Exam

M.Sc. in Quantum Fields and Fundamental Forces

TP.4 — Advanced Field Theory

2:00 - 5:00, Friday May 12, 2006

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Answer **THREE** out of the five questions

Use a separate booklet for each question. Make sure that each booklet carries your name, the course title, and the number of the question attempted.

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You may use the following results without proof:

- Loop integral in d dimensions (Minkowskian):

$$I_n(m^2) \equiv \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2 - m^2)^n} = (-1)^n \frac{im^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)}$$

- Gamma functions: $z\Gamma(z) = \Gamma(z+1)$ and

$$\Gamma\left(-m + \frac{\epsilon}{2}\right) = \frac{(-1)^m}{m!}\left(\frac{2}{\epsilon} + \sum_{p=1}^{m} \frac{1}{p} - \gamma + O(\epsilon)\right) \quad \text{for integer} \quad m \geq 0$$

- Feynman parameters:

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[xa + (1-x)b]^2}$$

- Dirac matrices:

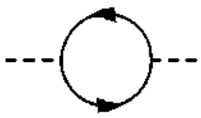
$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \mathrm{tr}\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$$

Question (1)

In the Yukawa theory,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \bar{\psi}(i\partial\!\!\!/ - m)\psi - g\bar{\psi}\psi\phi,$$

the leading quantum correction to the two-point function is given by the diagram



$$= -(-ig)^2\mu^{4-d}\int\frac{d^dp}{(2\pi)^d}\,\mathrm{tr}\left[\frac{i(p\!\!\!/ + k\!\!\!/ + m)}{(p+k)^2 - m^2}\frac{i(p\!\!\!/ + m)}{p^2 - m^2}\right], \tag{1.1}$$

where k is the momentum of the scalar.

(a) Explain the origin of the different factors that appear in Eq. (1.1).

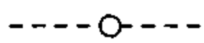
[5 marks]

(b) Show that the divergent $(1/\epsilon)$ part of the integral in $d = 4 - \epsilon$ dimensions is

$$-\frac{ig^2}{16\pi^2}\left(12m^2 - 2k^2\right)\frac{2}{\epsilon}.$$

[10 marks]

(c) In renormalised perturbation theory, there is also a counterterm diagram



$$= i(k^2\delta Z - \delta m^2).$$

What are the values of the counterterms δZ and δm^2 in the MS (minimal subtraction) scheme [or $\overline{MS}$ (modified minimal subtraction) scheme if you prefer]?

[5 marks]

[TOTAL 20 marks]

Question (2)

(a) Calculate the N-dimensional Grassmannian integral

$$\int \left(\prod_{i=1}^{N} d\theta_i^* d\theta_i \right) e^{-\theta_i^* B_{ij} \theta_j},$$

where θ_i and θ_i^* are complex Grassmannian numbers and B_{ij} are ordinary complex numbers.

[10 marks]

(b)) Use the result of (a) to write the Faddeev-Popov determinant

$$\det(i\partial^\mu D_\mu),$$

where $D_\mu^{ab} = \delta^{ab}\partial_\mu + g f^{abc} A_\mu^c$ is the covariant derivative of a non-Abelian gauge field theory, as a path integral over a ghost field c.

[5 marks]

(c) Discuss the properties of the ghost field in a non-Abelian SU(N) gauge field theory. Write down its interaction vertex or vertices.

[5 marks]

[TOTAL 20 marks]

Question (3)

The Lagrangian of scalar electrodynamics (theory of an electrically charged scalar field) is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2,$$

where ϕ is a complex scalar and A_μ is an Abelian gauge field, and we have used the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ and the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) What are the dimensionalities of the fields ϕ and A_μ in four dimensions?

[2 marks]

(b) Draw all the interaction vertices of the theory, and indicate which term in the Lagrangian each one corresponds to. (You do not have to write down the Feynman rules.)

[3 marks]

(c) List all the superficially divergent correlation functions indicating their superficial degrees of divergence. Draw the one-particle-irreducible one-loop diagrams that contribute to each one.

[5 marks]

(d) How many counterterms do you expect to need to renormalise the theory? Explain briefly why. Write the Lagrangian in terms of renormalised fields, couplings and counterterms, and explain how they are related to the bare fields and couplings. Are some of the counterterms related to each other? (You may assume that correlators with more than two photon legs either vanish or are finite.)

[10 marks]

[TOTAL 20 marks]

Question (4)

Consider a Euclidean scalar field theory with action

$$S_E = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \right)$$

and a momentum cutoff Λ. The quantum theory corresponds to the path integral

$$Z = \int_{0<k<\Lambda} \mathcal{D}\phi e^{-S_E},$$

where the subscript indicates that only the Fourier modes $\phi(k)$ with $0 < k < \Lambda$ are non-zero.

(a) Following the Wilsonian approach, one can derive an effective theory that has a lower cutoff $b\Lambda$ ($b < 1$) but describes the same physics at low energies. Its action $S_{\text{eff}}[\phi]$ is defined by requiring that

$$\int_{0<k<b\Lambda} \mathcal{D}\phi f[\phi] e^{-S_{\text{eff}}[\phi]} = \int_{0<k<\Lambda} \mathcal{D}\phi f[\phi] e^{-S_E[\phi]}$$

for any $f[\phi]$ that only depends on $k < b\Lambda$. Show that S_{eff} is given by an expression of the form

$$S_{\text{eff}}[\phi] = S_E[\phi] - \log \int_{b\Lambda<k<\Lambda} \mathcal{D}\phi_> e^{-\Delta S[\phi,\phi_>]}, \tag{4.1}$$

where, as the subscript shows, $\phi_>(k) = 0$ for $k < b\Lambda$ and for $k > \Lambda$. Write down ΔS.

[5 marks]

(b) What kinds of interaction vertices do you need to consider when you evaluate the integral in Eq. (4.1) perturbatively? Draw all the diagrams that contribute to S_{eff} to order λ^2. Do they give rise to terms that are not present in S_E?

[5 marks]

(c) Imagine being able to do the integral (4.1) to all orders in λ. What kinds of new terms would you expect to obtain?

[2 marks]

Consider now a theory with a non-renormalisable term,

$$\tilde{S}_E = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 + \frac{1}{6!}\kappa\phi^6 \right).$$

For simplicity, assume that you are close to the Gaussian fixed point so that the contribution from the integral in Eq. (4.1) is negligible, so that $\tilde{S}_{\text{eff}}[\phi] = \tilde{S}_E[\phi]$.

(d) Define new rescaled coordinates $x' = xb$. The momenta and the cutoff get rescaled as $k' = k/b$, $\Lambda' = (b\Lambda)/b = \Lambda$. Write the action $\tilde{S}_{\text{eff}}$ in the new coordinates.

[2 marks]

(e) Define a rescaled field ϕ' in such a way that the kinetic term has the canonical normalisation. How do the terms in $\tilde{S}_{\text{eff}}$ depend on b, when they are written in terms of x' and ϕ'? Which terms survive in the limit $b \to 0$?

[3 marks]

(f) Comment on the significance of your findings in parts (c) and (e) for the renormalisability of quantum field theories.

[3 marks]

[TOTAL 20 marks]

Question (5)

In terms of renormalised quantities, the QCD Lagrangian is

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) + \bar\psi_i \left(i\bar\partial - m\right)\psi_i \\ & +\frac{1}{2} g f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^{b\mu}A^{c\nu} - \frac{1}{4}g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu}A^{e\nu} \\ & - g t^a_{ij} A^a_\mu \bar\psi_i \gamma^\mu \psi_j \\ & -\frac{1}{4}\delta Z_A(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) + \bar\psi_i \left(i\delta Z_\psi \bar\partial - \delta Z_m m\right)\psi_i \\ & +\frac{1}{2}\delta Z_3 g f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^{b\mu}A^{c\nu} - \frac{1}{4}\delta Z_4 g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu}A^{e\nu} \\ & - \delta Z_2 g t^a_{ij} A^a_\mu \bar\psi_i \gamma^\mu \psi_j + \ldots \end{aligned} \tag{5.1}$$

(a) To leading order

$$\begin{aligned}\delta Z_A &= \frac{g^2}{16\pi^2}\left[\left(\frac{13}{6} - \frac{\xi}{2}\right)N - \frac{2}{3}N_f\right]\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{M^2} - \gamma\right), \\ \delta Z_\psi &= -\frac{g^2}{16\pi^2}\frac{N^2-1}{2N}\xi\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{M^2} - \gamma\right), \\ \delta Z_2 &= -\frac{g^2}{16\pi^2}\left[\frac{3}{4}N\left(\frac{1}{4}N + \frac{N^2-1}{2N}\right)\xi\right]\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{M^2} - \gamma\right), \end{aligned} \tag{5.2}$$

where N is the number of colours. Compute the leading term in the beta function

$$\beta(g) = M\left.\frac{\partial g}{\partial M}\right|_B, \tag{5.3}$$

where the B indicates that the bare quantities are kept fixed.

[8 marks]

(b) For comparison, the QED beta function is

$$\beta_{\mathrm{QED}}(e) = \frac{e^3}{12\pi^2}.$$

Solve Eq. (5.3) for $e(M)$ in QED and $g(M)$ in QCD. How do the solutions differ qualitatively from each other? What is the physical meaning of the renormalisation scale M, and how is the M-dependence of the coupling constant reflected in physical quantities such as the scattering amplitude?

[5 marks]

(c) What is meant by the Landau pole in QED? Discuss briefly its experimental significance and its implications for the existence of a continuum limit.

[3 marks]

(d) What is the analogue of the Landau pole in QCD? Discuss briefly its experimental significance and its implications for the existence of a continuum limit.

[4 marks]

[TOTAL 20 marks]