_{UVW}), \quad (36)$$

$$E_2(R, P_{W|U}) := \begin{cases} \min_{P_{\tilde{U}\tilde{V}\tilde{W}} \in \mathcal{T}_2(P_{UW}, P_V)} D(P_{\tilde{U}\tilde{V}\tilde{W}} \| Q_{UVW}) + (R - I_P(U; W|V)), & \text{if } I_P(U; W) > R, \\ \infty, & \text{otherwise,} \end{cases} \quad (37)$$

$$E_3(R, P_{W|U}, \tau) := \begin{cases} \min_{P_{\tilde{U}\tilde{V}\tilde{W}} \in \mathcal{T}_3(P_{UW}, P_V)} D(P_{\tilde{U}\tilde{V}\tilde{W}} \| Q_{UVW}) + (R - I_P(U; W|V)) \\ \quad + \tau E_r\left(\frac{R}{\tau}\right), & \text{if } I_P(U; W) > R, \\ \min_{P_{\tilde{U}\tilde{V}\tilde{W}} \in \mathcal{T}_3(P_{UW}, P_V)} D(P_{\tilde{U}\tilde{V}\tilde{W}} \| Q_{UVW}) + I_P(V; W) + \tau E_r\left(\frac{R}{\tau}\right), & \text{otherwise,} \end{cases} \quad (38)$$

$$E_4(R, P_{W|U}, \tau) := \begin{cases} D(P_V \| Q_V) + (R - I_P(U; W|V)) + E_s(R, \tau), & \text{if } I_P(U; W) > R, \\ D(P_V \| Q_V) + I_P(V; W) + E_s(R, \tau), & \text{otherwise,} \end{cases} \quad (39)$$

$$P_{UVW} = P_{UV}P_{W|U}, \quad Q_{UVW} = Q_{UV}P_{W|U}, \quad E_s(R, \tau) \geq \tau E_r\left(\frac{R}{\tau}\right), \quad (40)$$

$$\mathcal{T}_1(P_{UW}, P_{VW}) := \{P_{\tilde{U}\tilde{V}\tilde{W}} \in \mathcal{T}(\mathcal{U} \times \mathcal{V} \times \mathcal{W}) : P_{\tilde{U}\tilde{W}} = P_{UW}, P_{\tilde{V}\tilde{W}} = P_{VW}\},$$

$$\mathcal{T}_2(P_{UW}, P_V) := \{P_{\tilde{U}\tilde{V}\tilde{W}} \in \mathcal{T}(\mathcal{U} \times \mathcal{V} \times \mathcal{W}) : P_{\tilde{U}\tilde{W}} = P_{UW}, P_{\tilde{V}} = P_V, H_P(W|V) \leq H(\tilde{W}|\tilde{V})\},$$

$$\mathcal{T}_3(P_{UW}, P_V) := \{P_{\tilde{U}\tilde{V}\tilde{W}} \in \mathcal{T}(\mathcal{U} \times \mathcal{V} \times \mathcal{W}) : P_{\tilde{U}\tilde{W}} = P_{UW}, P_{\tilde{V}} = P_V\}.$$

The proof of Theorem 8 is given in Appendix C. Although the expression  $\kappa_s$  in Theorem 8 appears complicated, the terms  $E_1(P_{W|U})$  to  $E_4(R, P_{W|U}, \tau)$  can be understood to correspond to exponents caused due to distinct events that can possibly lead to a type 2 error. Note that  $E_1(P_{W|U})$  and  $E_2(R, P_{W|U})$  are the same terms appearing in the exponent achieved by the Shimokawa-Han-Amari scheme [7] for the noiseless channel setting, while  $E_3(R, P_{W|U}, \tau)$  and



$E_4(R, P_{W|U}, \tau)$  are additional terms introduced due to the channel.  $E_3(R, P_{W|U}, \tau)$  corresponds to the event when the encoder transmits a message  $M \neq 0$ , but the channel decoder outputs  $\hat{M} \neq M$  and  $g_S^{(k)}(\hat{M}, V^k) = H_0$ .  $E_4(R, P_{W|U}, \tau)$  is due to the error event when the error message  $M = 0$  is transmitted, the channel decoder outputs  $\hat{M} \neq M$  and  $g_S^{(k)}(\hat{M}, V^k) = H_0$ .

The term  $E_s(R, \tau)$  in (40) requires further explanation. This corresponds to the best error exponent in channel coding that can be achieved for the error message  $M = 0$  when the codewords for the remaining  $\lceil 2^{kR} \rceil$  messages in  $\tilde{\mathcal{C}}$  achieve the random coding error exponent  $\tau E_r(\frac{R}{\tau})$ . As mentioned above,  $E_s(R, \tau)$  is at least equal to  $\tau E_r(\frac{R}{\tau})$ . However,  $E_s(R, \tau)$  may be significantly higher in some cases, for instance, when the channel has positive zero error capacity [23] and it is possible to choose one error-free channel codeword while simultaneously achieving  $\tau E_r(\frac{R}{\tau})$  for the remaining  $\lceil 2^{kR} \rceil$  codewords, in which case  $E_s(R, \tau) = \infty$ .

We next show that the SHTCC scheme when specialized to the case of TACI recovers the result of Proposition 7, which implies its optimality (as the probability of type 1 error tends to zero). Towards that end, it is convenient to state the T2EE achievable by the SHTCC scheme for the case when additional side-information  $Z$  is available at the detector.

**Remark 9.** *The extension of Theorem 8 to the case when additional side information  $Z$  is available at the detector is straightforward. Let  $\hat{\kappa}_s$  denote the value of  $\kappa_s$  when,*

- (i)  $P_{UVW}$  and  $Q_{UVW}$  are replaced by  $P_{UVZW} = P_{UVZ}P_{W|U}$  and  $Q_{UVZW} = Q_{UVZ}P_{W|U}$ , respectively,
- (ii)  $D(P_{\tilde{U}\tilde{V}\tilde{W}}||Q_{UVW})$  and  $D(P_V||Q_V)$  are replaced by  $D(P_{\tilde{U}\tilde{V}\tilde{Z}\tilde{W}}||Q_{UVZW})$  and  $D(P_{VZ}||Q_{VZ})$ , respectively,
- (iii)  $I_P(U; W|V)$ ,  $I_P(U; W)$  and  $I_P(V; W)$  are replaced by  $I_P(U; W|V, Z)$ ,  $I_P(U; W|Z)$  and  $I_P(V; W|Z)$ , respectively,
- (iv) set  $\mathcal{B}$  is replaced by  $\mathcal{B}^s = \{P_{W|U} : I_P(U; W|V, Z) \leq R < \tau C, (V, Z) - U - W\}$ .
- (v) sets  $\mathcal{T}_1(P_{UW}, P_{VW})$ ,  $\mathcal{T}_2(P_{UW}, P_V)$  and  $\mathcal{T}_3(P_{UW}, P_V)$  are replaced, respectively, by

$$\mathcal{T}_1^s(P_{UW}, P_{VZW}) := \{P_{\tilde{U}\tilde{V}\tilde{Z}\tilde{W}} \in \mathcal{T}(\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \mathcal{Z}) : P_{\tilde{U}\tilde{W}} = P_{UW}, P_{\tilde{V}\tilde{Z}\tilde{W}} = P_{VZW}\},$$

$$\mathcal{T}_2^s(P_{UW}, P_{VZ}) := \{P_{\tilde{U}\tilde{V}\tilde{Z}\tilde{W}} \in \mathcal{T}(\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \mathcal{Z}) : P_{\tilde{U}\tilde{W}} = P_{UW}, P_{\tilde{V}\tilde{Z}} = P_{VZ},$$

$$H_P(W|V, Z) \leq H(\tilde{W}|\tilde{V}, \tilde{Z})\},$$

$$\mathcal{T}_3^s(P_{UW}, P_{VZ}) := \{P_{\tilde{U}\tilde{V}\tilde{Z}\tilde{W}} \in \mathcal{T}(\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \mathcal{Z}) : P_{\tilde{U}\tilde{W}} = P_{UW}, P_{\tilde{V}\tilde{Z}} = P_{VZ}\}.$$

It can be shown along the similar lines to the proof of Theorem 8 that  $\hat{\kappa}_s$  is an achievable T2EE for the GHTS problem.

The value of  $\hat{\kappa}_s$  when specialized to the case of TACI recovers the optimal T2EE given in Proposition 7 as follows. Let  $\mathcal{B}' := \{(P_{W|U}, R) : I(U; W|Z) \leq R < \tau C\}$ . Note that  $\mathcal{B}' \subseteq \mathcal{B}^s$  since  $I(U; W|V, Z) \leq I(U; W|Z)$ , which holds due to the Markov chain relation  $(Z, V) - U - W$ . Now, consider  $(P_{W|U}, R) \in \mathcal{B}'$ . For such  $W$ , we have

$$\begin{aligned} E_1(P_{W|U}) &= \min_{P_{\tilde{U}\tilde{V}\tilde{Z}\tilde{W}} \in \mathcal{T}_1^s(P_{UW}, P_{VWZ})} D(P_{\tilde{U}\tilde{V}\tilde{Z}\tilde{W}} || P_Z P_{U|Z} P_{V|Z} P_{W|U}) \\ &= I(V; W|Z), \end{aligned}$$

$$E_2(R, P_{W|U}) \geq I(U; W|Z) - I(U; W|V, Z) = I(V; W|Z),$$

$$\begin{aligned} E_3(R, P_{W|U}, \tau) &= \min_{P_{\tilde{U}\tilde{V}\tilde{Z}\tilde{W}} \in \mathcal{T}_3^s(P_{UW}, P_{VZ})} D(P_{\tilde{U}\tilde{V}\tilde{Z}\tilde{W}} || P_Z P_{U|Z} P_{V|Z} P_{W|U}) + I(V; W|Z) + \tau E_r\left(\frac{R}{\tau}\right) \\ &\geq I(V; W|Z), \end{aligned} \tag{41}$$

$$\begin{aligned} E_4(R, P_{W|U}, \tau) &= D(P_{VZ} || P_{VZ}) + I(V; W|Z) + E_s(R, \tau) \\ &\geq I(V; W|Z), \end{aligned} \tag{42}$$

where in (41) and (42), we used the non-negativity of  $D(\cdot || \cdot)$ ,  $E_r(\cdot)$  and  $E_s(R, \tau)$ .

Hence, we obtain that,

$$\begin{aligned} \kappa(\tau, \epsilon) &\geq \sup_{P_{W|U} \in \mathcal{B}^s} \min\{E_1(R, P_{W|U}), E_2(R, P_{W|U}), E_3(R, P_{W|U}, \tau), E_4(R, P_{W|U}, \tau)\} \\ &\geq \sup_{P_{W|U} \in \mathcal{B}^s} I(V; W|Z) \\ &\geq \sup_{P_{W|U} \in \mathcal{B}'} I(V; W|Z) \end{aligned} \tag{43}$$

$$= \sup_{P_{W|U} : I(W; U|Z) \leq \tau C} I(V; W|Z), \tag{44}$$

where in (43) we used the fact that  $\mathcal{B}' \subseteq \mathcal{B}^s$ ; and (44) follows since  $I(V; W|Z)$  and  $I(U; W|Z)$  are continuous functions of  $P_{W|U}$ .

The achievability of the optimal T2EE for TACI by using the SHTCC scheme shows the interesting fact that a noisy channel does not always degrade the performance of HT, in comparison to a noiseless channel of same capacity. Furthermore, it also shows that for TACI, the effect of the noisiness of the channel on the T2EE can be completely nullified by using a simple

separation based coding scheme. From (41) and (42), notice that this happens due to the fact that  $E_3(R, P_{W|U}, \tau)$  and  $E_4(R, P_{W|U}, \tau)$  are both larger than  $I(V; W|Z)$ . This can be explained intuitively as follows. For the scheme discussed in Appendix C that achieves a T2EE of  $\kappa^s$ , a type 2 error occurs only when the detector decodes a codeword  $\hat{W}^k$  that is jointly typical with the side information sequence  $(V^k, Z^k)$ . For the case of TACI, when  $H_1$  is the true hypothesis, then with high probability, the codeword  $W^k(J)$  chosen by the encoder is not jointly typical with  $V^k$ , i.e.,  $(W^k(J), V^k) \notin T_{[P_{VW}]_\delta}^k$ . Then, the above phenomenon corroborates the fact that given an error occurs at the channel decoder, the probability (averaged over randomly generated codebooks) that  $(\hat{W}^k, V^k) \in T_{[P_{VW|Z}]_\delta}^k$  decays exponentially as  $2^{-kI(V; W|Z)}$ .

### B. Local Decision Scheme (Zero-Rate Compression Scheme)

The SHTCC scheme described above is a two stage scheme in which the encoder at the observer communicates a compressed version of the actual observations using the optimal channel code of rate  $\frac{R}{\tau}$  bits per channel use ( $R \leq \tau C$ ), while the detector makes its decision on the hypothesis using an estimate of the compressed version based on the output of the channel decoder and side-information  $V^n$ . Now, suppose the observer makes the decision about the hypothesis locally using its observations  $U^n$  and transmits this 1 bit decision to the detector using a channel code (for  $|M| = 2$  messages over  $n$  channel uses), while the detector outputs its decision based on  $\hat{M}$  and  $V^n$ . Note that the rate of the code is  $R = \frac{1}{n}$  bits per channel use, which tends to zero asymptotically. The encoder  $f^{(k,n)} = f_C^{(k,n)} \circ f_S^{(k)}$  and decoder  $g^{(k,n)} = g_S^{(k)} \circ g_C^{(k,n)}$  are thus specified by the maps  $f_S^{(k)} : \mathcal{U}^k \rightarrow \{0, 1\}$  (0 and 1 denote local decisions of  $H_0$  and  $H_1$ , respectively),  $f_C^{(k,n)} : \{0, 1\} \rightarrow \mathcal{X}^n$ ,  $g_C^{(k,n)} : \mathcal{Y}^n \rightarrow \{0, 1\}$  and  $g_S^{(k)} : \mathcal{V}^k \times \{0, 1\} \rightarrow \{H_0, H_1\}$ . Consider the encoder at the observer with  $f_S^{(k)}$  and  $f_C^{(k,n)}$  defined as follows.

$$f_S^{(k)}(u^k) = \begin{cases} 0, & \text{if } P_{u^k} \in T_{[P_U]_\delta}, \\ 1, & \text{otherwise,} \end{cases}$$

and

$$f_C^{(k,n)}(f_S^{(k)}(u^k)) = \begin{cases} a^n, & \text{if } f_S^{(k)}(u^k) = 0, \\ b^n, & \text{otherwise.} \end{cases}$$

Here,  $a^n$  and  $b^n$  denote the codewords formed by repeating the symbols  $a$  and  $b$  from the channel input alphabet  $\mathcal{X}$ , which are chosen such that

$$E_c := D(P_{Y|X=a} || P_{Y|X=b}) = \arg \max_{x, x'} D(P_{Y|X=x} || P_{Y|X=x'}). \quad (45)$$

Notice that  $E_c$  is the best channel coding error exponent achievable for the message  $M = 1$  for a channel code with message set  $\mathcal{M} = \{0, 1\}$  (with equally probable messages 0 and 1), when the probability of decoding error goes to 0. Let the decoder be defined by

$$g_C^{(k,n)}(y^n) = \begin{cases} 0, & \text{if } y^n \in T_{[P_{Y|X=a}]_{\delta'}}^n, \\ 1, & \text{otherwise,} \end{cases}$$

where,  $\delta' > \delta$ , and

$$g_S^{(k)}(v^k, g_C^{(k,n)}(y^n)) = \begin{cases} 0, & \text{if } P_{v^k} \in T_{[P_V]_{\delta}}^k \text{ and } g_C^{(k,n)}(y^n) = 0, \\ 1, & \text{otherwise.} \end{cases}$$

By the law of large numbers, the probability of type 1 error goes to zero since,

$$\lim_{k \rightarrow \infty} \mathbb{P}(U^k \in T_{[P_U]_{\delta}}^k | H = H_0) = 1,$$

$$\lim_{k \rightarrow \infty} \mathbb{P}(V^k \in T_{[P_V]_{\delta}}^k | H = H_0) = 1,$$

$$\text{and } \lim_{n \rightarrow \infty} \mathbb{P}(Y^n \in T_{[P_{Y|X=a}]_{\delta'}}^n | H = H_0) = 1.$$

Note that a Type 2 error occurs only under two events. The first is when the observed sequences  $u^k$  and  $v^k$  are  $P_U$ -typical and  $P_V$ -typical, respectively, and there is no error at the channel decoder. The second event that causes an error is when the observed sequence  $v^k$  is  $P_V$ -typical, but  $u^k$  is not  $P_U$ -typical and the channel decoder  $f_C^{(k,n)}$  makes a decoding error. It is not hard to see that this scheme achieves a T2EE given by

$$\kappa_l = \min(D(P_U || Q_U), E_c) + D(P_V || Q_V). \quad (46)$$

Note that when the marginals of  $U$  and  $V$  are the same under both hypothesis (as in the case of TACI), we get  $\kappa_l = 0$ ; and hence, the SHTCC scheme performs strictly better than the local decision scheme. However, the performance of these two schemes are not comparable in general.

### C. Joint Hypothesis Testing and Channel Coding Scheme

Hybrid coding is a form of JSCC introduced in [26] for the lossy transmission of sources over noisy networks. As the name suggests, hybrid coding is a combination of the digital and analog (uncoded) transmission schemes. For simplicity, assume that  $\tau = 1$ , i.e.,  $k = n$ . In hybrid coding, the observed sequence  $U^n$  is first mapped to one of the codewords within a compression codebook. Then, a symbol-by-symbol function of the codeword,  $U^n$  and a time sharing sequence  $S^n$  (that is known a priori to the encoder and decoder) is transmitted as the channel codeword  $X^n$ . This procedure is reversed at the decoder, in which, the decoder first attempts to reconstruct the compression codeword using the channel output  $Y^n$ ,  $S^n$  and its own correlated side information  $V^n$ . Then, the reconstruction  $\hat{U}^n$  is obtained as a symbol-by-symbol function of the reconstructed codeword,  $Y^n$ ,  $V^n$  and  $S^n$ . In this subsection, we analyze the T2EE achieved by a *Joint Hypothesis Testing and Channel Coding* (JHTCC) scheme that utilizes hybrid coding for the communication between the observer and the detector. In the JHTCC scheme, we consider a slight generalization of hybrid coding in that, we allow the channel input  $X^n$  to be generated according to an arbitrary memoryless stochastic function, instead of a deterministic function. After reconstructing the quantization codeword, the detector performs the hypothesis test using the reconstructed codeword, channel output and side-information. As before, the side-information  $V^n$  and  $Z^n$  play symmetric roles; and hence, it is sufficient to state the result with only  $V^n$ .

**Theorem 10.** *Consider the GHT problem with  $L = 1$ , bandwidth ratio  $\tau = 1$  and channel  $P_{Y|X}$  with capacity  $C$ . Then,  $\kappa(\tau, \epsilon) \geq \kappa_h$  for  $0 < \epsilon \leq 1$ , where*

$$\kappa_h = \sup_{(P_{W|U}, P_{SX'}, P_{X|UWS}) \in \mathcal{B}_h} \min \left( E'_1(P_{W|U}, P_S, P_{X|UWS}), E'_2(P_{W|U}, P_S, P_{X|UWS}), E'_3(P_{SX'}) \right), \quad (47)$$

$$\mathcal{B}_h = \left\{ (P_{W|U}, P_{SX'}, P_{X|UWS}) : I_{\hat{P}}(U; W) < I_{\hat{P}}(W; Y, V|S) \right\},$$

$$E'_1(P_{W|U}, P_S, P_{X|UWS}) = \min_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \mathcal{T}'_1(\hat{P}_{UW}, \hat{P}_{VWSY})} D(P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} || \hat{Q}_{UVWSY}), \quad (48)$$

$$E'_2(P_{W|U}, P_S, P_{X|UWS}) = \min_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \mathcal{T}'_2(\hat{P}_{UW}, \hat{P}_{VWSY})} D(P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} || \hat{Q}_{UVWSY}) + I_{\hat{P}}(W; V, Y|S) - I_{\hat{P}}(U; W), \quad (49)$$

$$E'_3(P_{SX'}) = D(\hat{P}_{VSY} || \check{Q}_{VSY}) + I_{\hat{P}}(W; V, Y|S) - I_{\hat{P}}(U; W), \quad (50)$$

$$\hat{P}_{UVWSX'XY}(P_{W|U}, P_{SX'}, P_{X|UWS}) = P_{UV}P_{W|U}P_{SX'}P_{X|UWS}P_{Y|X}, \quad (51)$$

$$\hat{Q}_{UVWSX'XY}(P_{W|U}, P_{SX'}, P_{X|UWS}) = Q_{UV}P_{W|U}P_{SX'}P_{X|UWS}P_{Y|X}, \quad (52)$$

$$\check{Q}_{UVSX'XY}(P_{SX'}) = Q_{UV}P_{SX'}\mathbb{1}(X = X')P_{Y|X},$$

$$\mathcal{T}'_1(\hat{P}_{UW}, \hat{P}_{VWSY}) = \{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \mathcal{T}(\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \mathcal{S} \times \mathcal{Y}) : P_{\tilde{U}\tilde{W}} = \hat{P}_{UW}, P_{\tilde{V}\tilde{S}\tilde{Y}} = \hat{P}_{VWSY}\},$$

$$\mathcal{T}'_2(\hat{P}_{UW}, \hat{P}_{VWSY}) = \{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \mathcal{T}(\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \mathcal{S} \times \mathcal{Y}) : P_{\tilde{U}\tilde{W}} = \hat{P}_{UW}, P_{\tilde{V}\tilde{S}\tilde{Y}} = \hat{P}_{VSY},$$

$$H(\tilde{W}|\tilde{V}, \tilde{S}, \tilde{Y}) \geq H_{\hat{P}}(W|V, S, Y)\}.$$

**Corollary 11.**

$$\kappa(\tau, \epsilon) \geq \max(\kappa_h, \kappa_l, \kappa_s), \quad \forall \epsilon \in (0, 1]. \quad (53)$$

The proof of Theorem 10 is given in Appendix D. To contrast with Theorem 8, note that Theorem 10 has three competing factors in the T2EE characterization instead of four. Similar to the SHTCC scheme, these factors can be related to the various error events that occur in hybrid coding. More specifically, let the event that the encoder is unsuccessful in finding a codeword typical with  $U^n$  in the quantization codebook be referred to as the *encoding error*, and the event that happens when a wrong codeword (unintended by the encoder) is reconstructed at the detector be referred to as the *decoding error*. Then,  $E'_1(P_{W|U}, P_S, P_{X|UWS})$  is related to the event that neither the encoding nor the decoding error occurs, while  $E'_2(P_{W|U}, P_S, P_{X|UWS})$  and  $E'_3(P_{SX'})$  are related to the events that only the decoding error and both the encoding and decoding errors occur, respectively.

We next show that JHTCC scheme achieves the optimal T2EE for TACI. In Theorem 10, let  $W$  be of the form  $W = (W_c, X)$ , such that  $X$  is independent of  $(X', W_c, U, V)$ , and  $(U, V, W_c, S, X') - X - Y$ . Then, it follows that

$$\hat{P}_{UVW_cSX'XY}(P_{W_c|U}, P_X, P_{SX'}) = P_{UV}P_{W_c|U}P_{SX'}P_{X|S}P_{Y|X}, \quad (54)$$

$$\hat{Q}_{UVW_cSX'XY}(P_{W_c|U}, P_X, P_{SX'}) = Q_{UV}P_{W_c|U}P_{SX'}P_{X|S}P_{Y|X}, \quad (55)$$

$$\check{Q}_{UVSX'XY}(P_{SX'}) = Q_{UV}P_{SX'}\mathbb{1}(X = X')P_{Y|X}, \quad (56)$$

$$I_{\hat{P}}(W; V, Y|S) - I_{\hat{P}}(U; W) = I_{\hat{P}}(X; Y|S) - I_{\hat{P}}(U; W_c) + I_{\hat{P}}(V; W_c)$$

$$= I_{\hat{P}}(X; Y|S) - I_{\hat{P}}(U; W_c|V), \quad (57)$$

$$\mathcal{B}_h = \{P_{W_c|U}, P_{SX'}, P_{X|S} : I_{\hat{P}}(U; W_c|V) < I(X; Y|S)\}. \quad (58)$$

Using (54)-(58) in Theorem 10, we obtain

$$\begin{aligned} E'_1(P_{W|U}, P_S, P_{X|UWS}) &= E'_1(P_{W_c|U}, P_S, P_{X|S}) \\ &= \min_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \mathcal{T}'_1(\hat{P}_{UW}, \hat{P}_{VWSY})} D(P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} || \hat{Q}_{UVWSY}) \\ &\geq \min_{P_{\tilde{U}\tilde{V}\tilde{W}} \in \mathcal{T}_1(\hat{P}_{UW_c}, \hat{P}_{VW_c})} D(P_{\tilde{U}\tilde{V}\tilde{W}} || \hat{Q}_{UVW_c}) \\ &\quad + \min_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \mathcal{T}'_1(\hat{P}_{UW_c}, \hat{P}_{VW_cSY})} D(P_{\tilde{S}\tilde{Y}|\tilde{U}\tilde{V}\tilde{W}} || \hat{Q}_{SY|UW_c} | P_{\tilde{U}\tilde{V}\tilde{W}}) \\ &= E_1(P_{W_c|U}) + \min_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \mathcal{T}'_1(\hat{P}_{UW_c}, \hat{P}_{VW_cSY})} D(P_{\tilde{Y}|\tilde{U}\tilde{V}\tilde{W}\tilde{S}} || \hat{Q}_{Y|UW_cS} | P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}}), \end{aligned} \quad (59)$$

$$E'_2(P_{W|U}, P_S, P_{X|UWS}) = E'_2(P_{W_c|U}, P_S, P_{X|S}) \quad (60)$$

$$= \min_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \mathcal{T}'_2(\hat{P}_{UW}, \hat{P}_{VWSY})} D(P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} || \hat{Q}_{UVWSY}) + I_{\hat{P}}(W; V, Y|S) - I_{\hat{P}}(U; W) \quad (61)$$

$$= \min_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \mathcal{T}'_2(\hat{P}_{UW}, \hat{P}_{VWSY})} D(P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} || \hat{Q}_{UVWSY}) + I_{\hat{P}}(X; Y|S) - I_{\hat{P}}(U; W|V) \quad (62)$$

$$\begin{aligned} &\geq \min_{P_{\tilde{U}\tilde{V}\tilde{W}} \in \mathcal{T}_2(\hat{P}_{UW_c}, \hat{P}_V)} D(P_{\tilde{U}\tilde{V}\tilde{W}} || \hat{Q}_{UVW_c}) + I_{\hat{P}}(X; Y|S) - I_{\hat{P}}(U; W_c|V) \\ &\quad + \min_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \mathcal{T}'_2(\hat{P}_{UW_c}, \hat{P}_{VW_cSY})} D(P_{\tilde{S}\tilde{Y}|\tilde{U}\tilde{V}\tilde{W}} || \hat{Q}_{SY|UW_c} | P_{\tilde{U}\tilde{V}\tilde{W}}) \\ &\geq \min_{P_{\tilde{U}\tilde{V}\tilde{W}} \in \mathcal{T}_3(\hat{P}_{UW_c}, \hat{P}_V)} D(P_{\tilde{U}\tilde{V}\tilde{W}} || \hat{Q}_{UVW_c}) + I_{\hat{P}}(X; Y|S) - I_{\hat{P}}(U; W_c|V) \\ &\quad + \min_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \mathcal{T}'_2(\hat{P}_{UW_c}, \hat{P}_{VW_cSY})} D(P_{\tilde{Y}|\tilde{U}\tilde{V}\tilde{W}\tilde{S}} || \hat{Q}_{Y|UW_cS} | P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}}), \end{aligned} \quad (63)$$

$$\begin{aligned} E'_3(P_{SX'}) &= D(\hat{P}_{VSY} || \check{Q}_{VSY}) + I_{\hat{P}}(W; V, Y|S) - I_{\hat{P}}(U; W) \\ &= D(P_V || Q_V) + I_{\hat{P}}(X; Y|S) - I_{\hat{P}}(U; W_c|V) + D(\hat{P}_{Y|S} || \check{Q}_{Y|S} | P_S). \end{aligned} \quad (64)$$

Now, we specialize Theorem 10 for TACI by substituting  $V = (V', Z)$ , such that the joint distribution of the r.v.'s  $U, V'$  and  $Z$  is  $P_{UV'Z}$  and  $Q_{UV'Z} = P_Z P_{U|Z} P_{V'|Z}$  under  $H_0$  and  $H_1$ ,

respectively. Then, it follows from (59), (63) and (64) that

$$E'_1(P_{W_c|U}, P_S, P_{X|S}) \geq E_1(P_{W_c|U}) = I_{\hat{P}}(V'; W_c|Z), \quad (65)$$

$$E'_2(P_{W_c|U}, P_S, P_{X|S}) \geq I_{\hat{P}}(X; Y|S) - I_{\hat{P}}(U; W_c|V', Z), \quad (66)$$

$$E'_3(P_{S_{X'}}) \geq I_{\hat{P}}(X; Y|S) - I_{\hat{P}}(U; W_c|V', Z), \quad (67)$$

such that  $(P_{W_c|U}, P_{S_{X'}}, P_{X|S}) \in \mathcal{B}_h$ , where  $\mathcal{B}_h$  is as defined in (58). Let

$$\mathcal{B}'_h := \{(P_{W_c|U}, P_{S_{X'}}, P_{X|S}) : P_{X|S} = P_X, I_{\hat{P}}(U; W_c|Z) < I_{\hat{P}}(X; Y)\}. \quad (68)$$

Note that  $\mathcal{B}'_h \in \mathcal{B}_h$  due to the Markov chain  $(V', Z) - U - W_c$ . Thus, for any  $(P_{W_c|U}, P_{S_{X'}}, P_X) \in \mathcal{B}'_h$ , we have from (66) that,

$$\begin{aligned} & I_{\hat{P}}(X; Y) - I_{\hat{P}}(U; W_c|V', Z) \\ &= I_{\hat{P}}(X; Y) - I_{\hat{P}}(U; W_c|Z) + I_{\hat{P}}(V'; W_c|Z) \geq I_{\hat{P}}(V'; W_c|Z). \end{aligned} \quad (69)$$

This proves that  $E'_1(P_{W_c|U}, P_S, P_X)$ ,  $E'_2(P_{W_c|U}, P_S, P_X)$  and  $E'_3(P_{S_{X'}})$  are all greater than or equal to  $I_{\hat{P}}(V'; W_c|Z)$  for  $(P_{W_c|U}, P_S, P_X) \in \mathcal{B}'_h$ . Taking supremum over all  $P_X$  and noting that  $I_{\hat{P}}(X; Y) \leq C$ , it follows that the optimal T2EE given in Proposition 7 is achievable.

Since separate source and channel coding is a special case of hybrid coding, it is expected that JHTCC scheme can achieve a larger T2EE compared to SHTCC scheme. However, we do not have a proof of this claim. Note that since the KL-divergence is non-negative, we obtain from (59) that,  $E'_1(P_{W|U}, P_S, P_{X|UWS}) \geq E_1(P_{W_c|U})$ . Also, we can write

$$\begin{aligned} & \sup_{(P_{W_c|U}, P_{S_{X'}}, P_{X|S}) \in \mathcal{B}_h} [E'_2(P_{W|U}, P_S, P_{X|UWS}) - E_3(I_{\hat{P}}(X; Y|S), P_{W_c|U}, 1)] \\ &= \sup_{(P_{W_c|U}, P_{S_{X'}}, P_{X|S}) \in \mathcal{B}_h} \left[ \min_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \mathcal{T}'_2(\hat{P}_{UW_c}, \hat{P}_{VW_cSY})} D(P_{\tilde{Y}|\tilde{U}\tilde{V}\tilde{W}\tilde{S}} || Q_{Y|UW_cS} | P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}}) \right. \\ & \quad \left. - E_r(I_{\hat{P}}(X; Y|S)) \right], \end{aligned} \quad (70)$$

and

$$\sup_{(P_{W_c|U}, P_{S_{X'}}, P_{X|S}) \in \mathcal{B}_h} [E'_3(P_{S_{X'}}) - E_4(I_{\hat{P}}(X; Y|S), P_{W_c|U}, 1)]$$



$$= \sup_{(P_{W_c|U}, P_{S_X'}, P_{X|S}) \in \mathcal{B}_h} \left[ D(\hat{P}_{Y|S} || \check{Q}_{Y|S} | P_S) - E_s(I_{\hat{P}}(X; Y|S), 1) \right]. \quad (71)$$

The difficulty in the proof of the above claim lies in showing that the R.H.S. of (70) and (71) are non-negative, from which the result would immediately follow.

From a practical viewpoint, the complexity of implementing the SHTCC, local decision and JHTCC schemes are all different. Evidently, the local decision scheme is the simplest since it involves the communication of only a single bit of information between the observer and the detector. The SHTCC scheme is more complex than the local decision scheme; however, it still utilizes a modular approach, in which, the HT and channel coding are done separately by independent blocks, both at the encoder and the detector. On the other hand, channel coding and HT is done jointly in the JHTCC scheme, which makes it the most complex among the schemes studied here.

## V. CONCLUSIONS

In this paper, we have studied the T2EE achievable for the distributed HT problem over orthogonal DMC's with side information available at the detector. For the special case of TACI, single-letter upper and lower bounds are obtained for the T2EE, which are shown to be tight when there is a single observer in the system. It is interesting to note that the reliability function of the channel does not play a role in the T2EE for TACI. We also established single-letter lower bounds on the T2EE for the GHT problem with a single observer, first using the SHTCC scheme that performs independent HT and channel coding and then using the JHTCC scheme that utilizes hybrid coding for the communication between the observer and the detector.

## APPENDIX A

### T2EE USING JOINT TYPICALITY DETECTOR

Here, we provide the proof for the case  $L = 1$ . For given arbitrary positive integers  $k$  and  $n$  such that  $n \leq k\tau$ , fix  $f_1^{(k,n)} = P_{X_1^n|U_1^k}$ . For any integer  $j$  and sequence  $u_1^{kj}$ , the observer transmits  $X_1^{nj} = f_1^{(kj,nj)}(u_1^{kj})$  generated i.i.d. according to  $\prod_{j'=1}^j P_{X_1^n|U_1^k=u_1^k(j')}$ . The detector declares  $H_0 : P_{U_1VZ}$  if  $(Y_1^{nj}, V^{kj}, Z^{kj}) \in T_{[\tilde{Y}_1^n \tilde{V}^k \tilde{Z}^k]_{\delta_j}}^j$  (here  $\delta_j \rightarrow 0$  as  $j \rightarrow \infty$ ) where  $(\tilde{Y}_1^n, \tilde{U}_1^k, \tilde{V}^k, \tilde{Z}^k) \sim P_{Y_1^n|U_1^k} P_{U_1VZ}^{\otimes k}$  and  $H_1 : Q_{U_1VZ}$  otherwise. To simplify the exposition, we denote  $(Y_1^n, V^k, Z^k)$

and  $T_{[\tilde{Y}_1^n \tilde{V}^k \tilde{Z}^k]_{\delta_j}}^j$  by  $W_{k,n}$  and  $T_{[\tilde{W}_{k,n}]_{\delta_j}}^j$ , respectively. By the Markov lemma [24], type 1 error probability tends to zero as  $j \rightarrow \infty$ . The type 2 error probability is bounded by

$$\begin{aligned}
\beta' \left( kj, nj, f_1^{(kj, nj)}, \epsilon \right) &\leq Q_{Y_1^{nj} V^{kj} Z^{kj}} \left( T_{[\tilde{W}_{k,n}]_{\delta_j}}^j \right) \\
&\leq \sum_{\tilde{P} \in T_{[\tilde{W}_{k,n}]_{\delta_j}}^j} \sum_{w_{k,n}^j \in T_{\tilde{P}}} Q_{W_{k,n}^j} (w_{k,n}^j) \\
&\stackrel{(a)}{=} \sum_{\tilde{P} \in T_{[\tilde{W}_{k,n}]_{\delta_j}}^j} \sum_{w_{k,n}^j \in T_{\tilde{P}}} 2^{-j(H(\tilde{P}) + D(\tilde{P} \| Q_{W_{k,n}}))} \\
&\stackrel{(b)}{=} \sum_{\tilde{P} \in T_{[\tilde{W}_{k,n}]_{\delta_j}}^j} 2^{-jD(\tilde{P} \| Q_{W_{k,n}})} \stackrel{(c)}{\leq} (j+1)^{|\mathcal{W}_{k,n}|} 2^{-jB_{k,n}(j)},
\end{aligned}$$

where,

$$B_{k,n}(j) := \min_{\tilde{P} \in T_{[\tilde{W}_{k,n}]_{\delta_j}}^j} D(\tilde{P} \| Q_{W_{k,n}}),$$

and (a), (b) and (c) follow from Lemmas 2.3, 2.6 and 2.2 in [23], respectively. Hence,

$$\frac{\log \left( \beta' \left( kj, nj, f_1^{(kj, nj)}, \epsilon \right) \right)}{kj} \leq -\frac{B_{k,n}(j)}{k} + \delta'_{k,n}(j),$$

where  $\delta'_{k,n}(j) := \frac{|\mathcal{W}_{k,n}| \log(j+1)}{kj}$  and  $|\mathcal{W}_{k,n}| = |\mathcal{Y}|^n |\mathcal{V}|^k |\mathcal{Z}|^k$ . Note that for any  $k$  and  $n$ ,  $\delta'_{k,n}(j) \rightarrow 0$  as  $j \rightarrow \infty$ . Also, since  $\delta_j$  is chosen such that it tends to 0 as  $j \rightarrow \infty$ ,  $B_{k,n}(j)$  converges to  $D(P_{W_{k,n}} \| Q_{W_{k,n}})$  by the continuity of  $D(\tilde{P} \| Q_{W_{k,n}})$  in  $\tilde{P}$  for fixed  $Q_{W_{k,n}}$ . Since  $k$ ,  $n$  and  $f_1^{(k,n)}$  are arbitrary, it follows from (4) and (6) that  $\theta(\tau)$  is an achievable T2EE for any upper bound  $\epsilon$  on the type 1 error probability. It is easy to see that this scheme can be generalized to  $L > 1$ .

## APPENDIX B

### PROOF OF THEOREM 5

For the achievability part, consider the following scheme.

**Encoding:** Fix  $k, n \in \mathbb{Z}^+$  and  $P_{X_l^n | U_l^k}$  at encoder  $l$ ,  $l \in \mathcal{L}$ . For  $j \in \mathbb{Z}^+$ , upon observing  $u_l^{kj}$ , encoder  $l$  transmits  $X_l^{nj} = f_l^{(kj, nj)}(U_l^{kj})$  generated i.i.d. according to  $\prod_{j'=1}^j P_{X_l^{n'} | U_l^k = u_l^{k'}(j')}$ . Encoder  $L+1$  performs uniform random binning on  $V^k$ , i.e.,  $f_{L+1}^{kj} : \mathcal{V}^{kj} \rightarrow \mathcal{M} = \{1, 2, \dots, 2^{kjR}\}$ . By uniform random binning, we mean that  $f_{L+1}^{kj}(V^{kj}) = m$ , where  $m$  is selected uniformly at random from the set  $\mathcal{M}$ .

**Decoding:** Let  $M$  denote the received bin index, and  $\delta > 0$  be an arbitrary number. If there exists a unique sequence  $\hat{V}^{kj}$  such that  $f_{L+1}^{kj}(\hat{V}^{kj}) = M$  and  $(\hat{V}^{kj}, Y_{\mathcal{L}}^{nj}, Z^{kj}) \in T_{[V^k Y_{\mathcal{L}}^n Z^k]_{\delta}}^j$ , then the decoder outputs  $g^{(kj,nj)}(M, Y_{\mathcal{L}}^{nj}, Z^{kj}) = \hat{V}^{kj}$ . Else, an error is declared.

**Analysis of the probability of error:** The possible error events under the above encoding and decoding rules are:  $\mathcal{E}_1 = \{(V^{kj}, Y_{\mathcal{L}}^{nj}, Z^{kj}) \notin T_{[V^k Y_{\mathcal{L}}^n Z^k]_{\delta}}^j\}$  and

$$\mathcal{E}_2 = \left\{ \begin{array}{l} \exists \tilde{V}^{kj} \neq V^{kj}, f_{L+1}^{kj}(\tilde{V}^{kj}) = f_{L+1}^{kj}(V^{kj}) \\ (\tilde{V}^{kj}, Y_{\mathcal{L}}^{nj}, Z^{kj}) \in T_{[V^k Y_{\mathcal{L}}^n Z^k]_{\delta}}^j \end{array} \right\}.$$

By the joint typicality lemma [24],  $Pr(\mathcal{E}_1) \rightarrow 0$  as  $j \rightarrow \infty$ . Also,

$$\begin{aligned} \mathbb{P}(\mathcal{E}_2) &= \sum_{v^{kj}, y_{\mathcal{L}}^{nj}, z^{kj}} \mathbb{P}(v^{kj}, y_{\mathcal{L}}^{nj}, z^{kj}) \times \mathbb{P}\left(f_{L+1}^{kj}(\tilde{V}^{kj}) = f_{L+1}^{kj}(v^{kj}), (\tilde{V}^{kj}, y_{\mathcal{L}}^{nj}, z^{kj}) \in T_{[V^k Y_{\mathcal{L}}^n Z^k]_{\delta}}^j\right) \\ &= \sum_{v^{kj}, y_{\mathcal{L}}^{nj}, z^{kj}} \mathbb{P}(v^{kj}, y_{\mathcal{L}}^{nj}, z^{kj}) \sum_{v^{kj} \in T_{[V^k Y_{\mathcal{L}}^n Z^k]_{\delta}}^j} e^{-kjR} \\ &\leq e^{j(H(V^k|Y_{\mathcal{L}}^n, Z^k) + \delta)} e^{-kjR} \\ &= e^{kj\left(\frac{H(V^k|Y_{\mathcal{L}}^n, Z^k)}{k} - R\right)}. \end{aligned}$$

Hence,  $\mathbb{P}(\mathcal{E}_2) \rightarrow 0$  as  $j \rightarrow \infty$  if  $R > H(V^k|Y_{\mathcal{L}}^n, Z^k) + \delta$ ,  $(Z^k, V^k) - U_l^k - X_l^n - Y_l^n$ ,  $l \in \mathcal{L}$ . Since  $\delta > 0$  is arbitrary, this proves that  $R > \frac{H(V^k|Y_{\mathcal{L}}^n, Z^k)}{k}$  is an achievable rate.

For the converse, we have by Fano's inequality that  $H(V^k|f_{L+1}^k(V^k), Y_{\mathcal{L}}^n, Z^k) \leq \gamma_k$ , where  $\gamma_k \rightarrow 0$  as  $k \rightarrow \infty$ . Hence, we obtain

$$\begin{aligned} kR &= \log(|\mathcal{M}|) \geq H(M|Y_{\mathcal{L}}^n, Z^k) \\ &= H(M|Y_{\mathcal{L}}^n, Z^k) - H(V^k|M, Y_{\mathcal{L}}^n, Z^k) \\ &\quad + H(V^k|M, Y_{\mathcal{L}}^n, Z^k) \\ &\geq H(V^k, M|Y_{\mathcal{L}}^n, Z^k) - \gamma_k \\ &= H(V^k|Y_{\mathcal{L}}^n, Z^k) + H(M|V^k, Y_{\mathcal{L}}^n, Z^k) - \gamma_k \\ &\geq H(V^k|Y_{\mathcal{L}}^n, Z^k) - \gamma_k. \end{aligned}$$

This proves the converse by noting that  $(Z^k, V^k) - U_l^k - X_l^n - Y_l^n$ ,  $l \in \mathcal{L}$  holds for any communication scheme.

## APPENDIX C

### PROOF OF THEOREM 8

We first describe the encoding and decoding operations in the SHTCC scheme. The random coding method is used to analyze the type 1 and type 2 error probabilities achieved by this scheme, averaged over the ensemble of randomly generated codebooks. This guarantees the existence of at least one deterministic codebook that achieves same or lower type 1 and type 2 error probabilities. We mention here that unless specified otherwise, the mutual information and entropy terms appearing in the proof below, like for example,  $I(U; W)$ ,  $I(U; W|V)$ ,  $H(W|V)$  etc. are computed with respect to the joint distribution  $P_{UVW} = P_{UV}P_{W|U}$ .

#### *Codebook Generation:*

Fix  $0 \leq R < \tau C$ , conditional distribution  $P_{W|U}$  and positive numbers  $\delta'$ ,  $\delta''$ ,  $\delta$ ,  $\tilde{\delta} > 0$  (whose values will be specified later in the proof). Generate  $2^{k(I(U; W) + \delta')}$  independent sequences  $W^k(j)$ ,  $j \in [2^{k(I(U; W) + \delta')}]$  randomly according to the distribution  $\prod_{i=1}^k P_W(w_i)$  where

$$P_W(w) = \sum_{u \in \mathcal{U}} \sum_{w \in \mathcal{W}} P_U(u) P_{W|U}(w|u).$$

Denote this codebook by  $\mathcal{C}$  which is referred to as the *source codebook*. Next, the *channel codebook* used by  $f_C^{(k,n)}$  is obtained by independently generating  $\lceil 2^{kR} \rceil$  codewords  $X^n(m)$ ,  $m \in [1 : \lceil 2^{kR} \rceil]$  i.i.d. according to the distribution  $\prod_{i=1}^n P_X(x_i)$ , such that it achieves the random coding exponent  $E_r(\frac{R}{\tau})$  of the channel  $P_{Y|X}$  [27]. The codeword  $X^n(0)$  corresponding to  $M = 0$  is chosen such that it achieves an error probability with exponent  $E_s(R, \tau)$  at the channel decoder. Denote this collection of codewords by  $\tilde{\mathcal{C}}$ .

*Encoding:* If  $I(U; W) + \delta' > R$ , i.e., the number of codewords in the source codebook is larger than the number of codewords in the channel codebook, the encoder performs uniform random binning on the sequences  $W^k(j)$ ,  $j \in [2^{k(I(U; W) + \delta')}]$  in  $\mathcal{C}$ , i.e., for each codeword in  $\mathcal{C}$ , it selects an index uniformly at random from the set  $[2^{kR}]$ . Denote the bin index selected for  $W^k(j)$  by  $f_B(j)$ . If the observed sequence  $U^k$  is typical, i.e.,  $U^k \in T_{[U]_{\delta''}}^k$ , the source encoder  $f_S^{(k)}$  first looks for a sequence  $W^k(J)$  such that  $(U^k, W^k(J)) \in T_{[UW]_{\delta}}^k$ ,  $\delta > \delta''$ . If there exists multiple such codewords, it chooses one of the index  $J$  among them uniformly at random and outputs the bin-index  $M = f_B(J)$ ,  $M \in [1 : 2^{kR}]$  or  $M = J$  depending on whether  $I(U; W) + \delta' > R$  or otherwise. If  $U^k \notin T_{[U]_{\delta''}}^k$  or such an index  $J$  does not exist,  $f_S^{(k)}$  outputs the *error* message  $M = 0$ . The channel encoder  $f_C^{(k,n)}$  outputs the codeword  $X^n(M)$  from the codebook  $\tilde{\mathcal{C}}$ .

*Decoding:* At the decoder,  $g_C^{(k,n)}$  maps the received channel output  $Y^n$  to an estimate  $\hat{M}$  of the transmitted message  $M$ . If  $\hat{M} = 0$ ,  $H_1$  is declared. Else, given the side information sequence  $V^k$  and estimated bin-index  $\hat{M}$ ,  $g_S^{(k,n)}$  searches for a typical sequence  $\hat{W}^k = W^k(\hat{l}) \in T_{[W]_{\hat{\delta}}}^k$ ,  $\hat{\delta} = |\mathcal{U}|\delta$  in the codebook such that

$$\hat{l} = \arg \min_{\substack{l: \hat{M}=f_B(l), \\ W^k(l) \in T_{[W]_{\hat{\delta}}}^k}} H_e(W^k(l)|V^k).$$

The decoder declares  $H_0$  if  $(\hat{W}^k, V^k) \in T_{[WV]_{\tilde{\delta}}}^k$ , for  $\tilde{\delta} > \delta$ , else,  $H_1$  is declared.

We next analyze the Type 1 and Type 2 error probabilities achieved by the above scheme (in the limit  $\delta, \delta', \tilde{\delta} \rightarrow 0$ ).

**Analysis of Type 1 error:** A type 1 error occurs only if one of the following events happen.

$$\begin{aligned} \mathcal{E}_{CE} &= \left\{ g_C^{(k,n)}(Y^n) \neq X^n(M) \right\} \\ \mathcal{E}_{TE} &= \left\{ (U^k, V^k) \notin T_{[UV]_{\tilde{\delta}}}^k, \tilde{\delta} = \frac{\delta''}{|\mathcal{V}|} \right\} \\ \mathcal{E}_{EE} &= \left\{ \nexists j \in [1 : 2^{k(I(U;W)+\delta')}] : (U^k, W^k(j)) \in T_{[UW]_{\delta}}^k \right\} \\ \mathcal{E}_{ME} &= \left\{ (V^k, W^k(J)) \notin T_{[VW]_{\tilde{\delta}}}^k \right\} \\ \mathcal{E}_{DE} &= \left\{ \exists l \in [1 : 2^{k(I(U;W)+\delta')}] , l \neq J : f_B(l) = f_B(J), W^k(l) \in T_{[W]_{\tilde{\delta}}}^k, \right. \\ &\quad \left. H_e(W^k(l)|V^k) \leq H_e(W^k(J)|V^k) \right\} \end{aligned}$$

The probability of the event  $\mathcal{E}_{CE}$ , that an error occurs at the channel decoder  $g_C^{(k,n)}$  tends to 0 as  $n \rightarrow \infty$  since  $E_r(\frac{R}{\tau})$  is positive for  $R < \tau C$ .  $\mathcal{E}_{TE}$  tends to 0 asymptotically by the weak law of large numbers. Note that given  $\mathcal{E}_{TE}^c$  holds,  $U^k \in T_{[U]_{\delta''}}^k$  and by the covering lemma [23, Lemma 9.1], it is well known that  $\mathcal{E}_{EE}$  tends to 0 doubly exponentially for  $\delta > \delta''$  and  $\delta'$  appropriately chosen. Given  $\mathcal{E}_{EE}^c \cap \mathcal{E}_{TE}^c$  holds, it follows from the Markov chain relation  $V - U - W$  and the Markov lemma [24], that  $\mathbb{P}(\mathcal{E}_{ME})$  tends to zero as  $k \rightarrow \infty$  for  $\tilde{\delta} > \delta$  (appropriately chosen). Next, we consider the probability of the event  $\mathcal{E}_{DE}$ . Given that  $\mathcal{E}_{ME}^c \cap \mathcal{E}_{EE}^c \cap \mathcal{E}_{TE}^c$  holds, note that  $\lim_{k \rightarrow \infty} H_e(W^k(J)|V^k) \rightarrow H(W|V)$  as  $\tilde{\delta} \rightarrow 0$ . Thus, we have

$$\begin{aligned}
& \mathbb{P}(\mathcal{E}_{DE} | V^k = v^k, W^k(J) = w^k, \mathcal{E}_{ME}^c \cap \mathcal{E}_{EE}^c \cap \mathcal{E}_{TE}^c) \\
& \leq \sum_{\substack{l=1, \\ l \neq J}}^{2^{k(I(U;W)+\delta')}} \sum_{\substack{\tilde{w}^k \in T_{[W]_\delta}^k: \\ H_e(\tilde{w}^k|v^k) \leq H_e(w^k|v^k)}} \mathbb{P}\left(f_B(l) = f_B(J), W^k(l) = \tilde{w}^k | V^k = v^k, W^k(J) = w^k, \right. \\
& \quad \left. \mathcal{E}_{ME}^c \cap \mathcal{E}_{EE}^c \cap \mathcal{E}_{TE}^c\right) \\
& = \sum_{\substack{l=1, \\ l \neq J}}^{2^{k(I(U;W)+\delta')}} \sum_{\substack{\tilde{w}^k \in T_{[W]_\delta}^k: \\ H_e(\tilde{w}^k|v^k) \leq H_e(w^k|v^k)}} \mathbb{P}(W^k(l) = \tilde{w}^k | V^k = v^k, W^k(J) = w^k, \mathcal{E}_{ME}^c \cap \mathcal{E}_{EE}^c \cap \mathcal{E}_{TE}^c) \frac{1}{2^{kR}} \\
& \leq \sum_{\substack{l=1, \\ l \neq J}}^{2^{k(I(U;W)+\delta')}} \sum_{\substack{\tilde{w}^k \in T_{[W]_\delta}^k: \\ H_e(\tilde{w}^k|v^k) \leq H_e(w^k|v^k)}} 2 \cdot 2^{-kR} 2^{-k(H(W)-\delta_1)} \tag{72} \\
& \leq \sum_{\substack{l=1, \\ l \neq J}}^{2^{k(I(U;W)+\delta')}} (k+1)^{|\mathcal{V}||\mathcal{W}|} 2^{k(H(W|V)+\gamma_1(k))} \times 2 \cdot 2^{-kR} 2^{-k(H(W)-\delta_1)} \tag{73} \\
& \leq 2^{-k(R-I(U;W|V)-\delta_2^{(k)})},
\end{aligned}$$

where  $\delta_1 = O(\delta)$ ,  $\gamma_1(k) = |H_e(w^k|v^k) - H(W|V)|$  and  $\delta_2^{(k)} = \delta_1 + \frac{1}{k}|\mathcal{V}||\mathcal{W}| \log(2k+2) + \delta' + \gamma_1(k) \xrightarrow{(k)} 0$  as  $\tilde{\delta}, \delta', \delta \rightarrow 0$ . To obtain (72), we used the fact that

$$\begin{aligned}
& \mathbb{P}(W^k(l) = \tilde{w}^k | \mathcal{E}_{ME}^c \cap \mathcal{E}_{EE}^c \cap \mathcal{E}_{TE}^c, W^k(J) = w^k, V^k = v^k) \\
& \leq 2 \times \mathbb{P}(W^k(l) = \tilde{w}^k | \mathcal{E}_{ME}^c \cap \mathcal{E}_{EE}^c \cap \mathcal{E}_{TE}^c, V^k = v^k). \tag{74}
\end{aligned}$$

This follows similar to (94), which is discussed in the type 2 error analysis section below. In order to obtain the expression in (73), we first summed over the types  $P_{\tilde{W}}$  of sequences within the typical set  $T_{[W]_\delta}^k$  that have empirical entropy less than  $H_e(w^k|v^k)$  and used the facts that the number of sequences within such a type is upper bounded by  $2^{k(H(W|V)+\gamma_1(k))}$  and the total number of types is upper bounded by  $(k+1)^{|\mathcal{V}||\mathcal{W}|}$  [23]. Summing over all  $(w^k, v^k) \in T_{[VW]_\delta}^k$ , we obtain

$$\begin{aligned}
& \mathbb{P}(\mathcal{E}_{DE} | \mathcal{E}_{ME}^c \cap \mathcal{E}_{EE}^c \cap \mathcal{E}_{TE}^c) \\
& \leq \sum_{(w^k, v^k) \in T_{[WV]_{\tilde{\delta}}}^k} \mathbb{P}(W^k(J) = w^k, V^k = v^k | \mathcal{E}_{ME}^c \cap \mathcal{E}_{EE}^c \cap \mathcal{E}_{TE}^c) 2^{-k(R-I(U;W|V)-\delta_2^{(k)})} \\
& \leq 2^{-k(R-I(U;W|V)-\delta_2^{(k)})}.
\end{aligned} \tag{75}$$

Hence, if  $I(U;W|V) < R < \tau C$ , the probability of the events causing Type 1 error tends to zero asymptotically.

**Analysis of Type 2 error:** First, note that a type 2 error occurs only if  $V^k \in T_{[V]_{\delta'''}}^k$ ,  $\delta''' = |\mathcal{W}| \tilde{\delta}$  and hence we can restrict the type 2 error analysis to only such  $V^k$ . Denote the event that a type 2 error happens by  $\mathcal{D}_0$ . Let

$$\mathcal{E}_0 = \left\{ U^k \notin T_{[U]_{\delta''}}^k \right\}. \tag{76}$$

The type 2 error probability can be written as

$$\beta(k, \tau, \epsilon) = \sum_{(u^k, v^k) \in \mathcal{U}^k \times \mathcal{V}^k} \mathbb{P}(U^k = u^k, V^k = v^k | H_1) \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k). \tag{77}$$

Let  $\mathcal{E}_{NE} := \mathcal{E}_{EE}^c \cap \mathcal{E}_0^c$ . The last term in (77) can be upper bounded as follows.

$$\begin{aligned}
& \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k) \\
& = \mathbb{P}(\mathcal{E}_{NE} | U^k = u^k, V^k = v^k) \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}) \\
& \quad + \mathbb{P}(\mathcal{E}_{NE}^c | U^k = u^k, V^k = v^k) \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}^c) \\
& \leq \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}) + \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}^c).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
& \beta(k, \tau, \epsilon) \\
& \leq \sum_{(u^k, v^k) \in \mathcal{U}^k \times \mathcal{V}^k} \mathbb{P}(U^k = u^k, V^k = v^k | H_1) \left[ \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}) \right. \\
& \quad \left. + \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}^c) \right].
\end{aligned} \tag{78}$$

First, assume that  $\mathcal{E}_{NE}$  holds. Then,

$$\begin{aligned} \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}) &= \sum_{j=1}^{2^{k(I(U;W)+\delta')}} \sum_{m=1}^{2^{kR}} \mathbb{P}(J = j, f_B(J) = m | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}) \\ &\quad \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, J = j, f_B(J) = m, \mathcal{E}_{NE}). \end{aligned} \quad (79)$$

By the symmetry of the codebook generation, encoding and decoding procedure, the term  $\mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, J = j, f_B(J) = m, \mathcal{E}_{NE})$  in (79) is independent of the value of  $J$  and  $f_B(J)$ . Hence, w.l.o.g. assuming  $J = 1$  and  $f_B(J) = 1$ , we can write

$$\begin{aligned} &\mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}) \\ &= \sum_{j=1}^{2^{k(I(U;W)+\delta')}} \sum_{m=1}^{2^{kR}} \mathbb{P}(J = j, f_B(J) = m | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}) \\ &\quad \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, \mathcal{E}_{NE}) \\ &= \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, \mathcal{E}_{NE}) \\ &= \sum_{w^k \in \mathcal{W}^k} \mathbb{P}(W^k(1) = w^k | U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, \mathcal{E}_{NE}) \\ &\quad \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, W^k(1) = w^k, \mathcal{E}_{NE}). \end{aligned} \quad (80)$$

Given  $\mathcal{E}_{NE}$  holds,  $\mathcal{D}_0$  may occur in three possible ways (i) when the channel decoder makes an error and the codeword retrieved from the bin is jointly typical with  $V^k$  (ii) when an unintended wrong codeword is retrieved from the correct bin that is jointly typical with  $V^k$  and (iii) when there is no error at the channel decoder and the correct codeword is retrieved from the bin, that is also jointly typical with  $V^k$ . We refer to the event in case (i) as the *channel error event*  $\mathcal{E}_{CE}$  and that in case (ii) as the *binning error event*  $\mathcal{E}_{BE}$ , respectively. More specifically,

$$\mathcal{E}_{CE} = \{\hat{M} = g_C^{(k,n)}(Y^n) \neq M\} \text{ and} \quad (81)$$

$$\begin{aligned} \mathcal{E}_{BE} = \left\{ \exists l \in \left[ 1 : 2^{k(I(U;W)+\delta')} \right], l \neq J, f_B(l) = \hat{M}, W^k(l) \in T_{[P_W]_\delta}^k, \right. \\ \left. (V^k, W^k(l)) \in T_{[P_{VW}]_\delta}^k \right\}. \end{aligned} \quad (82)$$



Define the following events

$$\mathcal{F} = \{U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, W^k(1) = w^k, \mathcal{E}_{NE}\}, \quad (83)$$

$$\mathcal{F}_1 = \{U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, W^k(1) = w^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}\}, \quad (84)$$

$$\mathcal{F}_2 = \{U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, W^k(1) = w^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}^c\}, \quad (85)$$

$$\mathcal{F}_{21} = \{U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, W^k(1) = w^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}^c, \mathcal{E}_{BE}\}, \quad (86)$$

$$\mathcal{F}_{22} = \{U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, W^k(1) = w^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}^c, \mathcal{E}_{BE}^c\}. \quad (87)$$

The last term in (80) can be expressed as follows.

$$\mathbb{P}(\mathcal{D}_0|\mathcal{F}) = \mathbb{P}(\mathcal{E}_{CE}|\mathcal{F}) \mathbb{P}(\mathcal{D}_0|\mathcal{F}_1) + \mathbb{P}(\mathcal{E}_{CE}^c|\mathcal{F}) \mathbb{P}(\mathcal{D}_0|\mathcal{F}_2),$$

where

$$\mathbb{P}(\mathcal{D}_0|\mathcal{F}_2) = \mathbb{P}(\mathcal{E}_{BE}|\mathcal{F}_2) \mathbb{P}(\mathcal{D}_0|\mathcal{F}_{21}) + \mathbb{P}(\mathcal{E}_{BE}^c|\mathcal{F}_2) \mathbb{P}(\mathcal{D}_0|\mathcal{F}_{22}). \quad (88)$$

Since the channel encoder and decoder uses randomly generated codewords achieving the best random coding error exponent  $E_r(\frac{R}{\tau})$ , we have

$$\mathbb{P}(\mathcal{E}_{CE}|\mathcal{F}) = 2^{-nE_r(\frac{R}{\tau})} = 2^{-k\tau E_r(\frac{R}{\tau})}. \quad (89)$$

Next, consider the type 2 error event which happens when  $H_0$  is declared in spite of an error at the channel decoder. We need to consider two separate cases (i)  $I(U; W) > R$  and  $I(U; W) \leq R$ . Note that in the former case, binning is performed and type 2 error happens at the decoder only if a sequence  $W^k(l)$  exists in the wrong bin  $\hat{M} \neq M = f_B(J)$  such that  $(V^k, W^k(l)) \in T_{[VW]_{\delta}}^k$ . However as noted in [26], the calculation of probability of this event does not follow using the standard random coding argument usually encountered in achievability proofs due to the fact that the codeword  $W^k(J)$  chosen depends on the entire codebook. Following techniques similar to [26], we analyze the probability of this event (averaged over the codebooks  $\mathcal{C}$ ,  $\tilde{\mathcal{C}}$  and random binning) as follows. We first consider the case when  $I(U; W) > R$ .

$$\begin{aligned} \mathbb{P}(\mathcal{D}_0|\mathcal{F}_1) &\leq \mathbb{P}(\exists W^k(l) : f_B(l) = \hat{M} \neq 1, (W^k(l), v^k) \in T_{[WV]_{\delta}}^k | \mathcal{F}_1) \\ &\leq \sum_{l=2}^{2^{k(I(U;W)+\delta')}} \sum_{\hat{m} \neq 1} \mathbb{P}(\hat{M} = \hat{m} | \mathcal{F}_1) \mathbb{P}((W^k(l), v^k) \in T_{[WV]_{\delta}}^k : f_B(l) = \hat{m} | \mathcal{F}_1) \end{aligned}$$

$$\begin{aligned}
&= \sum_{l=2}^{2^{k(I(U:W)+\delta')}} \sum_{\hat{m} \neq 1} \mathbb{P}(\hat{M} = \hat{m} | \mathcal{F}_1) \sum_{(\tilde{w}^k, v^k) \in T_{[WV]_{\tilde{\delta}}}^{\tilde{w}^k}} \mathbb{P}(W^k(l) = \tilde{w}^k : f_B(l) = \hat{m} | \mathcal{F}_1) \\
&= \sum_{l=2}^{2^{k(I(U:W)+\delta')}} \sum_{\hat{m} \neq 1} \mathbb{P}(\hat{M} = \hat{m} | \mathcal{F}_1) \sum_{(\tilde{w}^k, v^k) \in T_{[WV]_{\tilde{\delta}}}^{\tilde{w}^k}} \mathbb{P}(W^k(l) = \tilde{w}^k | \mathcal{F}_1) \frac{1}{2^{kR}} \tag{90}
\end{aligned}$$

$$= \sum_{l=2}^{2^{k(I(U:W)+\delta')}} \sum_{(\tilde{w}^k, v^k) \in T_{[WV]_{\tilde{\delta}}}^{\tilde{w}^k}} \mathbb{P}(W^k(l) = \tilde{w}^k | \mathcal{F}_1) \frac{1}{2^{kR}}. \tag{91}$$

Let  $\mathcal{C}_l^- = \mathcal{C} \setminus \{W^k(1), W^k(l)\}$ . Then,

$$\mathbb{P}(W^k(l) = \tilde{w}^k | \mathcal{F}_1) = \sum_{\mathcal{C}_l^- = c} \mathbb{P}(\mathcal{C}_l^- = c) \mathbb{P}(W^k(l) = \tilde{w}^k | \mathcal{F}_1, \mathcal{C}_l^- = c). \tag{92}$$

The last term in (92) can be upper bounded as follows.

$$\begin{aligned}
&\mathbb{P}(W^k(l) = \tilde{w}^k | \mathcal{F}_1, \mathcal{C}_l^- = c) \\
&= \frac{\mathbb{P}(W^k(l) = \tilde{w}^k, U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, W^k(1) = w^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c)}{\mathbb{P}(U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, W^k(1) = w^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c)} \\
&= \mathbb{P}(W^k(l) = \tilde{w}^k | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c) \\
&\quad \frac{\mathbb{P}(J = 1, f_B(J) = 1, W^k(1) = w^k | W^k(l) = \tilde{w}^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c)}{\mathbb{P}(J = 1, f_B(J) = 1, W^k(1) = w^k | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c)} \\
&= \mathbb{P}(W^k(l) = \tilde{w}^k | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c) \\
&\quad \frac{\mathbb{P}(J = 1, f_B(J) = 1, W^k(1) = w^k | W^k(l) = \tilde{w}^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c)}{\mathbb{P}(J = 1, f_B(J) = 1, W^k(1) = w^k | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c)} \\
&= \mathbb{P}(W^k(l) = \tilde{w}^k | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c) \\
&\quad \frac{\mathbb{P}(W^k(1) = w^k | W^k(l) = \tilde{w}^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c)}{\mathbb{P}(W^k(1) = w^k | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c)} \\
&\quad \frac{\mathbb{P}(J = 1 | W^k(1) = w^k, W^k(l) = \tilde{w}^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c)}{\mathbb{P}(J = 1 | W^k(1) = w^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c)} \\
&\quad \frac{\mathbb{P}(f_B(J) = 1 | J = 1, W^k(1) = w^k, W^k(l) = \tilde{w}^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c)}{\mathbb{P}(f_B(J) = 1 | J = 1, W^k(1) = w^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c)}. \tag{93}
\end{aligned}$$

Since the codewords are generated independently of each other and the binning operation is done independent of the codebook generation, we have

$$\begin{aligned} \mathbb{P}(W^k(1) = w^k | W^k(l) = \tilde{w}^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c) \\ = \mathbb{P}(W^k(1) = w^k | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c), \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}(f_B(J) = 1 | J = 1, W^k(1) = w^k, W^k(l) = \tilde{w}^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c) \\ = \mathbb{P}(f_B(J) = 1 | J = 1, W^k(1) = w^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}_l^- = c). \end{aligned}$$

Consider the term in (93). Let  $N(u^k, \mathcal{C}_l^-) = |\{w^k(l') \in \mathcal{C}_l^- : l' \neq 1, l' \neq l, (w^k(l'), u^k) \in T_{[WU]_\delta}^k\}|$ . Recall that if there are multiple sequences in the codebook  $\mathcal{C}$  that are typical with the observed sequence  $U^k$ , then the encoder selects one of them uniformly at random. Thus if  $(\tilde{w}^k, u^k) \in T_{[WU]_\delta}^k$ , then

$$\begin{aligned} & \frac{\mathbb{P}(J = 1 | W^k(1) = w^k, W^k(l) = \tilde{w}^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}^- = c)}{\mathbb{P}(J = 1 | W^k(1) = w^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}^- = c)} \\ &= \left[ \frac{1}{N(u^k, \mathcal{C}^-) + 2} \right] \frac{1}{\mathbb{P}(J = 1 | W^k(1) = w^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}^- = c)} \\ &\leq \frac{N(u^k, \mathcal{C}^-) + 1}{N(u^k, \mathcal{C}^-) + 2} \leq 1. \end{aligned}$$

If  $(\tilde{w}^k, u^k) \notin T_{[WU]_\delta}^k$ , then

$$\begin{aligned} & \frac{\mathbb{P}(J = 1 | W^k(1) = w^k, W^k(l) = \tilde{w}^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}^- = c)}{\mathbb{P}(J = 1 | W^k(1) = w^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}^- = c)} \\ &= \left[ \frac{1}{N(u^k, \mathcal{C}^-) + 1} \right] \frac{1}{\mathbb{P}(J = 1 | W^k(1) = w^k, U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}^- = c)} \\ &\leq \frac{N(u^k, \mathcal{C}^-) + 2}{N(u^k, \mathcal{C}^-) + 1} \leq 2. \end{aligned}$$

Hence, the term in (92) can be upper bounded as

$$\begin{aligned} & \mathbb{P}(W^k(l) = \tilde{w}^k | \mathcal{F}_1) \\ & \leq \sum_{\mathcal{C}^- = c} \mathbb{P}(\mathcal{C}^- = c) 2 \mathbb{P}(W^k(l) = \tilde{w}^k | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}, \mathcal{C}^- = c) \\ & = 2 \mathbb{P}(W^k(l) = \tilde{w}^k | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}). \end{aligned} \tag{94}$$

Substituting (94) in (91), we obtain

$$\begin{aligned}
& \mathbb{P}(\mathcal{D}_0|\mathcal{F}_1) \\
& \leq \sum_{l=1}^{2^{k(I(U;W)+\delta')}} \sum_{\substack{\tilde{w}^k; \\ (\tilde{w}^k, v^k) \in T_{[WV]_\delta}^k}} 2 \mathbb{P}(W^k(l) = \tilde{w}^k | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}) \frac{1}{2^{kR}} \\
& = \sum_{l=1}^{2^{k(I(U;W)+\delta')}} \sum_{\substack{\tilde{w}^k; \\ (\tilde{w}^k, v^k) \in T_{[WV]_\delta}^k}} 2 \times 2^{-k(H(W)-\delta_1)} \frac{1}{2^{kR}} \\
& = 2 \times 2^{k(I(U;W)+\delta')} 2^{k(H(W|V)+\delta_3)} 2^{-k(H(W)-\delta_1)} \frac{1}{2^{kR}} \\
& \leq 2^{-k(R-I(U;W|V)-\delta_4^{(k)})}, \tag{95}
\end{aligned}$$

where  $\delta_4^{(k)} := \delta' + \delta_1 + \delta_3 + \frac{1}{k} \xrightarrow{k} 0$  as  $\delta, \delta' \rightarrow 0$ .

For the case  $I(U;W) \leq R$  (when binning is not done), the terms can be bounded similarly using (94) as follows.

$$\begin{aligned}
& \mathbb{P}(\mathcal{D}_0|\mathcal{F}_1) = \sum_{\hat{m} \neq 1} \mathbb{P}(\hat{M} = \hat{m}|\mathcal{F}_1) \mathbb{P}((W^k(\hat{m}), v^k) \in T_{[WV]_\delta}^k | \mathcal{F}_1) \\
& \leq \sum_{\hat{m} \neq 1} \mathbb{P}(\hat{M} = \hat{m}|\mathcal{F}_1) \sum_{\substack{\tilde{w}^k; \\ (\tilde{w}^k, v^k) \in T_{[WV]_\delta}^k}} 2 \mathbb{P}(W^k(\hat{m}) = \tilde{w}^k | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}) \\
& \leq \sum_{\hat{m} \neq 1} \mathbb{P}(\hat{M} = \hat{m}|\mathcal{F}_1) 2^{-k(I(V;W)-(\delta_1+\delta_3+\frac{1}{k}))} \leq 2^{-k(I(V;W)-\delta_4^{(k)})}. \tag{96}
\end{aligned}$$

Next, consider the event when the encoding and channel error does not happen, i.e.,  $\mathcal{E}_{NE} \cap \mathcal{E}_{CE}^c$  holds. For the case  $I(U;W) > R$ , the binning error event denoted by  $\mathcal{E}_{BE}$  happens when a wrong codeword  $W^k(l)$ ,  $l \neq J$  is retrieved from the bin with index  $M$  by the empirical entropy decoder such that  $(W^k(l), V^k) \in T_{[WV]_\delta}^k$ . Let  $P_{\tilde{U}\tilde{V}\tilde{W}}$  denote the type of  $P_{U^k V^k W^k(J)}$ . Note that  $P_{\tilde{U}\tilde{W}} \in \mathcal{T}_{[UW]_\delta}^k$  when  $\mathcal{E}_{NE}$  holds. If  $H(\tilde{W}|\tilde{V}) < H(W|V)$ , there exists a codeword in the bin with index  $M$  having empirical entropy strictly less than  $H(W|V)$ . Hence, the decoded codeword  $\hat{W}^k \notin T_{[WV]_\delta}^k$  (asymptotically) since  $(\hat{W}^k, V^k) \in T_{[WV]_\delta}^k$  necessarily implies that  $H_e(\hat{W}^k|V^k) := H(P_{\hat{W}^k}|P_{V^k}) \rightarrow H(W|V)$  as  $\delta \rightarrow 0$ . Consequently, a Type 2 error can happen under the event  $\mathcal{E}_{BE}$  only when  $H(\tilde{W}|\tilde{V}) \geq H(W|V)$ . The probability of the event  $\mathcal{E}_{BE}$  can be

upper bounded under this condition as follows.

$$\begin{aligned}
& \mathbb{P}(\mathcal{E}_{BE}|\mathcal{F}_2) \\
& \leq \mathbb{P}\left(\exists l \neq 1, l \in [1 : 2^{k(I(U;W)+\delta')}] : f_B(l) = 1 \text{ and } (W^k(l), v^k) \in T_{[WV]_{\tilde{\delta}}}^k|\mathcal{F}_2\right) \\
& \leq \sum_{l=2}^{2^{k(I(U;W)+\delta')}} \mathbb{P}\left((W^k(l), v^k) \in T_{[WV]_{\tilde{\delta}}}^k|\mathcal{F}_2\right) \mathbb{P}\left(f_B(l) = 1|\mathcal{F}_2, (W^k(l), v^k) \in T_{[WV]_{\tilde{\delta}}}^k\right) \\
& = \sum_{l=2}^{2^{k(I(U;W)+\delta')}} \mathbb{P}\left((W^k(l), v^k) \in T_{[WV]_{\tilde{\delta}}}^k|\mathcal{F}_2\right) 2^{-kR} \\
& \leq \sum_{l=2}^{2^{k(I(U;W)+\delta')}} \sum_{\substack{\tilde{w}^k: \\ (\tilde{w}^k, v^k) \in T_{[WV]_{\tilde{\delta}}}^k}} 2 \mathbb{P}(W^k(l) = \tilde{w}^k|U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}^c) 2^{-kR} \quad (97) \\
& = 2^{-k(R-I(U;W|V)-\delta_4^{(k)})}. \quad (98)
\end{aligned}$$

In (97), we used the fact that

$$\mathbb{P}(W^k(l) = \tilde{w}^k|\mathcal{F}_2) \leq 2 \mathbb{P}(W^k(l) = \tilde{w}^k|U^k = u^k, V^k = v^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}^c), \quad (99)$$

which follows in a similar way as (94). Also, note that, by definition,  $\mathbb{P}(\mathcal{D}_0|\mathcal{F}_{21}) = 1$ .

We proceed to analyze the R.H.S of (78) which upper bounds the type 2 error  $\beta(k, \tau, \epsilon)$ , in the limit  $k \rightarrow \infty$  and  $\delta, \delta', \tilde{\delta} \rightarrow 0$ . Towards this end, we first focus on the the case when  $\mathcal{E}_{NE}$  holds. From (80), it follows that

$$\lim_{k \rightarrow \infty} \lim_{\delta, \tilde{\delta}, \delta' \rightarrow 0} \sum_{(u^k, v^k) \in \mathcal{U}^k \times \mathcal{V}^k} \mathbb{P}(U^k = u^k, V^k = v^k|H_1) \mathbb{P}(\mathcal{D}_0|U^k = u^k, V^k = v^k, \mathcal{E}_{NE}) \quad (100)$$

$$\begin{aligned}
& = \lim_{k \rightarrow \infty} \lim_{\delta, \tilde{\delta}, \delta' \rightarrow 0} \sum_{(u^k, v^k) \in \mathcal{U}^k \times \mathcal{V}^k} \mathbb{P}(U^k = u^k, V^k = v^k|H_1) \\
& \quad \mathbb{P}(\mathcal{D}_0|U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, \mathcal{E}_{NE}). \quad (101)
\end{aligned}$$

Rewriting the summation in (101) as the sum over the types and sequences within a type, we

obtain

$$\begin{aligned}
& \mathbb{P}(\mathcal{D}_0 | \mathcal{E}_{NE}) \\
&= \sum_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}} \in \\ \mathcal{T}^k(\mathcal{U} \times \mathcal{V} \times \mathcal{W})}} \sum_{\substack{(u^k, v^k, w^k) \\ \in T_{P_{\tilde{U}\tilde{V}\tilde{W}}}}} \left[ \mathbb{P}(U^k = u^k, V^k = v^k | H_1) \mathbb{P}(\mathcal{D}_0 | \mathcal{F}) \right. \\
&\quad \left. \mathbb{P}(W^k(1) = w^k | U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, \mathcal{E}_{NE}) \right]. \quad (102)
\end{aligned}$$

Since  $H_1$  is the true underlying hypothesis, we have

$$\begin{aligned}
& \mathbb{P}(U^k = u^k, V^k = v^k | H_1) \mathbb{P}(W^k(1) = w^k | U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, \mathcal{E}_{NE}) \\
&= \left[ \prod_{i=1}^k Q_{UV}(u_i, v_i) \right] \mathbb{P}(W^k(1) = w^k | U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, \mathcal{E}_{NE}) \\
&\leq \left[ \prod_{i=1}^k Q_{UV}(u_i, v_i) \right] \frac{1}{|T_{P_{\tilde{W}|\tilde{U}}}|} \leq 2^{-k(H(\tilde{U}\tilde{V}) + D(P_{\tilde{U}\tilde{V}} \| Q_{UV}) + H(\tilde{W}|\tilde{U}) - \frac{1}{k}|\mathcal{U}||\mathcal{W}|\log(k+1))}, \quad (103)
\end{aligned}$$

where  $P_{\tilde{U}\tilde{V}\tilde{W}}$  denotes the type of the sequence  $(u^k, v^k, w^k)$ .

With (89), (95), (96), (98) and (103), we have the necessary machinery to analyze (102). First, consider that the event  $\mathcal{E}_{NE} \cap \mathcal{E}_{CE}^c \cap \mathcal{E}_{BE}^c$  holds. In this case,

$$\begin{aligned}
& \mathbb{P}(\mathcal{D}_0 | \mathcal{F}_{22}) = \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, W^k(1) = w^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}^c, \mathcal{E}_{BE}^c) \\
&= \begin{cases} 1, & \text{if } P_{u^k w^k} \in T_{[UW]_\delta}^k \\ & \text{and } P_{v^k w^k} \in T_{[VW]_{\tilde{\delta}}}^k, \\ 0, & \text{otherwise.} \end{cases} \quad (104)
\end{aligned}$$

Thus, the following terms in (102) can be simplified (in the limit  $\delta, \tilde{\delta} \rightarrow 0$ ) as

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \lim_{\delta, \tilde{\delta}, \delta' \rightarrow 0} \sum_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}} \in \\ \mathcal{T}^k(\mathcal{U} \times \mathcal{V} \times \mathcal{W})}} \sum_{\substack{(u^k, v^k, w^k) \\ \in T_{P_{\tilde{U}\tilde{V}\tilde{W}}}}} \left[ \mathbb{P}(U^k = u^k, V^k = v^k | H_1) \mathbb{P}(\mathcal{E}_{CE}^c | \mathcal{F}) \mathbb{P}(\mathcal{E}_{BE}^c | \mathcal{F}_2) \mathbb{P}(\mathcal{D}_0 | \mathcal{F}_{22}) \right. \\
&\quad \left. \mathbb{P}(W^k(1) = w^k | U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, \mathcal{E}_{NE}) \right] \\
&\leq \lim_{k \rightarrow \infty} \lim_{\delta, \tilde{\delta}, \delta' \rightarrow 0} \sum_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}} \in \\ \mathcal{T}^k(\mathcal{U} \times \mathcal{V} \times \mathcal{W})}} \sum_{\substack{(u^k, v^k, w^k) \\ \in T_{P_{\tilde{U}\tilde{V}\tilde{W}}}}} \left[ \mathbb{P}(U^k = u^k, V^k = v^k | H_1) \mathbb{P}(\mathcal{D}_0 | \mathcal{F}_{22}) \right]
\end{aligned}$$

$$\begin{aligned}
& \mathbb{P}(W^k(1) = w^k | U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, \mathcal{E}_{NE}) \Big] \\
& \leq \lim_{k \rightarrow \infty} (k+1)^{|\mathcal{U}||\mathcal{V}||\mathcal{W}|} \max_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}} \in \\ \mathcal{T}_1(P_{UW}, P_{VW})}} 2^{kH(\tilde{U}\tilde{V}\tilde{W})} 2^{-k(H(\tilde{U}\tilde{V}) + D(P_{\tilde{U}\tilde{V}} \| Q_{UV}) + H(\tilde{W}|\tilde{U}) - \frac{1}{k}|\mathcal{U}||\mathcal{W}| \log(k+1))} \\
& = \lim_{k \rightarrow \infty} 2^{-k\tilde{E}_{1k}}.
\end{aligned} \tag{105}$$

Here,

$$\begin{aligned}
\tilde{E}_{1k} &:= \min_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}} \in \\ \mathcal{T}_1(P_{UW}, P_{VW})}} H(\tilde{U}\tilde{V}) + D(P_{\tilde{U}\tilde{V}} \| Q_{UV}) + H(\tilde{W}|\tilde{U}) - H(\tilde{U}\tilde{V}\tilde{W}) \\
&\quad - \frac{1}{k}|\mathcal{U}||\mathcal{V}||\mathcal{W}| \log(k+1) - \frac{1}{k}|\mathcal{U}||\mathcal{W}| \log(k+1) \\
&= \min_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}} \in \\ \mathcal{T}_1(P_{UW}, P_{VW})}} \sum P_{\tilde{U}\tilde{V}\tilde{W}} \log \left( \frac{P_{\tilde{U}\tilde{V}}}{Q_{UV}} \frac{1}{P_{\tilde{U}\tilde{V}}} \frac{P_{\tilde{U}}}{P_{\tilde{U}\tilde{W}}} P_{\tilde{U}\tilde{V}\tilde{W}} \right) - o(1) \\
&= \min_{P_{\tilde{U}\tilde{V}\tilde{W}} \in \mathcal{T}_1(P_{UW}, P_{VW})} D(P_{\tilde{U}\tilde{V}\tilde{W}} \| Q_{UVW}) - o(1) \xrightarrow{(k)} E_1(P_{W|U}),
\end{aligned} \tag{106}$$

and  $Q_{UVW} := Q_{UV}P_{W|U}$ . To obtain (105), we used (103) and (104). This results in the term  $E_1(P_{W|U})$  in (36).

Next, consider the terms corresponding to the event  $\mathcal{E}_{NE} \cap \mathcal{E}_{CE}^c \cap \mathcal{E}_{BE}$  in (102). Note that given the event  $\mathcal{F}_{21} = \{U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, W^k(1) = w^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}^c, \mathcal{E}_{BE}\}$  occurs,  $P_{u^k w^k} \in T_{[UW]_\delta}^k$ . Also,  $\mathcal{D}_0$  can happen only if  $H_e(w^k|v^k) \geq H(W|V) - \gamma_2(\tilde{\delta})$  for some positive function  $\gamma_2(\tilde{\delta}) \in O(\tilde{\delta})$  and  $P_{v^k} \in T_{[V]_{\delta'''}}^k$ . Using these facts to simplify the terms corresponding to the event  $\mathcal{E}_{NE} \cap \mathcal{E}_{CE}^c \cap \mathcal{E}_{BE}$  in (102), we obtain

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \lim_{\delta', \delta, \tilde{\delta} \rightarrow 0} \sum_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}} \in \\ \mathcal{T}^k(\mathcal{U} \times \mathcal{V} \times \mathcal{W})}} \sum_{\substack{(u^k, v^k, w^k) \\ \in T_{P_{\tilde{U}\tilde{V}\tilde{W}}}^k}} \left[ \mathbb{P}(U^k = u^k, V^k = v^k | H_1) \mathbb{P}(\mathcal{E}_{CE}^c | \mathcal{F}) \mathbb{P}(\mathcal{E}_{BE} | \mathcal{F}_2) \mathbb{P}(\mathcal{D}_0 | \mathcal{F}_{21}) \right. \\
& \quad \left. \mathbb{P}(W^k(1) = w^k | U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, \mathcal{E}_{NE}) \right] \\
& \leq \lim_{k \rightarrow \infty} \lim_{\delta', \delta, \tilde{\delta} \rightarrow 0} \sum_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}} \in \\ \mathcal{T}^k(\mathcal{U} \times \mathcal{V} \times \mathcal{W})}} \sum_{\substack{(u^k, v^k, w^k) \\ \in T_{P_{\tilde{U}\tilde{V}\tilde{W}}}^k}} \left[ \mathbb{P}(U^k = u^k, V^k = v^k | H_1) \mathbb{P}(\mathcal{E}_{BE} | \mathcal{F}_2) \mathbb{P}(\mathcal{D}_0 | \mathcal{F}_{21}) \right. \\
& \quad \left. \mathbb{P}(W^k(1) = w^k | U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, \mathcal{E}_{NE}) \right] \\
& \leq \lim_{k \rightarrow \infty} \max_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}} \in \\ \mathcal{T}_2(P_{UW}, P_V)}} 2^{kH(\tilde{U}\tilde{V}\tilde{W})} 2^{-k(H(\tilde{U}\tilde{V}) + D(P_{\tilde{U}\tilde{V}} \| Q_{UV}) + H(\tilde{W}|\tilde{U}) + R - I(U; W|V))} \\
& \quad 2^{(|\mathcal{U}||\mathcal{V}||\mathcal{W}| \log(k+1) + |\mathcal{U}||\mathcal{W}| \log(k+1))}
\end{aligned}$$

$$= \lim_{k \rightarrow \infty} 2^{-k\tilde{E}_{2k}}, \quad (107)$$

where,

$$\begin{aligned} \tilde{E}_{2k} := & \min_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}} \in \\ \mathcal{T}_2(P_{UV}, P_V)}} H(\tilde{U}\tilde{V}) + D(P_{\tilde{U}\tilde{V}} \| Q_{UV}) + H(\tilde{W}|\tilde{U}) + R - I(U; W|V) \\ & - \frac{1}{k} |\mathcal{U}| |\mathcal{V}| |\mathcal{W}| \log(k+1) - \frac{1}{k} |\mathcal{U}| |\mathcal{W}| \log(k+1) \\ & \xrightarrow{(k)} E_2(R, P_{W|U}). \end{aligned} \quad (108)$$

Note that  $\mathcal{E}_{BE}$  occurs only when  $I(U; W) > R$ .

Next, consider that the event  $\mathcal{E}_{NE} \cap \mathcal{E}_{CE}$  holds. As in the case above, note that given  $\mathcal{F}_1 = \{U^k = u^k, V^k = v^k, J = 1, f_B(J) = 1, W^k(1) = w^k, \mathcal{E}_{NE}, \mathcal{E}_{CE}\}$ ,  $P_{u^k w^k} \in T_{[UV]_\delta}^k$  and  $\mathcal{D}_0$  occurs only if  $P_{v^k} \in T_{[V]_{\delta'''}}^k$ . Using these facts and eqns. (95), (96) and (89), it can be shown that the terms corresponding to this event in (102) result in the factor  $E_3(R, \tau)$  given in (38).

Finally, we analyze the case when the event  $\mathcal{E}_{NE}^c$  occurs. Since the encoder declares  $H_1$  if  $\hat{M} = 0$ , it is clear that  $\mathcal{D}_0$  occurs only when the channel error event  $\mathcal{E}_{CE}$  happens. Thus, we have

$$\begin{aligned} \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}^c) &= \mathbb{P}(\mathcal{E}_{CE} | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}^c) \\ &= \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}^c \cap \mathcal{E}_{CE}). \end{aligned} \quad (109)$$

From the coding scheme, it follows that

$$\mathbb{P}(\mathcal{E}_{CE} | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}^c) \leq 2^{-kE_s(R, \tau)}. \quad (110)$$

When binning is performed at the encoder,  $\mathcal{D}_0$  occurs only if there exists a sequence  $\hat{W}^k$  in the bin  $\hat{M} \neq 0$  such that  $(\hat{W}^k, V^k) \in T_{[WV]_\delta}^k$ . Also, recalling that the encoder sends the error message  $M = 0$  independent of the source codebook  $\mathcal{C}$ , it can be shown using standard arguments that for such  $v^k \in T_{[V]_{\delta'''}}^k$ ,

$$\mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}^c \cap \mathcal{E}_{CE}) \leq 2^{-k(R - I(U; W|V) - \delta_5)}, \quad (111)$$

where  $\delta_5 = \delta_1 + \delta_3 + \delta'$ . Thus, from (109), (110) and (111), we obtain



$$\begin{aligned}
& \lim_{\delta, \delta', \tilde{\delta} \rightarrow 0} \sum_{u^k, v^k} \mathbb{P}(U^k = u^k, V^k = v^k | H_1) \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}^c \cap \mathcal{E}_{CE}) \\
& \leq 2^{-kD(P_V || Q_V)} \times 2^{-kE_s(R, \tau)} \times 2^{-k(R - I(U; W|V))}.
\end{aligned} \tag{112}$$

On the other hand, when binning is not performed,  $\mathcal{D}_0$  occurs only if  $(W^k(\hat{M}), V^k) \in T_{[WV]_{\tilde{\delta}}}^k$  and in this case, we obtain

$$\begin{aligned}
& \lim_{\delta, \delta', \tilde{\delta} \rightarrow 0} \sum_{u^k, v^k} \mathbb{P}(U^k = u^k, V^k = v^k | H_1) \mathbb{P}(\mathcal{D}_0 | U^k = u^k, V^k = v^k, \mathcal{E}_{NE}^c \cap \mathcal{E}_{CE}) \\
& \leq 2^{-kD(P_V || Q_V)} \times 2^{-kE_s(R, \tau)} \times 2^{-kI(V; W)}.
\end{aligned} \tag{113}$$

This results in the factor  $E_4(R, \tau)$  in (39). Since the T2EE is lower bounded by the minimal value of the exponent due to the various type 2 error events, this completes the proof of the theorem.

## APPENDIX D

### PROOF OF THEOREM 10

We only give a sketch of the proof as the intermediate steps follow similar to that given in the proof of Theorem 8.

*Codebook Generation:*

Fix distributions  $(P_{W|U}, P_{SX'}, P_{X|UWS}) \in \mathcal{B}_h$  and let

$$\hat{P}_{UVWSX'XY}(P_{W|U}, P_{SX'}, P_{X|UWS}) = P_{UV} P_{W|U} P_{SX'} P_{X|UWS} P_{Y|X}.$$

Generate the quantization codebook  $\mathcal{C} = \{W^n(j), j \in [2^{n(I_{\hat{P}}(U; W) + \delta')}] \}$ , where each codeword  $W^n(j)$  is generated independently according to the distribution  $\prod_{i=1}^n \hat{P}_W$ , where  $\hat{P}_W = \sum_{u \in \mathcal{U}} P_U(u) P_{W|U}(w|u)$ . Also, generate a sequence  $S^n$  (for time-sharing) i.i.d. according to the distribution  $\prod_{i=1}^n P_S$ . The realization  $S^n = s^n$  is revealed to both the encoder and detector.

*Encoding:* If the observed sequence  $U^n$  is typical, i.e.,  $U^n \in T_{[U]_{\delta''}}^n$ , the encoder first looks for a sequence  $W^n(J)$  such that  $(U^n, W^n(J)) \in T_{[\hat{P}_{UW}]_{\delta}}^n$ ,  $\delta > \delta''$ . If there exists multiple such codewords, it chooses one among them uniformly at random. The encoder transmits  $X^n$  over the channel, where  $X^n$  is generated according to the distribution  $\prod_{i=1}^n P_{X|UWS}(x_i | u_i, w_i, s_i)$ . If

$U^n \notin T_{[U]_{\delta''}}^k$  or such an index  $J$  does not exist, the encoder generates the channel input  $X^n$  randomly according to  $\prod_{i=1}^n P_{X'|S}(x_i|s_i)$ .

**Decoding:** The detector first checks if  $(V^n, S^n, Y^n) \in T_{[\hat{P}_{VSY}]_{\tilde{\delta}}}^n$ ,  $\tilde{\delta} > \delta$ . If the check is unsuccessful,  $H_1$  is declared. Else, searches for a typical sequence  $\hat{W}^n = W^n(\hat{l}) \in T_{[W]_{\tilde{\delta}}}^k$ ,  $\hat{\delta} = |\mathcal{W}|\delta$  in the codebook such that

$$W^n(\hat{l}) = \arg \min_{W^n(l) \in T_{[\hat{P}_W]_{\hat{\delta}}}^n} H_e(W^n(l)|V^n, S^n, Y^n).$$

If  $(V^n, S^n, Y^n, \hat{W}^n) \in T_{[\hat{P}_{VSYW}]_{\hat{\delta}}}^n$ ,  $H_0$  is declared, else  $H_1$  is declared.

### Analysis of Type 1 error:

A type 1 error occurs only if one of the following events happen.

$$\begin{aligned} \tilde{\mathcal{E}}_{TE} &= \left\{ (U^n, V^n, S^n) \notin T_{[\hat{P}_{UVS}]_{\tilde{\delta}}}^n, \tilde{\delta} = \frac{\delta''}{|\mathcal{V}||\mathcal{S}|} \right\} \\ \tilde{\mathcal{E}}_{EE} &= \left\{ \nexists j \in [1 : 2^{n(I(U;W)+\delta')}] : (U^n, W^n(j)) \in T_{[UW]_{\tilde{\delta}}}^n \right\} \\ \tilde{\mathcal{E}}_{ME} &= \left\{ (V^n, W^n(J), S^n) \notin T_{[\hat{P}_{VWS}]_{\tilde{\delta}}}^n \right\} \\ \tilde{\mathcal{E}}_{CE} &= \left\{ (V^n, W^n(J), S^n, Y^n) \notin T_{[\hat{P}_{VWSY}]_{\tilde{\delta}}}^n \right\} \\ \tilde{\mathcal{E}}_{DE} &= \left\{ \exists l \in [1 : 2^{n(I(U;W)+\delta')}] , l \neq J, W^n(l) \in T_{[\hat{P}_W]_{\tilde{\delta}}}^n, \right. \\ &\quad \left. H_e(W^n(l)|V^n, S^n, Y^n) \leq H_e(W^n(J)|V^n, S^n, Y^n) \right\} \end{aligned}$$

By the weak law of large numbers,  $\tilde{\mathcal{E}}_{TE}$  tends to 0 asymptotically with  $n$  for any  $\tilde{\delta} > 0$ . The covering lemma guarantees that  $\tilde{\mathcal{E}}_{EE} \cap \tilde{\mathcal{E}}_{TE}^c$  tends to 0 doubly exponentially for  $\tilde{\delta} < \delta$  and  $\delta'$  appropriately chosen. Given  $\tilde{\mathcal{E}}_{EE}^c \cap \tilde{\mathcal{E}}_{TE}^c$  holds, it follows from the Markov lemma and the weak law of large numbers, respectively, that  $\mathbb{P}(\tilde{\mathcal{E}}_{ME})$  and  $\mathbb{P}(\tilde{\mathcal{E}}_{CE})$  tends to zero asymptotically for  $\tilde{\delta} > \delta$  (appropriately chosen). Next, we consider the probability of the event  $\tilde{\mathcal{E}}_{DE}$ . Given that  $\tilde{\mathcal{E}}_{CE}^c \cap \tilde{\mathcal{E}}_{ME}^c \cap \tilde{\mathcal{E}}_{EE}^c \cap \tilde{\mathcal{E}}_{TE}^c$  holds, note that  $\lim_{n \rightarrow \infty} H_e(W^n(J)|V^n, S^n, Y^n) \rightarrow H(W|V, S, Y)$  as  $\tilde{\delta} \rightarrow 0$ . Hence, similar to that shown in Appendix C, it can be shown that

$$\mathbb{P}(\tilde{\mathcal{E}}_{DE} | \tilde{\mathcal{E}}_{CE}^c \cap \tilde{\mathcal{E}}_{ME}^c \cap \tilde{\mathcal{E}}_{EE}^c \cap \tilde{\mathcal{E}}_{TE}^c) \leq 2^{-n(I_{\hat{P}}(W;Y,V,Q) - I_{\hat{P}}(U;W) - \delta_6^{(n)})}.$$

where  $\delta_6^{(n)} \xrightarrow{(n)} 0$  as  $\tilde{\delta}, \delta' \rightarrow 0$ . Hence, if  $I_{\hat{P}}(U;W) < I_{\hat{P}}(W;Y,V,Q)$ , the probability of the events causing Type 1 error tends to zero asymptotically.

**Analysis of Type 2 error:** The analysis of the T2EE is very similar to that of the SHTCC scheme given in Appendix C. Hence, only a sketch of the proof is provided, with the differences from the proof of the SHTCC scheme highlighted.

Let  $\mathcal{E}_0$  be defined as in (76). Then, as in Appendix C, the type 2 error probability can be written as

$$\begin{aligned} & \beta(n, 1, \epsilon) \\ & \leq \sum_{(u^n, v^n) \in \mathcal{U}^n \times \mathcal{V}^n} \mathbb{P}(U^n = u^n, V^n = v^n | H_1) \left[ \mathbb{P}(\tilde{\mathcal{E}}_{EE} \cap \mathcal{E}_0^c | U^n = u^n, V^n = v^n) \right. \\ & \quad \left. + \mathbb{P}(\mathcal{D}_0 | U^n = u^n, V^n = v^n, \tilde{\mathcal{E}}_{NE}) + \mathbb{P}(\mathcal{D}_0 | U^n = u^n, V^n = v^n, \mathcal{E}_0) \right], \end{aligned} \quad (114)$$

where,  $\tilde{\mathcal{E}}_{NE} := \tilde{\mathcal{E}}_{EE}^c \cap \mathcal{E}_0^c$ . As before, it is sufficient to restrict the analysis to the events  $\tilde{\mathcal{E}}_{NE}$  and  $\mathcal{E}_0$  that dominate the type 2 error. Define the events

$$\begin{aligned} \tilde{\mathcal{E}}_{T2} = \left\{ \exists l \in \left[ 1 : 2^{n(I(U;W)+\delta')} \right], l \neq J, W^n(l) \in T_{[\hat{P}_W]_\delta}^n, \right. \\ \left. (V^n, W^n(l), S^n, Y^n) \in T_{[\hat{P}_{VWSY}]_\delta}^n \right\}, \end{aligned} \quad (115)$$

$$\tilde{\mathcal{F}} = \{U^n = u^n, V^n = v^n, J = 1, W^n(1) = w^n, S^n = s^n, Y^n = y^n, \tilde{\mathcal{E}}_{NE}\}, \quad (116)$$

$$\tilde{\mathcal{F}}_1 = \{U^n = u^n, V^n = v^n, J = 1, W^n(1) = w^n, S^n = s^n, Y^n = y^n, \tilde{\mathcal{E}}_{NE}, \tilde{\mathcal{E}}_{T2}^c\}, \quad (117)$$

$$\tilde{\mathcal{F}}_2 = \{U^n = u^n, V^n = v^n, J = 1, W^n(1) = w^n, S^n = s^n, Y^n = y^n, \tilde{\mathcal{E}}_{NE}, \tilde{\mathcal{E}}_{T2}\}. \quad (118)$$

By the symmetry of the codebook generation, encoding and decoding procedure, the term  $\mathbb{P}(\mathcal{D}_0 | U^n = u^n, V^n = v^n, J = j, \tilde{\mathcal{E}}_{NE})$  is independent of the value of  $J$ . Hence, w.l.o.g. assuming  $J = 1$ , we can write

$$\begin{aligned} & \mathbb{P}(\mathcal{D}_0 | U^n = u^n, V^n = v^n, \tilde{\mathcal{E}}_{NE}) \\ & = \sum_{j=1}^{2^{n(I(U;W)+\delta')}} \mathbb{P}(J = j | U^n = u^n, V^n = v^n, \tilde{\mathcal{E}}_{NE}) \mathbb{P}(\mathcal{D}_0 | U^n = u^n, V^n = v^n, J = 1, \tilde{\mathcal{E}}_{NE}) \\ & = \mathbb{P}(\mathcal{D}_0 | U^n = u^n, V^n = v^n, J = 1, \tilde{\mathcal{E}}_{NE}) \\ & = \sum_{\substack{(w^n, s^n, y^n) \\ \in \mathcal{W}^n \times \mathcal{S}^n \times \mathcal{Y}^n}} \mathbb{P}(W^n(1) = w^n, S^n = s^n, Y^n = y^n | U^n = u^n, V^n = v^n, J = 1, \tilde{\mathcal{E}}_{NE}) \\ & \quad \mathbb{P}(\mathcal{D}_0 | U^n = u^n, V^n = v^n, J = 1, W^n(1) = w^n, S^n = s^n, Y^n = y^n, \tilde{\mathcal{E}}_{NE}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{(w^n, s^n, y^n) \\ \in \mathcal{W}^n \times \mathcal{S}^n \times \mathcal{Y}^n}} \mathbb{P}(W^n(1) = w^n, S^n = s^n, Y^n = y^n | U^n = u^n, V^n = v^n, J = 1, \tilde{\mathcal{E}}_{NE}) \\
&\quad \mathbb{P}(\mathcal{D}_0 | \tilde{\mathcal{F}}).
\end{aligned} \tag{119}$$

The last term in (119) can be upper bounded using the events in (116)-(118) as follows.

$$\mathbb{P}(\mathcal{D}_0 | \tilde{\mathcal{F}}) \leq \mathbb{P}(\mathcal{D}_0 | \tilde{\mathcal{F}}_1) + \mathbb{P}(\tilde{\mathcal{E}}_{T2} | \tilde{\mathcal{F}}) \mathbb{P}(\mathcal{D}_0 | \tilde{\mathcal{F}}_2).$$

We next analyze the R.H.S of (114), which upper bounds the type 2 error  $\beta(n, 1, \epsilon)$ , in the limit  $n \rightarrow \infty$  and  $\delta, \delta', \tilde{\delta} \rightarrow 0$ . First, note that, Also,

$$\mathbb{P}(\mathcal{D}_0 | \tilde{\mathcal{F}}_1) = \begin{cases} 1, & \text{if } P_{u^n w^n} \in T_{[\hat{P}_{UW}]_\delta}^n \\ & \text{and } P_{v^n w^n s^n y^n} \in T_{[\hat{P}_{VWSY}]_{\tilde{\delta}}}^k, \\ 0, & \text{otherwise.} \end{cases} \tag{120}$$

Hence, the terms corresponding to the event  $\tilde{\mathcal{F}}_1$  in (114) can be upper bounded (in the limit  $\delta, \tilde{\delta} \rightarrow 0$ ) as

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \lim_{\delta', \delta, \tilde{\delta} \rightarrow 0} \sum_{\substack{(u^n, v^n, w^n, s^n, y^n) \\ \in \mathcal{U}^n \times \mathcal{V}^n \times \mathcal{W}^n \times \mathcal{S}^n \times \mathcal{Y}^n}} \left[ \mathbb{P}(\mathcal{D}_0 | \tilde{\mathcal{F}}_1) \mathbb{P}(U^n = u^n, V^n = v^n | H_1) \right. \\
&\quad \left. \mathbb{P}(W^n(1) = w^n, S^n = s^n, Y^n = y^n | U^n = u^n, V^n = v^n, J = 1, \tilde{\mathcal{E}}_{NE}) \right] \\
&\leq \lim_{n \rightarrow \infty} \lim_{\delta', \delta, \tilde{\delta} \rightarrow 0} \sum_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \\ \mathcal{T}^n(\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \mathcal{S} \times \mathcal{Y})}} \sum_{\substack{(u^n, v^n, w^n, s^n, y^n) \\ \in T_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}}}^n}} \left[ \mathbb{P}(\mathcal{D}_0 | \tilde{\mathcal{F}}_1) \mathbb{P}(U^n = u^n, V^n = v^n | H_1) \right. \\
&\quad \mathbb{P}(S^n = s^n, Y^n = y^n | U^n = u^n, J = 1, W^n(1) = w^n, \tilde{\mathcal{E}}_{NE}) \\
&\quad \left. \mathbb{P}(W^n(1) = w^n | U^n = u^n, J = 1, \tilde{\mathcal{E}}_{NE}) \right] \\
&\leq \lim_{n \rightarrow \infty} \lim_{\delta', \delta, \tilde{\delta} \rightarrow 0} \sum_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \\ \mathcal{T}^n(\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \mathcal{S} \times \mathcal{Y})}} \sum_{\substack{(u^n, v^n, w^n, s^n, y^n) \\ \in T_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}}}^n}} \left[ \mathbb{P}(\mathcal{D}_0 | \tilde{\mathcal{F}}_1) \times 2^{-n(H(\tilde{U}\tilde{V}) + D(P_{\tilde{U}\tilde{V}} || Q_{UV}))} \right. \\
&\quad \left. 2^{-n(H(\tilde{W}|\tilde{U}) - \frac{1}{n}|\mathcal{U}||\mathcal{W}|\log(n+1))} 2^{-n(H(\tilde{S}\tilde{Y}|\tilde{U}\tilde{W}) + D(P_{\tilde{S}\tilde{Y}|\tilde{U}\tilde{W}} || \hat{P}_S \hat{P}_Y | UWS | P_{\tilde{U}\tilde{W}}))} \right] \\
&\leq \lim_{n \rightarrow \infty} \max_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \\ \mathcal{T}'_1(\hat{P}_{UW}, \hat{P}_{VWSY})}} \left[ 2^{-n(H(\tilde{U}\tilde{V}) + D(P_{\tilde{U}\tilde{V}} || Q_{UV}))} 2^{-n(H(\tilde{W}|\tilde{U}) - \frac{1}{n}|\mathcal{U}||\mathcal{W}|\log(n+1))} \right. \\
&\quad \left. 2^{-n(H(\tilde{S}\tilde{Y}|\tilde{U}\tilde{W}) + D(P_{\tilde{S}\tilde{Y}|\tilde{U}\tilde{W}} || \hat{P}_S \hat{P}_Y | UWS | P_{\tilde{U}\tilde{W}}))} \times 2^{-n(H(\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}) - \frac{1}{n}|\mathcal{U}||\mathcal{V}||\mathcal{W}||\mathcal{S}||\mathcal{Y}|\log(n+1))} \right]
\end{aligned}$$

$$= \lim_{n \rightarrow \infty} 2^{-nE_{1n}^*}, \quad (121)$$

where,

$$\begin{aligned} E_{1n}^* &= \min_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \\ \mathcal{T}'_1(\hat{P}_{UW}, \hat{P}_{VWSY})}} \left[ H(\tilde{U}\tilde{V}) + D(P_{\tilde{U}\tilde{V}} \| Q_{UV}) + H(\tilde{W}|\tilde{U}) + H(\tilde{S}\tilde{Y}|\tilde{U}\tilde{W}) \right. \\ &\quad \left. + D(P_{\tilde{S}\tilde{Y}|\tilde{U}\tilde{W}} \| \hat{P}_S \hat{P}_Y |_{UWS} | P_{\tilde{U}\tilde{W}}) - H(\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}) - \frac{1}{n}(|\mathcal{U}||\mathcal{W}| + |\mathcal{U}||\mathcal{V}||\mathcal{W}||\mathcal{S}||\mathcal{Y}|) \log(n+1) \right] \\ &= \min_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \\ \mathcal{T}'_1(\hat{P}_{UW}, \hat{P}_{VWSY})}} \left[ \sum_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \log \left( \frac{1}{P_{\tilde{U}\tilde{V}}} \frac{P_{\tilde{U}\tilde{V}}}{Q_{UV}} \frac{P_{\tilde{U}}}{P_{\tilde{U}\tilde{W}}} \frac{1}{P_{\tilde{S}\tilde{Y}|\tilde{U}\tilde{W}}} \frac{P_{\tilde{S}\tilde{Y}|\tilde{U}\tilde{W}}}{\hat{P}_{SY|UW}} P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \right) - o(1) \right] \\ &= \min_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \\ \mathcal{T}'_1(\hat{P}_{UW}, \hat{P}_{VWSY})}} \left[ D(P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} | Q_{UV} P_{\tilde{W}|\tilde{U}} \hat{P}_{SY|UW}) - o(1) \right] \\ &\xrightarrow{(n)} E'_1(P_{W|U}, P_S, P_{X|UWS}). \end{aligned} \quad (122)$$

Here, (122) follows from the fact that  $P_{\tilde{W}|\tilde{U}} \rightarrow P_{W|U}$  given  $\tilde{\mathcal{E}}_{NE}$ , as  $\delta \rightarrow 0$ .

Next, consider the terms corresponding to the event  $\tilde{\mathcal{F}}_2$  in (114). Given  $\tilde{\mathcal{F}}_2$ ,  $P_{\tilde{U}\tilde{W}} \in T_{[UW]_\delta}^n$  and  $\mathcal{D}_0$  occurs only if  $(V^n, S^n, Y^n) \in T_{[\hat{P}_{VSY}]_{\delta'''}}^n$ ,  $\delta''' = |\mathcal{W}|\tilde{\delta}$ , and  $H(\tilde{W}|\tilde{V}, \tilde{S}, \tilde{Y}) \geq H_{\hat{P}}(W|V, S, Y) - \gamma_2(\tilde{\delta})$ , for some  $\gamma_2(\tilde{\delta}) \in O(\tilde{\delta})$ . Thus, we have,

$$\begin{aligned} &\lim_{n \rightarrow \infty} \lim_{\delta', \delta, \tilde{\delta} \rightarrow 0} \sum_{\substack{(u^n, v^n, w^n, s^n, y^n) \\ \in \mathcal{U}^n \times \mathcal{V}^n \times \mathcal{W}^n \times \mathcal{S}^n \times \mathcal{Y}^n}} \left[ \mathbb{P}(\mathcal{D}_0 | \tilde{\mathcal{F}}_2) \times \mathbb{P}(\tilde{\mathcal{E}}_{T2} | \tilde{\mathcal{F}}) \times \mathbb{P}(U^n = u^n, V^n = v^n | H_1) \right. \\ &\quad \left. \mathbb{P}(W^n(1) = w^n, S^n = s^n, Y^n = y^n | U^n = u^n, V^n = v^n, J = 1, \tilde{\mathcal{E}}_{NE}) \right] \\ &\leq \lim_{n \rightarrow \infty} \lim_{\delta', \delta, \tilde{\delta} \rightarrow 0} \sum_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \\ \mathcal{T}^n(\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \mathcal{S} \times \mathcal{Y})}} \sum_{\substack{(u^n, v^n, w^n, s^n, y^n) \\ \in T_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}}}^n}} \left[ \mathbb{P}(\mathcal{D}_0 | \tilde{\mathcal{F}}_2) \times \mathbb{P}(\tilde{\mathcal{E}}_{T2} | \tilde{\mathcal{F}}) \right. \\ &\quad \mathbb{P}(U^n = u^n, V^n = v^n | H_1) \mathbb{P}(W^n(1) = w^n | U^n = u^n, J = 1, \tilde{\mathcal{E}}_{NE}) \\ &\quad \left. \mathbb{P}(S^n = s^n, Y^n = y^n | U^n = u^n, J = 1, W^n(1) = w^n, \tilde{\mathcal{E}}_{NE}) \right] \\ &\leq \lim_{n \rightarrow \infty} \lim_{\delta', \delta, \tilde{\delta} \rightarrow 0} \sum_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \\ \mathcal{T}^n(\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \mathcal{S} \times \mathcal{Y})}} \sum_{\substack{(u^n, v^n, w^n, s^n, y^n) \\ \in T_{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}}}^n}} \left[ \mathbb{P}(\mathcal{D}_0 | \tilde{\mathcal{F}}_2) \cdot 2 \cdot 2^{-n(I_{\hat{P}}(W; V, S, Y) - I_{\hat{P}}(U; W) - \delta_7)} \right. \\ &\quad 2^{-n(H(\tilde{U}\tilde{V}) + D(P_{\tilde{U}\tilde{V}} \| Q_{UV}))} 2^{-n(H(\tilde{W}|\tilde{U}) - \frac{1}{n}|\mathcal{U}||\mathcal{W}| \log(n+1))} \\ &\quad \left. 2^{-n(H(\tilde{S}\tilde{Y}|\tilde{U}\tilde{W}) + D(P_{\tilde{S}\tilde{Y}|\tilde{U}\tilde{W}} \| \hat{P}_S \hat{P}_Y |_{UWS} | P_{\tilde{U}\tilde{W}}))} \right] \end{aligned} \quad (123)$$

$$\begin{aligned}
&\leq \lim_{n \rightarrow \infty} \max_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \\ \mathcal{T}'_2(\hat{P}_{UW}, \hat{P}_{VWSY})}} \left[ 2^{-n(H(\tilde{U}\tilde{V}) + D(P_{\tilde{U}\tilde{V}} \| Q_{UV}))} \times 2^{-n(H(\tilde{W}|\tilde{U}) - \frac{1}{n}|\mathcal{U}||\mathcal{W}|\log(n+1))} \right. \\
&\quad \left. 2^{-n(I_{\hat{P}}(W;V,S,Y) - I_{\hat{P}}(U;W) - \delta_7 - \frac{1}{n})} \right. \\
&\quad \left. 2^{-n(H(\tilde{S}\tilde{Y}|\tilde{U}\tilde{W}) + D(P_{\tilde{S}\tilde{Y}|\tilde{U}\tilde{W}} \| \hat{P}_S \hat{P}_Y | UWS) | P_{\tilde{U}\tilde{W}}))} \times 2^{-n(H(\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}) - \frac{1}{n}|\mathcal{U}||\mathcal{V}||\mathcal{W}||\mathcal{S}||\mathcal{Y}|\log(n+1))} \right] \\
&= \lim_{n \rightarrow \infty} 2^{-nE_{2n}^*}, \tag{124}
\end{aligned}$$

where,

$$\begin{aligned}
E_{2n}^* &= \min_{\substack{P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} \in \\ \mathcal{T}'_2(\hat{P}_{UW}, \hat{P}_{VWSY})}} \left[ D(P_{\tilde{U}\tilde{V}\tilde{W}\tilde{S}\tilde{Y}} | Q_{UV} P_{\tilde{W}|\tilde{U}} \hat{P}_{SY|UW}) + I_{\hat{P}}(W;V,S,Y) - I_{\hat{P}}(U;W) - o(1) \right] \\
&\xrightarrow{(n)} E'_2(P_{W|U}, P_S, P_{X|UWS}). \tag{125}
\end{aligned}$$

In (123), we used the fact that

$$\mathbb{P}(\tilde{\mathcal{E}}_{T2} | \tilde{\mathcal{F}}) \leq 2 \times 2^{-n(I_{\hat{P}}(W;V,S,Y) - I_{\hat{P}}(U;W) - \delta_7)},$$

which follows from

$$\mathbb{P}(W^n(l) = \tilde{w}^n | \mathcal{F}_2) \leq 2 \mathbb{P}(W^k(l) = \tilde{w}^k | U^n = u^n, V^n = v^n, S^n = s^n, Y^n = y^n, \tilde{\mathcal{E}}_{NE}, \tilde{\mathcal{E}}_{T2}). \tag{126}$$

Eqn. (126) can be proved similar to (94).

Finally, we consider the case when  $\mathcal{E}_0$  holds.

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \lim_{\delta, \delta', \tilde{\delta} \rightarrow 0} \sum_{u^n, v^n} \mathbb{P}(U^n = u^n, V^n = v^n | H_1) \mathbb{P}(\mathcal{D}_0 | U^n = u^n, V^n = v^n, \mathcal{E}_0) \\
&= \lim_{n \rightarrow \infty} \lim_{\delta, \delta', \tilde{\delta} \rightarrow 0} \sum_{u^n, v^n} \mathbb{P}(U^n = u^n, V^n = v^n | H_1) \\
&\quad \sum_{s^n, y^n} \mathbb{P}(S^n = s^n, Y^n = y^n, \mathcal{D}_0 | U^n = u^n, V^n = v^n, \mathcal{E}_0) \\
&= \lim_{n \rightarrow \infty} \lim_{\delta, \delta', \tilde{\delta} \rightarrow 0} \sum_{u^n, v^n} \mathbb{P}(U^n = u^n, V^n = v^n | H_1) \left[ \sum_{s^n, y^n} \mathbb{P}(S^n = s^n, Y^n = y^n | U^n = u^n, V^n = v^n, \mathcal{E}_0) \right. \\
&\quad \left. \mathbb{P}(\mathcal{D}_0 | U^n = u^n, V^n = v^n, S^n = s^n, Y^n = y^n, \mathcal{E}_0) \right] \\
&= \lim_{n \rightarrow \infty} \lim_{\delta, \delta', \tilde{\delta} \rightarrow 0} \sum_{u^n, v^n} \mathbb{P}(U^n = u^n, V^n = v^n | H_1) \left[ \sum_{s^n, y^n} \mathbb{P}(S^n = s^n, Y^n = y^n | \mathcal{E}_0) \right]
\end{aligned}$$

$$\begin{aligned}
& \mathbb{P}(\mathcal{D}_0 | V^n = v^n, S^n = s^n, Y^n = y^n, \mathcal{E}_0) \Big] \\
&= \lim_{n \rightarrow \infty} \lim_{\delta, \delta', \tilde{\delta} \rightarrow 0} \sum_{v^n, s^n, y^n} \mathbb{P}(V^n = v^n | H_1) \mathbb{P}(S^n = s^n, Y^n = y^n | \mathcal{E}_0) \\
& \mathbb{P}(\mathcal{D}_0 | V^n = v^n, S^n = s^n, Y^n = y^n, \mathcal{E}_0). \tag{127}
\end{aligned}$$

Again,  $\mathcal{D}_0$  occurs only if there exists a sequence  $(W^n(l), V^n, S^n, Y^n) \in T_{[\hat{P}_{WVSY}]_{\tilde{\delta}}}^n$  for some  $l \in [2^{n(I_{\hat{P}}(U;W)+\delta')}]$ . Noting that the quantization codebook is independent of the  $(V^n, S^n, Y^n)$  given that  $\mathcal{E}_0$  holds, it can be shown using standard arguments that

$$\mathbb{P}(\mathcal{D}_0 | V^n = v^n, S^n = s^n, Y^n = y^n, \mathcal{E}_0) \leq 2^{-n(I_{\hat{P}}(W;V,S,Y)-I_{\hat{P}}(U;W)-\delta_7)}. \tag{128}$$

Also,

$$\mathbb{P}(S^n = s^n, Y^n = y^n | \mathcal{E}_0) \leq 2^{-n(H(\tilde{S}\tilde{Y})+D(P_{\tilde{S}\tilde{Y}}||\check{Q}_{SY}))}. \tag{129}$$

Hence, using (128) and (129) in (127), we obtain

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \lim_{\delta, \delta', \tilde{\delta} \rightarrow 0} \sum_{v^n, s^n, y^n} \mathbb{P}(V^n = v^n | H_1) \mathbb{P}(S^n = s^n, Y^n = y^n | \mathcal{E}_0) \\
& \mathbb{P}(\mathcal{D}_0 | V^n = v^n, S^n = s^n, Y^n = y^n, \mathcal{E}_0) \\
& \leq \lim_{n \rightarrow \infty} \lim_{\delta, \delta', \tilde{\delta} \rightarrow 0} \sum_{v^n, s^n, y^n} 2^{-n(H(\tilde{V})+D(P_{\tilde{V}}||Q_V))} 2^{-n(H(\tilde{S}\tilde{Y})+D(P_{\tilde{S}\tilde{Y}}||\check{Q}_{SY}))} 2^{-n(I_{\hat{P}}(W;V,S,Y)-I_{\hat{P}}(U;W)-\delta_7)} \\
& \leq \lim_{n \rightarrow \infty} (n+1)^{|\mathcal{V}||\mathcal{S}||\mathcal{Y}|} \max_{P_{\tilde{V}\tilde{S}\tilde{Y}}=\hat{P}_{VSY}} 2^{nH(\tilde{V}\tilde{S}\tilde{Y})} 2^{-n(H(\tilde{V})+D(P_{\tilde{V}}||Q_V))} 2^{-n(H(\tilde{S}\tilde{Y})+D(P_{\tilde{S}\tilde{Y}}||\check{Q}_{SY}))} \\
& \quad 2^{-n(I_{\hat{P}}(W;V,S,Y)-I_{\hat{P}}(U;W))} \\
& = \lim_{n \rightarrow \infty} 2^{-nE_{3n}^*},
\end{aligned}$$

where,

$$\begin{aligned}
E_{3n}^* &= \min_{P_{\tilde{V}\tilde{S}\tilde{Y}}=\hat{P}_{VSY}} D(P_{\tilde{V}\tilde{S}\tilde{Y}}||\check{Q}_{VSY}) + I_{\hat{P}}(W;V,S,Y) - I_{\hat{P}}(U;W) - |\mathcal{V}||\mathcal{S}||\mathcal{Y}| \log(n+1) \\
& \xrightarrow{(n)} E'_3(P_{SX'}).
\end{aligned}$$

Since the T2EE is lower bounded by the minimal value of the exponent due to the various type 2 error events, this completes the proof of the theorem.

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