General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.
You may use the following results without proof:

- Loop integral in $d$ dimensions (Minkowskian):
  \[
  I_n(m^2) \equiv \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2)^n} = (-1)^n \frac{im^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)}
  \]

- Gamma functions: $z \Gamma(z) = \Gamma(z + 1)$ and
  \[
  \Gamma\left(-m + \frac{\epsilon}{2}\right) = \frac{(-1)^m}{m!} \left(\frac{2}{\epsilon} + \sum_{p=1}^{m} \frac{1}{p} - \gamma + O(\epsilon)\right) \quad \text{for integer } m \geq 0
  \]

- Feynman parameters:
  \[
  \frac{1}{ab} = \int_{0}^{1} \frac{dx}{[xa + (1-x)b]^2}
  \]

- Gaussian integral
  - Normal
    \[
    \frac{\int d^N q_i q_j e^{-\frac{1}{2}q^T M q}}{\int d^N q e^{-\frac{1}{2}q^T M q}} = (M^{-1})_{ij},
    \]
    where $q = (q_1, \ldots, q_N)$ and $M$ is an $N \times N$ matrix.
  - Grassmannian
    \[
    \frac{\int d^N c^* d^N c c_i c_j^* e^{-c^T M c}}{\int d^N c^* d^N c e^{-c^T M c}} = (M^{-1})_{ij},
    \]
    where $c = (c_1, \ldots, c_N)$ and $M$ is an $N \times N$ matrix.

- Gaussian two-point function
  - Normal
    \[
    \frac{\int D\phi \phi(x)\phi(y)e^{-\frac{1}{2} \int d^4xd^4y\phi(x)M(x,y)\phi(y)}}{\int D\phi e^{-\frac{1}{2} \int d^4xd^4y\phi(x)M(x,y)\phi(y)}} = M^{-1}(x,y).
    \]
  - Grassmannian
    \[
    \frac{\int D\bar{\psi}\psi(x)\bar{\psi}(y)e^{-\frac{1}{2} \int d^4xd^4y\bar{\psi}(x)M_{\alpha\beta}(x,y)\psi_{\beta}(y)}}{\int D\bar{\psi}D\psi e^{-\frac{1}{2} \int d^4xd^4y\bar{\psi}(x)M_{\alpha\beta}(x,y)\psi_{\beta}(y)}} = (M^{-1})_{\alpha\beta}(x,y).
    \]

- Gamma matrices:
  \[
  \text{tr}\gamma^\mu = 0, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},
  \]
  \[
  \text{tr}\gamma^5 = 0, \quad (\gamma^5)\dagger = \gamma^5, \quad \{\gamma^5, \gamma^\mu\} = 0.
  \]
1. (i) Show that in $4 - \epsilon$ dimensions

$$\mu^\epsilon \int \frac{d^{4-\epsilon}p}{(2\pi)^{4-\epsilon} p^2 - m^2} \frac{1}{p^2 - m^2} = \frac{im^2}{16\pi^2} \left( \frac{2}{\epsilon} + \ln \frac{4\pi \mu^2}{m^2} + 1 - \gamma + O(\epsilon) \right),$$

and

$$\mu^\epsilon \int \frac{d^{4-\epsilon}p}{(2\pi)^{4-\epsilon} (p^2 - m^2)^2} = \frac{i}{16\pi^2} \left( \frac{2}{\epsilon} + \ln \frac{4\pi \mu^2}{m^2} - \gamma + O(\epsilon) \right).$$

(ii) Consider the integral

$$I_{2\mu\nu\rho}^{\mu\nu\rho} = \mu^\epsilon \int \frac{d^{4-\epsilon}q}{(2\pi)^{4-\epsilon} ((k + q)^2 - m^2)(q^2 - m^2)}.$$  

(a) Show that the divergent part is

$$I_{2\mu\nu\rho}^{\mu\nu\rho} = -\frac{i}{64\pi^2} \left[ (k^\mu g^{\nu\rho} + k^\nu g^{\rho\mu} + k^\rho g^{\mu\nu}) \left( m^2 - \frac{k^2}{6} \right) + k^\mu k^\nu k^\rho \right] \frac{2}{\epsilon} + \text{finite}.$$  

(b) Use the integrals

$$\int_0^1 dx \ x \ln(1 - 2x)^2 = -1,$$

$$\int_0^1 dx \ x^2 \ln(1 - 2x)^2 = \frac{5}{9},$$

$$\int_0^1 dx \ x^3 \ln(1 - 2x)^2 = -\frac{1}{3},$$

to show that for $k^2 = 4m^2$, the full integral, including finite parts, is

$$I_{2\mu\nu\rho}^{\mu\nu\rho} = -\frac{im^2}{192\pi^2} (k^\mu g^{\nu\rho} + k^\nu g^{\rho\mu} + k^\rho g^{\mu\nu}) \left( \frac{2}{\epsilon} + \ln \frac{4\pi \mu^2}{m^2} - \gamma + \frac{5}{3} \right)$$

$$-\frac{im^2}{64\pi^2} (k^\mu k^\nu k^\rho) \left( \frac{2}{\epsilon} + \ln \frac{4\pi \mu^2}{m^2} - \gamma + \frac{4}{3} \right).$$  

[6 marks]

[Total 20 marks]
2. The Lagrangian of scalar electrodynamics (theory of an electrically charged scalar field) is

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\kappa} (\partial_{\mu} A^\mu)^2 + (D_{\mu} \phi)^* (D^\mu \phi) - m^2 \phi^* \phi - \frac{\lambda}{4} (\phi^* \phi)^2, \]

where \( \phi \) is a complex scalar, \( A^\mu \) is an Abelian gauge field, \( D_{\mu} = \partial_{\mu} + ieA_{\mu} \) is a covariant derivative, and \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) is the field strength tensor.

(i) Show that the Feynman rules for the interaction vertices are

\[ \leftrightarrow -i\lambda, \quad (1) \]

\[ \leftrightarrow -ie(k_{\mu} + k'_{\mu}), \quad (2) \]

\[ \leftrightarrow 2ie^2 g_{\mu\nu}. \quad (3) \]

Indicate also the momenta of the legs and show how momentum is conserved at each vertex. [6 marks]

(ii) Draw the 1PI diagrams that contribute to the scalar and photon two-point functions, at one-loop level. [3 marks]

(iii) Use the Feynman gauge propagators

\[ \leftrightarrow -i g^{\mu\nu} \frac{k^2}{k^2 - m^2}, \]

\[ \leftrightarrow i \frac{k^2}{k^2 - m^2}. \]

to write down the integrals corresponding to the 1PI diagrams in (ii) in dimensional regularisation. [7 marks]

(iv) Using the expressions given in Question 1.(i), calculate the photon two-point function at \( k = 0 \). Explain the significance of your result. [4 marks]

[Total 20 marks]
3. Consider again scalar electrodynamics (see Question 2)

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* (D^\mu \phi) - m^2 \phi^* \phi - \frac{\lambda}{4} (\phi^* \phi)^2, \]

this time in \( d \) spacetime dimensions.

(i) Find the dimensionalities of the fields \( A_\mu \) and \( \phi \) and parameters \( e, \lambda \) and \( m \). [3 marks]

(ii) Show that the dimensionality of the momentum-space 1PI correlator \( \tilde{\Gamma}_{E_A,E_\phi} \) with \( E_A \) external photon legs and \( E_\phi \) external scalar legs is

\[ [\tilde{\Gamma}_{E_A,E_\phi}] = d - (E_A + E_\phi) \frac{d-2}{2}. \]

[4 marks]

(iii) Using momentum conservation or other arguments, show that the number of loops in a diagram with \( E_A \) external photon legs and \( E_\phi \) external scalar legs, and \( V_1, V_2 \) and \( V_3 \) vertices of the types (1), (2) and (3), respectively (see Question 2) is

\[ L = V_1 + \frac{1}{2} V_2 + V_3 - \frac{1}{2} (E_A + E_\phi) + 1. \]

[4 marks]

(iv) Using dimensional analysis, show that the superficial degree of divergence of the diagram is

\[ D = 4 - (E_A + E_\phi) + (d - 4) L. \]

[4 marks]

(v) Use the previous result to explain how the renormalisability of the theory depends on \( d \)? [3 marks]

(vi) What counterterms do you need to introduce to renormalise the theory in \( d = 3 \) dimensions? Explain. [2 marks]

[Total 20 marks]
4. Consider the Georgi-Glashow model, with an SU(2) gauge field $A_\mu$ and a scalar field $\Phi$ in the adjoint representation. The Lagrangian is

$$\mathcal{L} = -\frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \text{Tr} (D_\mu \Phi)(D^{\mu} \Phi) - \lambda \left( \text{Tr} \Phi^2 - v^2 \right)^2,$$

with field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ and covariant derivative $D_\mu \Phi = \partial_\mu \Phi + ig[A_\mu, \Phi]$. This Lagrangian is invariant under the SU(2) gauge symmetry

$$A_\mu \rightarrow A_\mu^U = U A_\mu U^\dagger, \quad \Phi \rightarrow \Phi^U = U \Phi U^\dagger.$$

(i) Express the kinetic terms of the gauge field $A_\mu$ in terms of its components $A_a^\mu$ defined as $A_\mu = A_a^\mu t^a$, where the $2 \times 2$ matrices $T^a$ satisfy $[t^a, t^b] = (i/2) \epsilon^{abc} t^c$ and $\text{Tr} \ t^a t^b = \delta^{ab}/2$. Show that without gauge fixing the tree-level two-point function $\langle A_a^\mu(k) A_b^\nu(q) \rangle$ is not well defined, and explain why. [6 marks]

(ii) Explain why any gauge-invariant expectation value can be calculated using the modified Lagrangian

$$\mathcal{L}_\xi = \mathcal{L} - \frac{1}{2\xi} (\partial^\mu A_a^\mu)^2 + \partial^\mu c^a \partial_\mu c^a - g \epsilon^{abc} \partial^\mu c^a A_b^\mu c^c.$$

What are the properties of the field $c$, and what type of particle does it describe? How is the value of the parameter $\xi$ determined? [11 marks]

(iii) Show that with Lagrangian $\mathcal{L}_\xi$, the tree-level gauge-field two-point function is

$$\langle A_a^\mu(k) A_b^\nu(q) \rangle = (2\pi)^4 \delta(k + q) - \frac{i \delta^{ab}}{k^2} \left[ \eta_{\mu\nu} - (1 - \xi) \frac{k_{\mu} k_{\nu}}{k^2} \right].$$

[3 marks]

[Total 20 marks]