INTRODUCTION, OUTLOOK AND REVIEW

1. Outline and goals

The course attempts to address two questions:

- What is QFT? What is renormalization?
- How do you calculate? In particular in the standard model with $SU(3) \times SU(2) \times U(1)$ gauge symmetry.

We will give a final answer to the first question and all the details to the second. Our main tool will be path integrals.

Outline:

1. Introduction, review and outlook.

2. Path integrals:
   - Formalism: QM, free theories
   - Quantization of gauge theories: ghosts and BRST symmetry
   - Feynman rules and perturbation theory

3. Renormalization:
   - In perturbation theory
   - Renormalisation group and Wilsonian effective picture
   - Renormalisation of QED

We will be following a long and distinguished history:

- Path integrals:
  - for QM: Dirac (1933)
  - for QFT: Feynman (1949)
  - Renormalization: (1950s)
     - Schwinger, Feynman, Tomonaga, Dyson, ...
QFT as effective theory and renormalization group


renormalisation of gauge theories: asymptotic freedom

'84: 't Hooft and Veltman (1971, Nobel 1999 for electroweak)
Gross, Politzer, and Wilczek (1973, Nobel 2004, asympt. freedom)

Books:

main text: "An Introduction to Quantum Field Theory"
Peskin and Schroeder
(Errata: http://www.slac.stanford.edu/xorg/peschro/QFT.html)

see also:

- "Quantum Field Theory": Srednicki
- "Quantum Field Theory": Ryder
- "The Quantum Theory of Fields": Weinberg
- "Quantum Field Theory": Itzykson & Zuber
- "Quantum Field Theory in a Nutshell": Zee

1.2 Ingredients of quantum field theory

The usual assumptions are: a quantum field theory is based on
two ingredients: in order to describe a relativistic quantum theory

1. Hilbert space \( \mathcal{H} \)
2. Field operator \( \phi(x) \)

Recall we have the Poincaré group of symmetries:

\[ x'^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu \]

Lorentz transformations.

This must be realized as a unitary representation on \( \mathcal{H} \):

\[ \varphi (x) \rightarrow U(\xi) \varphi (x) = U(x) \varphi (x) \]

\( U^\dagger = U^{-1} \)
such that we have a homomorphism:
\[ U(ra)U(r'a) = U(rr', a+a') \]
where \( \Lambda^a \) and \( \Lambda'^a \) are, and \( a^m = \Lambda^a v_\alpha + a^\alpha \). However, by Wigner's theorem we know that the irreducible, unitary representations are labelled by \( \mathbb{R}_{+}^{m} \) and \( \mathbb{R}_{+}^{n} \).

Thus we have:
\[ H = \bigoplus_{ \mathbb{R}_{+}^{m} } H_{m, n} \]

where acting with the momentum operator \( \hat{p}^m \) (generates translations):
\[ |p^m, \rangle \in H_{m, n}, \quad \hat{p}^m |p^m, \rangle = p^m |p^m, \rangle, \quad p^m = m^2, \quad p^+ > 0 \]

This decomposition gives the spectrum of the QFT.

A) Hilbert space \( H \)

Admits a unitary representation of the Poincaré group
\[ U(ra): H \rightarrow H \]
such that:
- eigenstates of \( \hat{p}^m \) lie in the positive light-cone.
- there is a vacuum state \( |0\rangle \) such that
\[ U(ra) |0\rangle = |0\rangle \]
i.e. invariant

A typical spectrum with a mass gap, plotting total momentum \( (E, P) \):

For single-particle states, mass \( m \), spin \( s \):
\[ |p^m, \rangle \in H_{m, s}, \quad \text{normalized by } \langle q^m | p^m \rangle = \frac{1}{(2\pi)^3 2E_p \delta^{0}(p^0-q^0)} \]
We also have, for local operators $\mathcal{O}(x)$: 

\[ U(x,a) \mathcal{O}(x) U(x,a)^{-1} = \mathcal{O}(x+4a) \] 

(rl scalar op.)

transformation under Poincaré group, and they satisfy micro-causality:

\[ [\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0 \quad \text{if} \quad (x-y)^2 > 0 \quad (\text{space-like}) \]

meaning we can simultaneously measure $\mathcal{O}_1(x)$ and $\mathcal{O}_2(y)$. This is physically important because here if $x, y$ are spatially separated (spatially separated there is no way of communicating between $x$ to $y$ that we are choosing to make a measurement).

In conventional field theory it is assumed there is one special local operator called "the field."

8) Field operator $\phi(x)$

\[ \phi(x) \text{ is an operator in the Hilbert space } \mathcal{H} \text{ such that:} \]

* it has definite transformation under Poincaré:

\[ \phi(x) \rightarrow \phi(x+4a) \]

eg: for a scalar: \[ U(x,a) \phi(x) U(x,a)^{-1} = \mathcal{O}(x+4a) \]

* it satisfies micro-causality:

\[ [\phi(x), \phi(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0 \]

(For fermionic fields, this is an anti-commutator.)

* all local operators $\mathcal{O}(x)$ are built from $\phi(x)$.

These assumptions A) & B) are called the "Wightman axioms."

(Actually the definitions are slightly more subtle - need to consider the
domain of $\phi(x)$, etc., but these are the ideas). They were an
attempt to define QFT ("axiomatic field theory"). (See eg.
Schröder & Wightman: "QFT, spin and all that"). However
actually this approach is not very helpful -

> only known examples satisfying the axioms are

free theories and so $1+1$-d theories!
since it's hard to define a theory and the show it supports the axioms.

(For example, perturbation expansions don't converge so they are a bad way of defining a QFT...) However this doesn't mean the axioms are wrong...

Given a field we can define in- and out-states. Let's assume for now \( \phi(x) \) is spin-0 and we have a norm. By Lorentz covariance we have, for the single particle state:

\[
\langle q | \phi(x) | p \rangle = \sqrt{2} \frac{e^{-ip\cdot x}}{\sqrt{2\pi}} \frac{p^2 + m^2}{\text{content}}
\]

where we have fixed the normalization:

\[
\langle q | p \rangle = (2\pi)^3 2E_p \delta^{(3)}(\vec{p} - \vec{q}) \quad \text{(Lorentz covariant)}
\]

Assuming there is a mass-gap we can consider multiply scattered states at early time (relative to \( m^{-1} \)). We define a set of states:

\[
\{ |0\> , |p_1\> , |p_2\> , |p_1, p_2\> , \ldots , |p_1, \ldots, p_n\> | m \}
\]

by requiring:

\[
\langle 0 | T \phi(x_1) \cdots \phi(x_n) | p_1, \ldots, p_n | m \rangle
\]

\[
= \begin{cases} 
0 & \text{if } \sum p_\alpha^2 \neq m^2 \\
(2\pi)^n \frac{e^{-ip_1 x_1} \cdots e^{-ip_n x_n}}{\text{content}} & \text{if } m^2 = \sum p_\alpha^2
\end{cases} \quad \text{for } n \neq r \quad \text{as } \xi_0 \to \infty
\]

is exactly the expression one would have in the free theory but with the extra \( \sqrt{2} \) factor. (For \( n > r \) one gets the same expression with correlation functions.)

Then these states define

\[
\text{in-states in Hilbert space}
\]

\[
\text{kin} = \{ |0\> , |p\> , |p_1, p_2, \ldots, p_n\> | m \} \quad \text{and the same for } \xi_0 \to -\infty
\]
out Hilbert space
\[ \mathcal{H} = \{ \{ \phi \} : \langle \phi \rangle \} \]

Then we have the assumption of

asymptotic completeness: \[ \mathcal{H} = \mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}} \]

and we can define the S-matrix

\[ S = \sum \text{a (unitary) change of basis.} \]

Note how:

- for many theories, these two are not connected, and so:
  - states can then be no S-matrix (only correlating functions).

- Having introduced the field operator, we can then consider correlating functions.

\[ G_{\omega}(x_1, ..., x_n) = \langle 0 | T\Phi(x_1) \cdots \Phi(x_n) \rangle_{\omega} | 0 \rangle \]

which physically is just

"the average value of a quantum average over \( T\Phi(x_1) \cdots \Phi(x_n) \)

in the ground state".

The analog is Green's function \( G(x, t) \)

\[ < q \Phi(x) > \]

\[ < q \Phi(x) > \]

\( < q \Phi(x) > \) - correlating between particle at \( t_0 \) and at \( t_0 + t \)

An important field theory case is useful results is:

- Wightman reconstruction theorem:
  - The set of \( G_{\omega}(x_1, ..., x_n) \) completely defines the QFT. In
    other words they are equivalent to knowing the Hilbert space \( \mathcal{H} \) and the field \( \Phi(x) \).

Of course we don't know which QFTs satisfy the axioms at - but
we still always assume that knowing the correlating functions
defines the theory.
1.3 Why do we need the field \( \phi(x) \)? What is the dynamics?

There are usually two different paths to QFT that are presented:

1. Start with relativistic quantum mechanics
   
   \( \approx \) problems with interpretation (rev. energy states, prob. density)
   
   \( \approx \) doesn't capture particle products e.g. \( e^+ e^- \rightarrow 2\gamma \)

2. Start with classical field theory (e.g. EM)
   
   \( \approx \) how do you quantize?

In the picture we have developed so far the first problem is related to the need for local operators. To see this let's go back to the one-particle, Fock space (again, focusing on sqrt 0):

\[ |\text{He} \rangle = |p \rangle \quad p^2 = m^2 \]

A general state is given by:

\[ |\psi \rangle = \int \frac{dp}{(2\pi)^2 2E_p} \psi(p) \, 1p \rangle \]

In trying to interpret this state we note that defining:

\[ \Psi(q) = \int \frac{dp}{(2\pi)^2 2E_p} \psi(p) e^{-i p \cdot q} \]

gives a function satisfying the Klein-Gordon equation:

\[ (\partial^2 - m^2) \psi = 0 \quad \text{with} \]

\[ \psi(x) \]

Thus

\[ |\text{He} \rangle \quad \Rightarrow \quad \text{positive energy solutions of KG.} \]

be then have the natural conjecture:

- \( \psi(q) \) is the relativistic wavefunction describing a free

  particle at positiv \( x^\mu q^\mu \)

However, there is one basic problem with this interpretation: \( \psi(q) \)

can propagate faster than light.
If we only allow massive energies, e.g. try the eikonal a $\delta$-function:

$$\eta(p) = \begin{cases} e^{i\frac{p^0}{2m}x^0} & \text{if } t > t_0 \\ \text{zero} & \text{otherwise} \end{cases}$$

gives $\delta(t - t_0)$ at $t = t_0$.

A wavefunction that was initially localized inside the light cone can evolve to one with a tail outside the light cone. This can be corrected by including negative energy states, but then we have an instability to decaying to the negative energy states. This problem is not unique to the spin=0y variable; e.g. Breuer spin-1/2 wavefunction is equivalent to solutions of the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \Rightarrow (\gamma^\mu \partial_\mu - m^+ )\psi(x) = 0$$

so we have the same problem (in fact under wave-equations of any spin always satisfying the KG equation).

The problem here is that we are interpreting $\psi \sqrt{q^\mu}$ as the spectral

function: In QFT this is not the case. Recall the free theory:

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2E_p} \left[ a_p e^{-ipx} + a_p^* e^{ipx} \right]$$

The one-particle states are $|p\rangle = \sqrt{2E_p} a_p^* |0\rangle$ and

$$|\psi\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2E_p} \eta(p) \sqrt{2E_p} a_p^* |0\rangle$$

contributes to a linear combination of created quantum excitations of Fourier modes of the field with wave number $p^\mu$. The spectral variable enters as the argument of $\phi(x)$ not as the conjugate variable $q$. 
Thus, all notions of locality and causality are in the field
• the introduction of local operators $\phi(x)$ gives a way of defining locality in the relativistic theory.
• although we interpreted $|\psi>$ as single-particle states, this breaks down at energies near $E \sim m^2$: target states where we can start pair-producing particles. This is why the causality appears to go wrong.

But why is there one “special” local operator $\phi(x)$? Following Wigner, this is really just a property of having a particle interpretation — this allows us to define $\phi(x)$. For example,
• assume the existence of the thin decouplings:
  \[ \phi(x) \rightarrow \{ |\psi_1>, |\psi_2>, |\psi_3>, \ldots, |\psi_n>, |\psi_{n+1}>, \ldots \} \]
• define $a_{\psi}^\dagger, a_{\psi}^\dagger$ as creating an annihilation operator:
  \[ |\psi_1, \psi_2, \psi_3, \ldots, n, n+1, \ldots \rangle \]
\[ = |\psi_1, \psi_2, \psi_3, \ldots, n, n+1, \ldots \rangle \]
• define the field operator $\phi(x)$ by its asymptotic form as $x \rightarrow -\infty$
\[ \phi(x) \rightarrow \frac{1}{\sqrt{2\pi}} e^{-i\phi_n} \left( a_{\psi}^\dagger e^{-i\phi_n} + a_{\psi} e^{i\phi_n} \right) \]

The dynamics (i.e. Hamiltonian) will then determine $\phi(x)$ at all later times.

In this picture it is the fact that the in-states form a complete basis for $|\psi>$ which means that all operators can be built out of $\phi(x)$. However it is important to note that
• theories without a scale (mass-gap) have no in- and out-states, so it’s less clear that they should be described by a single field $\phi(x)$.

(And we could be, of course, equally well use these bases to define $\phi(x)$ — it would be equivalent.)
What about the dynamics? In general we expect some Hamiltonian determining the evolution of the Heisenberg picture:

$$\frac{\partial \hat{O}}{\partial t} = i \{ H, \hat{O} \}$$

This is really just the formal rearrangement commutate of the field operator again:

$$U(t/a) \hat{O}(x) U(t/a)^{-1} = \hat{O}(x+t)$$

so for translations: generated by momentum operator $\hat{P}_x$.

$$U(t/a) = e^{i \hat{P}_x t/a}$$

so if infinitesimally:

$$\alpha \hat{P}_x \hat{O} = [i \hat{P}_x, \hat{O}] \quad \text{or} \quad \partial_x \hat{O} = i \{ \hat{P}_x, \hat{O} \} \quad \hat{P}_0 = H$$

But what fun can it take? Neverless focuss on the "cluster decoupling principle". In terms of correlation functions it means:

**Cluster decoupling**

If we see probability separate off separately at point $x$ have a set of the operator, the correlation function factorizes:

$$\lim_{\lambda \to \infty} \langle \sqrt{Z} \, \text{Tr} \, \hat{O}(x_1) \ldots \hat{O}(x_n) \, \text{Tr} \, \hat{O}(y_1 + t a) \ldots \hat{O}(y_m + t a) \, \text{Tr} \rangle = \langle \sqrt{Z} \, \text{Tr} \, \hat{O}(x_1) \ldots \hat{O}(x_n) \rangle \langle \text{Tr} \hat{O}(y_1) \ldots \hat{O}(y_m) \rangle$$

If $x$ and $y$ separate. (Just decay as $e^{-\lambda t}$ for theory at mass gap.)

be for the corresponding S-matrix elements (via LSZ). It means in the corresponding S-matrix elements (via LSZ) it mean in.

Two sets of scatterings are independent. This is very reasonable and can be derived for the axioms. Weinberg argues that the only way this can happen is if:

the Hamiltonian $H$ is derived from a local Lagrangian density $\hat{H}$ in the field $\phi$:

$$H = \int \hat{\phi} \, \mathcal{L}(x)$$
The claim may be that the evidence is consistent with
the proposed scenario or even to assume that it is
described for a Lagrangian. But this is not a priori necessary
(though all our examples work this way). Note that Weinberg’s
argument is essentially perturbative — it is to do with what
kinds of more singularities in momenta can appear in Feynman
diagrams.

In conclusion:

- We have seen good evidence that QFTs should be
described by sets of quantum fields with fixed
Lagrangian dynamics in a QFT.

However, there are cautions:

1. For theories without a mass-gap one might not be
able to first define the quantum field (note we can’t
identify individual particle states). Instead one
might consider the set of all local quantum
(TM formalism or conformal field theories)

2. There might not even be a Lagrangian. If there is a
classical limit then one expects fields and Lagrangians to
emerge (e.g. electromagnetism) but if the theory is
always strongly coupled maybe this need not happen.

There are candidates for examples of non-Lagrangian theories (so-called
"M5-branes"). Also theories can exhibit duality-symmetries
giving to completely different interpretations as QFTs — that suggest
the notion of field is not really fundamental.
To review what we already knew about QFT examples, let's consider the Rosen model (often referred to as the basic example).

- **Scalar LQG**:

\[
L = \frac{1}{2} (\partial \phi)(\partial \phi) - \frac{m^2}{2} \phi^2 - \frac{1}{2} \lambda_0 \phi^4
\]

free Lagrangian \( \lambda_0 \) interaction

- **gauge**

\[
L = \frac{1}{4} (F_{\mu \nu} F^{\mu \nu}) - \frac{1}{4} \mu \phi^4
\]

free Lagrangian \( \mu \) interaction

-gauge constant \( \mu \) is dimensionless

Focusing first on \( \lambda_0 \); before calculating we need to calculate the correlation function:

\[
G_n(x_1, \ldots, x_n) = \langle \phi_{\mu}(x_1) \cdots \phi_{\mu}(x_n) \rangle_{\text{free}}
\]

where:

- \( \langle \cdots \rangle_{\text{free}} \) : ground state
- \( \phi_{\mu}(x) \) : field operator in Heisenberg picture

We do this perturbatively in the bare bare coupling constant \( \lambda_0 \)

by using the interaction picture:

\[
\phi_{\mu} \rightarrow \phi_{\mu} = i \left[ H_0, \phi_{\mu} \right]
\]

\( H_0 \) : free Hamiltonian in the interaction picture.

We then have:

\[
G_n(x_1, \ldots, x_n) = \lim_{\tau \to \infty} \langle 0 \left| \text{Tr} \phi_{\mu}(x_1) \cdots \phi_{\mu}(x_n) e^{-i \int_0^\tau \int dx^0 \int d^3 x \, \phi^4} \right| 0 \rangle
\]

where:

\[
\int_T d^4 x \, \text{Int.} \, \tau = -\frac{i}{4} \lambda_0 \int_T dx^0 \int d^3 x \, \phi^4(x)
\]

and

\( |0\rangle \) is ground state for \( H_{0,\tau} \)
The point of this formula is that:

* Correlator functions at the Interacting theory can be calculated in terms of correlator functions at the free theory.

To calculate $\text{C}^{(n)}(x_1, \ldots, x_n)$ we need to know

$$\text{C}^{(n)}_{\phi}(x_1, \ldots, x_n) = \langle 0 | \text{T} \phi(x_1) \ldots \phi(x_n) | 0 \rangle$$

[Note: we will usually drop the 'I' index - if the expectation value is for $\langle 0| \ldots |0 \rangle$ the field operators are in the interaction picture; if the expectation value is for $\langle \phi | \ldots | \phi \rangle$ then they are in the Herometer picture]. But we can calculate $\text{C}^{(n)}_{\phi}(x_1, \ldots, x_n)$ are the same as in the free theory and hence we can calculate them using Wick's theorem: (different notation for same object)

$$\text{C}_{\phi}^{(2)}(x, y) = \langle 0 | \text{T} \phi(x) \phi(y) | 0 \rangle = \phi_0(x) \phi_0(y)$$

$$D_\phi(x-y) = \frac{1}{(2\pi)^d} \int \frac{d^d p}{2\pi} e^{-ip(x-y)}$$

"Feynman propagator"

and

$$\text{C}_{\phi}^{(1)}(x_{\text{free}}) = \delta(x) \quad \text{for } n \text{ odd}$$

and for $n_{\text{even}}$ we sum over all possible pairings:

$$\text{C}_{\phi}^{(2)}(x_{\text{free}}) = \sum_{\text{pairings}} D_\phi(x_1 - x_2) \cdots D_\phi(x_n - x_n)$$

So for example

$$\text{C}_{\phi}^{(4)}(x_1, x_2, x_3, x_4) = \frac{x_1 - x_2}{x_3 - x_4} \frac{x_3 - x_4}{x_1 - x_2}$$
Expanding the exponential $e^{-i \int dx \, \phi^* \lambda \phi}$ gives the Feynman rules:

1. each line: $\phi^* \phi = D_\mu (x) - \phi^* \phi$

2. each vertex: $\phi^* \phi \phi = -i \lambda_e \int d^4x$

3. possible symmetry factors (keep track of combinatorics)

such that

\[
\langle \mathcal{M} \Phi (x_1) \phi (x_2) \phi (x_3) \lambda (x_4) \rangle = \frac{1}{2} \text{ (connected diagrams w/ n external legs)}
\]

for example:

\[
\langle \mathcal{M} \Phi (x_1) \phi (x_2) \phi (x_3) \rangle
\]

\[
= \sum \phi^* \phi \phi + \phi^* \phi \phi + \phi^* \phi \phi + \ldots
\]

If we take a Fourier transform we get the momentum space rule:

1. each line $k \cdot n = i$

2. each vertex $\sum k = 0$

3. integrate over undetermined momenta $\int d^4k / (2\pi)^4$

4. symmetry factor

5. delta function $(2\pi)^4 \delta (0)$ (momentum) for overall momentum conservation.

Before we calculate anything, what do we expect for the spectrum?

Consider the free theory:

- 3-particle continuum
- 2-particle continuum
- Single particle states $|p^\mu \rangle$

Diagram showing the spectrum with various continua and states.
For the continuum above 2m, consider:

\[ |p_1, p_2> = \sqrt{2E_{p_1}} \sqrt{2E_{p_2}} \alpha_{p_1} \alpha_{p_2} |0> \]

In this sector, the mass term:

\[ p^2 = (E - \mu)^2 \]

so the total momentum is:

\[ P^2 = \not{p}^2 + \not{m}^2 \]

we also have by definition:

\[ \langle 0 | \phi_\nu(x) | p_\nu > = e^{-\nu \cdot x} \]

and we are assuming that

the free-harmonic states \( |0>, |p>, ... \) form a complete basis for the remaining theory Hilbert space \( \mathcal{H} \).

Just as in quantum mechanics, under the perturbation, the eigenstates and eigenvalues shift. This is renormalization:

\[
\begin{align*}
|p_1 p_2> & \rightarrow |E_0 + 2m_0> \\
|p> & \rightarrow |E_0 + m_0> \\
|0> & \rightarrow |E_0>
\end{align*}
\]

"have mass \( m_0 \)"

\[ p^2 = m_0^2 \]

\[ \langle 0 | \phi_\nu(x) | p > = e^{-\nu \cdot x} \]

\[ \langle 21 | \phi(x) | p > = \sqrt{2} e^{-\nu \cdot x} \]

The bare and the added strength of the interaction \( \lambda_0 \) also renormalizes.

Thus the basic idea of renormalization is very natural—it always happens in perturbative theory. The "problem" is that namely the renormalization are infinite.

eg diverges, giving \( \infty \) connecting to \( m, \bar{\epsilon} \)
For QED the structure is the same; we have 3 fields:

\( O(x) \) \( \equiv \phi(x) \), \( \psi(x) \), \( A_\mu(x) \)

and we can calculate correlation functions perturbatively:

\[
\langle O(x) O(y) \rangle \approx \langle O(x) O(y) \rangle e^{i \int^x L_{\text{int}} dy \cdot \frac{d^4k}{(2\pi)^4} \int dx \cdot \frac{d^4k}{(2\pi)^4}}
\]

where:

\[
L_{\text{int}} = \int \frac{d^4k}{(2\pi)^4} \phi_i \phi_j \frac{4\pi}{\sqrt{\gamma}} \psi \psi
\]

We now have 2 types of propagators in the free theory:

\[
\langle O(x) O(y) \rangle = S_F(x-y)
\]

\[
= \int \frac{d^4k}{(2\pi)^4} \frac{1}{\gamma_\mu \nu m^2 + i\epsilon} e^{-i\vec{k} \cdot \vec{x}} e^{-i\vec{k} \cdot \vec{y}}
\]

(Recall \( S_F(x-y) \) is a matrix in terms of spinor indices.) We also have:

\[
\langle O(x) A_\mu(y) \rangle = A_\mu(x) A_\nu(y)
\]

\[
= D^\mu_{\nu}(x-y) - i\gamma_\mu \gamma_\nu D_\phi (x-y)
\]

Previously, we only had a somewhat incomplete argument for this expression — when we come to path integrals we will give a more complete argument.

This gives the Feynman rules:

1. propagator:
   \[
   \begin{align*}
   &\langle x \mid y \rangle = D^\mu_{\nu}(x-y) \\
   &\begin{array}{c}
   \hline
   \text{for} \quad A_\mu \\
   \hline
   \end{array} \\
   &\langle x \mid y \rangle = S_F(x-y)_a^b
   \end{align*}
   \]

2. vertex:
   \[
   \langle x \mid y \rangle = \int d^4\xi (-i \gamma_\mu \gamma_\nu) a^b
   \]

3. overall minus sign for each closed fermion loop.
and again:

\[ <\Sigma T\psi(x)\cdots\Sigma T\psi(x_n)|\Omega> = \mathbb{Z} \text{ (connectted diagrams)} \]

An important property of QED is gauge invariance: \( A_\mu \rightarrow A_\mu + \partial_\mu \chi \).

The interaction term

\[ -ie_0 \int d^4x \psi^\dagger \gamma^\mu A_\mu \psi \]

is invariant only if the current \( j^\mu = \partial_\mu \phi \) is conserved,

\[ \partial_\mu j^\mu = 0. \]

In time-ordered correlators function this operator equals

\[ \frac{2}{\mathcal{Z}^2} \langle \Sigma T j^\mu(x_1)\cdots j^\mu(x_n)\psi(x_1)\cdots\psi(x_n) |\Omega\rangle \]

\[ = \frac{1}{\mathcal{Z}^2} \sum_{\sigma} \left( \delta^{(n)}(x_1-y_1) - \delta^{(n)}(x-x_2) \right) \]

for example: after a Fourier transform,

\[ i\alpha \mu \langle \Sigma T j^\mu(t)\psi(\mathbf{p}_1)\cdots \psi(\mathbf{p}_n) |\Omega\rangle \]

\[ = -ie_0 \langle \Sigma T \psi(\mathbf{p}_1 + \mathbf{k})\cdots \psi(\mathbf{p}_n) |\Omega\rangle + e_0 \langle \Sigma T \psi(\mathbf{p}_1)\cdots \psi(\mathbf{p}_n + \mathbf{k}) |\Omega\rangle \]

or:

\[ i\alpha \mu \left( \begin{array}{c} \mu \\ \mu \end{array} \right) = e_0 \left( \begin{array}{c} \mu \\ \mu \end{array} \right) \]

Note that the terms on the RHS of the WT relation come from taking derivatives of the \( O(x_1^0 - x_2^0) \) step function in the time-ordered.

Again the theory renormalizes:

\[ m_0 \rightarrow m, \quad \mathbf{p} \rightarrow \mathbf{p} + \mathbf{k}, \quad e_0 \rightarrow e \]

and one has naive divergences in calculating these corrections.

A key point to recall is that neither the \( x_0 \) nor the GED perturbation expansion converge. The simplest argument is due...
to Ogura. Consider the simpler model of the QM anharmonic oscillator:

\[ H = \frac{1}{2} \dot{q}^2 + \frac{1}{2} \omega^2 q^2 + \frac{1}{4} \lambda q^4 \]

Here

**Hint**

We can solve it as the SHO. Now consider the perturbation in \( \lambda \).

The problem is that for \( \lambda < 0 \) the potential is unstable.

\[ \lambda < 0 \]

This means we get an infinite number of states with negative energy which cannot be perturbative at the SHO oscillation scale. Hence:

- The perturbative series cannot converge for \( \lambda < 0 \).

**Hint:** Since the perturbation converges, it doesn't distinguish \( \lambda \).

- Radius of convergence around \( \lambda = 0 \) is zero.

and the perturbation expansion does not converge. It is an asymptotic series (terms get smaller, but can then grow again).

For \( \lambda > 0 \) the argument is not exactly the same:

\[ V(q) = \frac{1}{2} \omega^2 q^2 + \frac{1}{4} \lambda q^4 \] is unstable for \( \lambda < 0 \).

For QED:

- Exponent in \( \alpha^2 / 4 \pi^2 \): fine structure constant.

If \( \alpha \) is negative, these opposite charges attract and opposite charges repel. Thus the vacuum is unstable to pair production — actually the \( e^- \) 's will bind to form a "cooper pair" (as in superconductivity) and we get a completely different ground state — not perturbatively close to the free ground state. So:

- \( \lambda > 0 \) and QED both have zero radius of convergence
  as perturbation series.
II PATH INTEGRALS

2.1 Path integrals in QM

The easiest way to see the appearance of path integrals is in quantum mechanics (see Wiener for diffusion / Brownian motion; Drake for QM (1933)).

As we will see, they are not defined formally but give a very interesting re-interpretation of QM. The extremum or field theory is essentially trivial. They can always be viewed as a formal device to keep track of manipulations one makes in correlation functions. They are ideally suited to:

1) Implementing symmetries of the theory
2) Exploring non-perturbation effects.
3) Controlling finite temperature.

Let's start with a simple non-relativistic QM theory

\[ H = \frac{\hbar}{2m} \hat{p}^2 + V(q) \]

\[ [q, \hat{p}] = i\hbar \]

We would like to calculate:

\[ U(q_a, q_b; T) = \text{amplitude for to go from position } q_a \]

\[ \text{to position } q_b \text{ in time } T \]

Let \( q (t) \) be position space wavefunctions. Then, in the Schrödinger picture, as a function of time:

\[ |q_a(t)\rangle = e^{iHt/\hbar} |q_a\rangle \]

and

\[ U(q_a, q_b; T) = \langle q_b | e^{-iHt/\hbar} |q_a\rangle \]

If we define discrete time steps small interval \( E = T/N \)

\[ e^{-i\Delta t} = e^{-i\frac{E}{N}} \]

\[ \Delta t = \frac{T}{N} \]
\[ \varepsilon = \frac{1}{N} \]
\[ e^{-iHt/N} = e^{-i\mathcal{H}t/\hbar} \cdots e^{-i\mathcal{H}t/\hbar} \]

Using the complete set of position eigenvalues:

\[ A = \int dq \; 1 |q_x < q_y| \]

and taking \( q_n = q_0 \) and \( q_0 = q_0 \) we have:

\[ U(q_0, q_0; T) = \frac{1}{\sqrt{N!}} \int dq_1 \cdots dq_{N-1} \frac{1}{2\pi} \left< q_1 | e^{-i\mathcal{H}t/\hbar} | q_2 \right> \cdots \frac{1}{2\pi} \left< q_{N-1} | e^{-i\mathcal{H}t/\hbar} | q_0 \right> \]

We also have a complete set of momentum states:

\[ A = \int dp \; 1 |p_x < p_y| \]

with \( <p|x> = \frac{1}{\sqrt{2\pi\hbar}} e^{-i\mathcal{H}t/\hbar} \)

For small \( \varepsilon \), we can then expand the exponential:

\[ \left< q_k | e^{-i\mathcal{H}t/\hbar} | q_{k-1} \right> = \left< q_k | \left( 1 - i\hbar \mathcal{H} t \right) | q_{k-1} \right> + O(\varepsilon^2) \]

\[ = \int dp_k \left< q_k | \left( 1 - i\hbar \mathcal{H} (q_k, p_k) \right) | p_k \right> \left< p_k | q_{k-1} \right> + O(\varepsilon^2) \]

\[ = \int dp_k \left( 1 - i\hbar \mathcal{H} (q_k, p_k) \right) e^{ip_k(q_k-q_{k-1})/\hbar} + O(\varepsilon^2) \]

\[ = \int dp_k \frac{2}{\pi \hbar} e^{-i\mathcal{H} t/\hbar} e^{ip_k(q_k-q_{k-1})/\hbar} + O(\varepsilon^2) \]

where we have used:

\[ \left< q_k | \mathcal{H} | p_k \right> = \left< q_k | 2m p_k^2 + V(q_k) \right> \]

\[ = \frac{1}{2m} p_k^2 + V(q_k) = \mathcal{H}(q_k, p_k) \]

(The more general \( \mathcal{H} \) there may be operator ordinary ambiguous since \( \hat{q}, \hat{p} \) do not commute.)

We can now substitute back into \( U(q_0, q_0; T) \) ...
Putting everything together gives:

$$U(q_x, q_y, T) = \int \frac{dq_x \cdots dq_{N-1}}{(2\pi \hbar)^{N/2}} \exp \left[ \frac{i}{\hbar} \sum_{k=1}^{N} (p_k (q_k - q_{k-1}) - \frac{\hbar}{\varepsilon} V(q_k)) \right]$$

If we focus just on the momentum integral, we have

$$H(p_x, q_x) = \frac{\hbar^2}{2m} p_x^2 + V(q_x),$$

thus the momentum integrals are of Gaussian form:

$$\int dp_x \exp \left[ \frac{i}{\hbar} \left( p_x (q_x - q_{x-1}) - \frac{\hbar}{\varepsilon} p_x^2 \right) \right] \approx \sqrt{\frac{2\pi \hbar}{\text{a}}} e^{-\frac{\hbar}{4\varepsilon} (p_x^2)} \frac{p_x}{\varepsilon} = I(a, b)$$

where:

$$a = \frac{\pi}{\varepsilon/\hbar}, \quad b = \epsilon (q_x - q_{x-1})/\hbar$$

Now completing the square:

$$I(a, b) = \sqrt{\frac{2\pi \hbar}{\text{a}}} \int dp_x \exp \left[ -\frac{\hbar}{4\varepsilon} (p_x - b/a)^2 \right]$$

$$= \sqrt{\frac{2\pi \hbar}{\text{a}}} \int dp_x' \exp \left[ -\frac{\hbar}{4\varepsilon} b^2 \right]$$

$$= \frac{1}{\sqrt{2\pi \hbar \varepsilon}} e^{\frac{\hbar^2 b^2}{4\varepsilon}} \exp \left[ \frac{\hbar}{4\varepsilon} (i\varepsilon/a) \mu (q_x - q_{x-1})^2 / \varepsilon^2 \right]$$

So

$$U(q_x, q_{x-1}, T) = \left( \frac{\hbar}{2\pi \hbar \varepsilon} \right)^{N/2} \int dq_x \cdots dq_{x-1} \exp \left[ \frac{i}{\hbar} \sum_{k=1}^{N} (p_k (q_k - q_{k-1}) - \frac{\hbar}{\varepsilon} V(q_k)) \right]$$

If we take the limit: $\varepsilon \to 0$ such that

$$\frac{q_k - q_{k-1}}{\varepsilon} \to q_k$$

and

$$\frac{p_k}{\varepsilon} \to p_k$$

we obtain the thermal Green's function.
and use the notation for integrate over paths:

\[ \int \mathcal{O}(q(t)) \ e^{-\frac{i}{\hbar} \int_0^T L(q,q) \ dt} \]

then:

\[ U(q_f,q_0,T) = \int \mathcal{O}(q(t)) \ e^{\frac{i}{\hbar} \int_0^T L(q,q) \ dt} \]

we can view this an integrating \( \mathcal{O}(q(t)) \) over all paths for \( q_0 \) to \( q_f \)
with a weight \( e^{-\frac{i}{\hbar} \int_0^T L(q,q) \ dt} \)

integrate over each yarn with weight \( e^{-\frac{i}{\hbar} \int_0^T L(q,q) \ dt} \)

This is analogous to the double slit experiment where you propagate the particles and then add the amplitudes for going through each slit.

However, one has to remember that the integrals over \( \mathcal{O} \) didn't really converge. Neither will the integrals over \( q_\alpha \), thus the wave function is not really well defined. Also there is a factor of \( \frac{i}{\hbar} \) in the volume \( \mathcal{O}(q(t)) \) along \( q \). To make better sense of the integral one goes to the Euclidean path integral:

\[ \int e^{\frac{i}{\hbar} \int_0^T \left( \frac{1}{2} m (dq/dt)^2 + V(q(t)) \right) dt} \]

so

\[ U(q_f,q_0,T) = \int \mathcal{O}(q(t)) \ e^{-\frac{i}{\hbar} \int_0^T \left( \frac{1}{2} m (dq/dt)^2 + V(q(t)) \right) dt} \]

\[ = \int \mathcal{O}(q(t)) \ e^{-\frac{S}{\hbar}} \]

"Euclidean action"
This expression in p. 20 makes sense mathematically: one can make a sensible notion of what $\delta q(t)$ means: when $\varepsilon \to 0$,

- general points are continuous but not necessarily differentiable
- the "$\delta q(t)$" is really short-hand for a limit of $q(t) - q(t')$
  and $|q(t) - q(t')| \approx a |t-t'|^{1/2}$ as $t \to t'$ (in practice)

There is also a physical interpretation of the Euclidean path integral
as doing statistical mechanics. Recall we have the partition function:

$$ Z(\beta) = \sum_{\text{states}} e^{-\beta E} / \beta = \frac{1}{\beta Z} \prod_{i}$$

For the bath example: a basis for the state is given by $\{q\}$

$$ Z(\beta) = \sum_{\text{states}} e^{-\beta E} / \beta = \frac{1}{\beta Z} \prod_{i}$$

but recall we had:

$$ U(q_{i}, q_{j}; T) = \langle q_{i} | e^{-iHT/\hbar} | q_{j} \rangle$$

so if we identify $T = -\beta \hbar$, we have

$$ Z(\beta) = \sum_{\text{states}} e^{-\beta E} / \beta = \frac{1}{\beta Z} \prod_{i}$$

Thus we have

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the Euclidean path integral evaluated on closed loops of

length $\beta$ calculates the partition function.
But in fact, as we have studied before, we did the p-terms and taken a limit \( \epsilon \to 0 \), \( N \to \infty \) giving:

\[ S = \int (p^2 - H) dt \]

\[ U(q_0, q_1; t) = \int_0^t dp \, e^{iS/L} \]

"Hamiltonian path integral"

where now:

\[ \int dq_1 \ldots dq_{n-1} \to \int dq \]

\[ \int \frac{dp_1 \ldots dp_n}{2\pi \hbar} \to \int dp \]

which differ from the usual used below - it is clear that there is some ambiguity in the definition of \( dq \). The usefulness of this form is that it applies to any Hamiltonian (not only those with \( p^{1/2} q \) terms).

Finally, note the classical mechanical has a very simple explanation in the path integral:

\[ U(q_0, q_1; t) = \int_{q(0)=q_0}^{q(T)=q_1} Dq(t) \, e^{iS/L} \]

Typically the phase \( iS/L \) varies quickly as one varies the path and so the contributions tend to interfere and cancel out. However, in near \( \Delta S / \Delta q = 0 \) the phase varies slowly and so contribution tend to add up, so in the limit \( \hbar \to 0 \):

- only stationary conjugate quantities:

\[ \Delta S / \Delta q = \frac{\partial}{\partial \phi} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \]

- thus path integral localized on the classical conjugate

(this is called the "stationary phase approximation").
11. Coupled kicks in QM or path integrals

As for QFT we can define QM via the correlation functions

$$G_{n}(t_{1},...,t_{n}) = \langle \{ \hat{T}_{\eta_{1}} \hat{a}(t_{1}) \hat{a}(t_{2}) ... \hat{a}(t_{n}) \} \rangle$$

These can be written in a simple form using path integrals.

Consider first just kicks \( \langle \{ \hat{T}_{\eta_{1}} \hat{a}(t_{1}) \hat{a}(t_{2}) \rangle \) assuming \( b_{1} > b_{2} \) and define:

\[ A(q_{0}, t_{1}) = \langle q_{0} \hat{a}^{\dagger}(t_{1}) \hat{a}(t_{1}) e^{iHt_{1}} | q_{0} \rangle \]

If we change to the Schrödinger picture (assuming Hermiticity and Schrödinger quanta agree at time \( t=0 \)) we have:

\[ \hat{a}(t) = e^{-itH_{0}} \hat{a} e^{itH_{0}} \]

so:

\[ A(q_{0}, t_{1}) = \langle q_{0} e^{-iH_{0}t_{1}} \hat{a} e^{iH_{0}t_{1}} \hat{a} e^{-iH_{0}t_{1}} \hat{a} e^{iH_{0}t_{1}} | q_{0} \rangle \]

If we again break the time into intervals \( t = \frac{2\pi}{N} \)

Suppose \( b_{1} \) is between \((i-1)\)th and \(i\)th.

Suppose \( b_{2} \) is between \((i-1)\)th and \(j\)th.

Then:

\[ q_{0} = q_{i}, \quad q_{n} = q_{j} \]

\[ A(q_{0}, q_{j}, t) = \int dq_{1} ... dq_{i-1} <q_{j} \hat{a} e^{-iH_{0}t_{j}} | q_{j} > < q_{i} \hat{a} e^{-iH_{0}t_{i}} | q_{i} > ... < q_{1} \hat{a} e^{-iH_{0}t_{1}} | q_{1} > \]

\[ = \int dq_{1} ... dq_{i-1} \prod_{k=i}^{j} < q_{k} | e^{-iH_{0}t_{k}} | q_{k} > \]

The calculation then goes through exactly as before. Note that the expansion into integrals is automatically time-ordered. Hence in the

limit \( t \rightarrow 0 \) we get:

\[ A(q_{0}, q_{j}, t) = \langle q_{0} \hat{a} e^{-iH_{0}t_{j}} \hat{a}(q_{j}) e^{-iH_{0}t_{j}} | q_{0} \rangle \]

\[ = \int_{q_{j}=q_{0}}^{q_{j}=q_{0}} q_{j}(t_{1}) q_{j}(t_{2}) e^{-iS} \]

To get the vacuum expectation value...
We consider the many-level system. \( H^N = E_1 |1\rangle \langle 1 | \).

\[
\lim_{T \to 0(1 - \varepsilon)} e^{-i H T} |q_0\rangle = \lim_{T \to 0(1 - \varepsilon)} \frac{1}{n} \sum_{n} e^{-i E_n T} |n\rangle \langle n | q_0\rangle
\]

\[
= \lim_{T \to 0(1 - \varepsilon)} \frac{e^{-i E_1 T}}{n} \langle n | q_0\rangle |1\rangle
\]

\[
= e^{i E_1 \varepsilon} \lim_{T \to 0(1 - \varepsilon)} e^{-i E_1 T} \langle q_0 | 1\rangle
\]

Here, \( \varepsilon \) means higher energy states are summed. Similarly:

\[
\lim_{T \to 0(1 - \varepsilon)} \langle q_1 | e^{-i H T} |q_0\rangle = \lim_{T \to 0(1 - \varepsilon)} e^{-i E_1 T} \langle q_1 | 1\rangle
\]

Hence:

\[
\langle \varphi | T q_0(1) q_1(1) | 1\rangle
\]

\[
= \lim_{T \to 0(1 - \varepsilon)} \frac{e^{-i E_1 T}}{n} \langle n | q_0\rangle \int \frac{d^3 q_{q_1} e^{i \varphi}}{q_{q_1} q_{q_0}} \langle q_{q_1} \rangle q_{q_0} e^{-i \varphi}
\]

Making the same change of variable for \( \langle 1 | 2 \rangle \) gives:

\[
\langle \varphi | T q_1(1) q_0(1) | 1\rangle
\]

\[
= \lim_{T \to 0(1 - \varepsilon)} \frac{\int d^3 q \langle q_1 | q_0 \rangle e^{i \varphi}}{\int d^3 q e^{i \varphi}}
\]

Now generally for any covalent function:

\[
\langle \varphi | T \langle q_1 | q_{1N} \rangle | 1\rangle = \lim_{T \to 0(1 - \varepsilon)} \frac{\int d^3 q \langle q_1 | q_{1N} \rangle e^{i \varphi}}{\int d^3 q e^{i \varphi}}
\]

or for any set of quantum states \( \sigma \):

\[
\langle \varphi | T \sigma(q_1) | 1\rangle = \lim_{T \to 0(1 - \varepsilon)} \frac{\int d^3 q \sigma(q_1) e^{i \varphi}}{\int d^3 q e^{i \varphi}}
\]

In the two-peak stationary phase analysis, we are taking thermal averages of two excitations.

This means that the covalent function are instantaneously leveled, momentum and time-reversal come naturally.
2.3 Path integral for scalar field theory and generating functionals

Everything we have done for QM can be taken over to scalar field theory.

\( q(t) \rightarrow q(x, \tau) = \phi(x^\tau) \)

such that the correlation functions are given by

\[
\langle \phi(x_1) \cdots \phi(x_n) \rangle = \lim_{T \to \infty} \frac{\int d\phi(x) \phi(x_1) \cdots \phi(x_n) e^{iS[\phi]}}{\int d\phi(x) e^{iS[\phi]}}
\]

where the action is given by

\[ S[\phi] = \int dx \left( \frac{1}{2} \delta^2 \phi + V(\phi) \right) \]

The simplest "derivation" is to use a lattice. For simplicity, consider 1+1 dimensional space-time: \( \phi(x^\tau) = \phi(t, x) \)

\[
\delta x \quad \ldots \quad \phi^m, \phi^m, \ldots
\]

Recall the Fock read

\( [\phi(t, x), \pi(t, y)] = i \hbar \delta(x-y) \)

to get the canonical fun

\( [\phi^m(t), p^m(t)] = i \hbar \delta^{m} \)

\( p^m(t) = \pi(t, x_0 + m \delta x) \cdot \delta x \)

we obtain how

\[
H = \int dx \left( \frac{1}{2} \pi^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + V(\phi) \right)
\]

\[
\Rightarrow \sum_{m} \delta x \left[ \frac{1}{2} \left( \frac{\partial \phi^m}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi^m}{\partial x} - \phi^{m-1} \right)^2 + V(\phi^m) \right]
\]

\[
= \sum_{m} \frac{p^m \cdot p^m}{2\hbar} + W(p^m)
\]

where

\[
W = \sum_{m} \frac{1}{2\hbar} (\phi^m - \phi^{m-1})^2 + \delta x V(\phi^m)
\]

but this is just an infinite number of particles \( \phi^m \) so we can use
our quantum mechanics result:
\[ U(q_1(x), q_2(y); T) = \int Dq \prod_\gamma Dq^\gamma(x) e^{-\frac{i}{\hbar} \int L(x, y)} \]

where
\[ L = \frac{\hbar}{2} \dot{q} \left( \frac{1}{2} (\dot{q}^2 - \frac{1}{2}(\frac{\partial^2 q}{\partial x^2})^2 - V(q) \right) \]

then taking the limit \( dx \to 0 \), \( \frac{\hbar}{2} \dot{q} \to \frac{\partial q}{\partial x} \) gives:
\[ U(q_1(x), q_2(y); T) = \int Dq \, e^{-\frac{i}{\hbar} L(q, x)} \]

and the argument for the exponent for the correlation function goes through equivalently.

Again, one can consider:
- Euclidean path integral \( \int Dq(x) e^{-S_E(q)} \)
- whether this defines a "good measure" (in analogy with measure sense mathematically defined as fuzzy and number of spaces dimensioned)

For example
\[ L = \frac{1}{2} (\dot{q}^2 - k q^2 - \frac{\partial^2 q}{\partial x^2}) \]
\[ d = 0 \times 1 + 1 \times 2 + 1 \] well defined
\[ d = 1 \times 2 + 1 \times 2 \] only defined if \( m^2 - 1 \log e^{-1} \)
\[ d = 3 \times 1 + 1 \times 2 \] only defined if \( m^2 - 1 \times 2 \)

(That is the golden dimentional questions ...).

Finally we note that there is another way of recasting the correlation function. Consider the generating function (function called partition function, by analogy with statistical mechanics); given a function \( S(x) \) define
\[ Z[S] = \int Dq(x) e^{i S[q(x)] - \frac{i}{\hbar} L[q(x)]} \]

we note that
\[ \frac{\delta Z}{\delta S[q]} = \int Dq(x) :i \dot{q}(x) \cdot i \delta q \cdot e^{i S[q(x)] + i \frac{1}{2} \int \delta q \delta q(x)} \]
where we are using the functional derivative:

\[ \frac{\delta}{\delta \phi(y)} \int d^4x \, \mathcal{L}(x) \phi(x) = \phi(y) \]

To see this, consider varying \( \mathcal{L}(x) \) to \( \mathcal{L}(x) + \delta \mathcal{L}(x) \) not \( \phi(x) \):

\[ \delta \int d^4x \, \mathcal{L}(x) \phi(x) = \int d^4x \, \delta \mathcal{L}(x) \phi(x) = \int d^4x \, \phi(x) \delta \mathcal{L}(x) \]

so

\[ \frac{\delta}{\delta \phi(y)} \int d^4x \, \mathcal{L}(x) \phi(x) = \int d^4x \, \phi(x) \delta \mathcal{L}(x) = \phi(y) \]

We then have:

\[ \frac{1}{i \mathcal{Z}(0)} \left. \frac{\delta \mathcal{Z}}{\delta \phi(y)} \right|_{\phi=0} = \frac{\int d^4x \, \phi(x) e^{\frac{i S[\phi]}{\hbar}}}{\int d^4x \, e^{\frac{i S[\phi]}{\hbar}}} = \langle 0 | \phi(y) | 0 \rangle \]

More generally:

\[ \langle 0 | \prod_{i=1}^{n} \phi(x_i) \prod_{j=1}^{m} \phi(x_j) | 0 \rangle = \left( \frac{i}{2\hbar} \right)^n \sum_{\text{permutations}} \frac{\delta^n}{\delta \phi(x_1) \ldots \delta \phi(x_n)} \left. \frac{\delta^m}{\delta \phi(x_1) \ldots \delta \phi(x_m)} \mathcal{Z} \right|_{\phi=0} \]

In statistical mechanics, we can interpret \( \mathcal{Z}(0) \) as the \( \xi^2 \) term in the function given some background source \( \mathcal{J}(x) \) (for example a magnetic field).

2. Free scalar field propagator

Let's try and calculate \( \mathcal{Z}(0) \) for the free scalar field:

\[ \mathcal{Z} = e^{\frac{i}{\hbar} \int d^4x \, \mathcal{L}(x) \phi(x)} \]

(\( \text{In the } \xi^2 \text{ term there is no renormalization here.} \)). We first note that our action can be written as, by integrating by parts,

\[ S[\phi] = \int d^4x \, \left[ \frac{1}{2} \left( \partial_\mu \phi \right) \left( \partial^\mu \phi \right) - m^2 \phi^2 \right] \]

\[ = -\frac{i}{2} \int d^4x \, \left( \partial_\mu \phi \right) \left( \partial^\mu \phi \right) \phi \]

\[ = -\frac{i}{2} \int d^4x \, d^4y \, \phi(x) \mathcal{M}(x,y) \phi(y) \]

where we have the operator:

\[ \mathcal{M}(x,y) = \frac{i}{\hbar} \delta^4(x-y) (\partial_0^2 + m^2) \]
Hence to calculate $Z[J]$ we have:

$$Z[J] = \int \mathcal{D}q(x) \ e^{-\frac{1}{2} \int \text{d}x \text{d}y \left( q(x)M(x,y)q(y) + \int \text{d}q \ J(x)q(y) \right)}$$

(again in the limit $T \to \infty$ if $\omega \to 0$).

To evaluate this instead consider the finite-dimensional ansatz:

$$q(x) \to q^j$$

replacing continuum variable $x$ by a discrete one $i$. (It will be a lattice).

and consider:

$$Z[J] = \int \text{d}q^1 \cdots \text{d}q^n \ e^{-\frac{1}{2} \sum_{i,j} q_i^T M_{ij} q_j + \sum_i J_i q^i}$$

where $M_{ij}$ is symmetric. $q_i^k, J_i$ are vectors.

If we diagonalize $\Lambda \in O(n)$

$$\Lambda^T M \Lambda = \begin{pmatrix} \Lambda' & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \Lambda' & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \Lambda' \end{pmatrix} \quad \tilde{q} = \Lambda q \quad \tilde{J} = \Lambda J$$

then:

$$\det \Lambda^T M \Lambda = 1$$

$$Z[J] = \int \text{d}q^1 \cdots \text{d}q^n \ e^{\frac{1}{2} \sum_{i,j} \tilde{q}_i \tilde{M}_{ij} \tilde{q}_j + \sum_i \tilde{J}_i \tilde{q}^i}$$

$$= \prod_i \int \text{d}q^i \ e^{-\frac{1}{2} \sum \tilde{q}_i^2 + \sum \tilde{J}_i \tilde{q}_i}$$

$$= \prod_i \sqrt{4\pi \mu_0} \ e^{-\tilde{J}_i^2/2\mu_0}$$

$$= (2\pi)^{n/2} (\det \tilde{M})^{-1/2} e^{\frac{1}{2} \int \tilde{J}^T \tilde{M}^{-1} \tilde{J}}$$

$$= (2\pi)^{n/2} (\det M)^{-1/2} e^{\frac{1}{2} \int J^T M^{-1} J}$$

Thus formally we arrive in the field theory case. (No funny at $i$ in $J$)

$$Z[J] = \int \mathcal{D}q(x) \ e^{-\frac{1}{2} \int \text{d}x \text{d}y \left( q(x)M^{-1}(x,y)q(y) \right)}$$

The easiest way to calculate the inverse of the operator $M$ is to go to momentum space: we evaluate $M^{-1}$ for every frequency and use the field
\[ \Phi(x) = \int \frac{d\Phi}{(2\pi)^2} \Phi(p) e^{-ixp} \]
\[ \Phi^*(p) = \Phi^{*\dagger(p)}, \text{ for reality.} \]

Hence:
\[ S[\varphi] = -\frac{i}{2} \int d^4x \frac{d\varphi}{(2\pi)^2} \frac{d\varphi}{(2\pi)^2} \Phi^*(p) e^{ip \cdot x} \delta^4(p - q) \Phi(q) e^{-iq \cdot x} \]
\[ = \frac{i}{2} \int d^4x \frac{d\varphi}{(2\pi)^2} \frac{d\varphi}{(2\pi)^2} \Phi^*(q^2 - m^2) \Phi(q) \Phi(q) e^{-iq \cdot x} \]

Now recall we really need
\[ \lim_{T \to \infty} \int_0^T dx : C(x) e^{-i(q^0 - p^0)x} \]
\[ = \lim_{T \to \infty} \int_0^T dx : C(x) e^{-i(q^0 - p^0)(1 - \epsilon)x} \]

We need to change variables: \( x^0 = (1 - \epsilon)x^0 \)
\[ \rho^0 = \rho^0 (1 - \epsilon) = \rho^0 (1 + \epsilon) \]

Then: doing \( x \) and \( q \) integrals:
\[ S[\varphi] = \frac{1}{2} \int d^4p \frac{d\varphi^*(p) \Phi^*(p)}{(2\pi)^2} \left( \rho^0 (1 + \epsilon) - \rho^0 = \rho^0 (1 - \epsilon) \right) \Phi(p) \]
\[ = \frac{1}{2} \int d^4p \frac{d\varphi^*(p) \Phi^*(p)}{(2\pi)^2} \left( \rho^0 - m^2 + i\epsilon \right) \Phi(p) \]
\[ e^{ixp} \]

And we can now:
\[ Z[J] = \int \exp \left[ \int d^4p \left( \frac{1}{2} \right) \left( \rho^0 - m^2 + i\epsilon \right) \Phi^*(p) \Phi(p) + i \bar{\Phi}(p) J(p) \right] \]

Note that the \( i \) actually give a \( -i \mathcal{S}[\cdot] \) contribution which makes the integral well defined: we get:
\[ \bar{\Phi} (p, q) = -2i \left( p^0 - m^2 + i\epsilon \right) \delta^{(4)}(p - q) \]

So:
\[ \bar{\Phi}^{-1} (p, q) = \frac{i}{p^2 - m^2 + i\epsilon} \delta^{(4)}(p - q) \]

\[ \mathcal{M}(p, q) \bar{\Phi}^{-1}(q, q) = (2\pi)^4 \delta^{(4)}(q - q) \]

\[ \mathcal{M}^{-1}(p, q) = \frac{i}{p^2 - m^2 + i\epsilon} \delta^{(4)}(p - q) \]

Hence:
\[ \frac{1}{2 \mathcal{M}} \mathcal{M}^{-1}(p, q, q) = \frac{1}{2} \frac{d\Phi}{(2\pi)^2} \frac{d\varphi}{(2\pi)^2} \mathcal{J}(\rho) \mathcal{M}^{-1}(p, q) \mathcal{J}(q) \]

And:
\[ \mathcal{Z}[J] = \mathcal{Z}[\mathcal{M}] \exp \left[ -\frac{1}{2} \int d^4x \int d^4y \int d^4p \bar{\mathcal{J}}(x) \mathcal{M}^{-1}(\rho) \mathcal{J}(y) \right] \]
where:
\[
D_p(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}
\]

Equation propagator

Explicating:
\[
\langle \Omega |T \phi(x_1)\phi(x_2)|\Omega \rangle = \frac{(-i)^n}{2^n n!} \frac{\delta^n}{\delta \phi(x_1) \cdots \delta \phi(x_n)} \left[ \int \frac{d^4y}{T} D_p(x-y) \phi(y) \right]_{y=0}
\]

\[
= -\frac{1}{2^n n!} \frac{\delta^n}{\delta \phi(x_1) \cdots \delta \phi(x_n)} \left[ \int \frac{d^4y}{\text{det}J} D_p(x-y) \phi(y) \right]_{y=0}
\]

\[
= D_p(x-y) ^n
\]

More generally, since \( D_p(x) \) is analytic quadratically in the exponential in \( x \), for \( n \) even:
\[
\langle \Omega |T \phi(x_1) \cdots \phi(x_n)|\Omega \rangle = \sum_{\text{permutations}} D_p(x_1-x_2) \cdots D_p(x_{n-1}-x_n)
\]

and vanish for \( n \) odd.

2.5 Free Dirac propagator

In considering Feynman fields we have the basic problem of the classical limit: we have anti-commutative relations:
\[
\{ \psi_{\alpha}(x), \psi_{\beta}(y) \} = \frac{\delta_{\alpha\beta}}{2} \delta^{(4)}(x-y)
\]

\[
\{ \bar{\psi}_{\alpha}(x), \bar{\psi}_{\beta}(y) \} = \frac{i}{\hbar} \delta_{\alpha\beta} \delta^{(4)}(x-y)
\]

In the limit \( \hbar \to 0 \), fields anti-commute — so cannot be represented by ordinary functions. Same is true for eigenvalues of fermion operators — and these are what appear in the denominators of the path integral. We need a new type of "number": Grassmann numbers:
\[
\frac{\partial}{\partial \psi} = -\gamma_\alpha \partial \quad \text{anticomute}
\]
Under a single Grassmann number θ, we have:
\[ \theta \cdot \theta = -\theta \cdot \theta = -\theta^2 = 0 \]
and we can define a Grassmann algebra \( \Lambda = \{ a + b\theta : a, b \in \mathbb{R} \} \) such that:
\[ (a + b\theta)(c + d\theta) = (a + c) + (b + d)\theta \]
\[ (a + b\theta)(c + d\theta) = ac + (bc + ad)\theta \]

More generally, we can consider an n-dimensional vector space \( V \) of Grassmann numbers with a product:
\[ v^2 = 0 \quad \text{for all} \ v \in V \]
and further multiply \( 0 = (vw)\theta = v + vw - tw + w = vw + tw \).

Given a basis \( v = v_1 \theta^i \in V \), we then define the Grassmann or exterior algebra:
\[ \Lambda(V) = \{ a_1 + a_2 \theta^i + \ldots + a_n \theta^i \theta^j + \cdots : a_i, a_j \in \mathbb{R} \} \]
where the coefficients \( a_i \) are totally antisymmetric since we have \( \theta^i \theta^j = -\theta^j \theta^i \). (Note \( (\theta^i \theta^j)(\theta^k \theta^l) = (\theta^i \theta^k)(\theta^j \theta^l) \) symmetric!)

Returning to the one-dimensional case as defined by Taylor expansions, a general function \( f \) as:
\[ f(\theta) = a + b\theta \]
and hence the derivative:
\[ \frac{df}{d\theta} f(\theta) = b \]

We then 2 Grassmann numbers we have:
\[ \frac{d}{d\theta} (y \theta) = -\frac{d}{d\theta} (\theta y) = -y \]

ie the derivate act acting on only when \( \theta \) and not \( y \). We can also define the analogue of \( \int dx f(x) \). The key point is that:
\[ \int dx f(x) = \int dy f(y + \alpha) \quad \text{where} \ x = y + \alpha \]

Hence we need, using \( \theta = \theta' + \chi \), we need:
\[ \int d\theta f(\theta) = \int d\theta' f(\theta' + \chi) \]
we also expect linearly:
\[ \int df(x + i \phi(x)) = \int df(x) + i \int df(x) \phi(x) \]

Given \( f(x) = a + ib \), so \( f(x + i) = a + ib \), we have:
\[ \int df(x) = a \]

and are also from the normality:
\[ \int df(x) = 1 \text{ etc.} \]
\[ \int df(x) + \int df(x + i \phi(x)) = \frac{1}{2} \int df(x) \phi(x) \]

Integrating is the same as differentiation!

We can also minimize complex (minimum upper)
\[ \theta = \frac{1}{2} (\theta_R + \imath \theta_I) \quad \theta^* = \frac{1}{2} (\theta_R - \imath \theta_I) \]

And
\[ \int \theta^\ast \theta = \frac{1}{2} \int (\theta_R + \imath \theta_I) (\theta_R - \imath \theta_I) \]

Generally we can now evaluate a Gaussian integral:
\[ \int \theta^\ast \theta = \frac{1}{2} \int (\theta_R + \imath \theta_I) (\theta_R - \imath \theta_I) \]

\[ = \int \theta^\ast \theta \left[ 1 - \theta^\ast \theta \right] = \int \theta^\ast \theta \left[ 1 - \theta^\ast \theta \right] \]

\[ = b \]

And hence for n-dimensional case:
\[ I = \int \theta^\ast \theta \ldots \theta_{n-1} \theta_n e^{-\frac{1}{2} \theta^\ast M \theta} \]
\[ = \int \theta^\ast \theta \ldots \theta_{n-1} \theta_n e^{\frac{1}{2} \theta^\ast M \theta} \]
\[ = \int \theta^\ast \theta \ldots \theta_{n-1} \theta_n \sum_{k=0}^{\infty} \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{1}{k!} (\theta^\ast \theta)^k \]
\[ = \int \theta^\ast \theta \ldots \theta_{n-1} \theta_n \frac{1}{k!} (\theta^\ast \theta)^k \]

Making sure we used exactly one \( \theta_i \) in \( \theta^\ast \) for each integral, leading:
\[ \theta_1 \ldots \theta_n = \xi_1 \ldots \xi_n (\theta_1 \ldots \theta_n) \]

We have
\[ I = \int \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{k!} \left( \frac{1}{2\pi} \right)^{\frac{n}{2}} \frac{1}{k!} (\theta^\ast \theta)^k \xi_1 \ldots \xi_n \]

\[ I = \frac{1}{n!} \int \theta^\ast \theta \ldots \theta_{n-1} \theta_n \theta (\theta \ldots \theta) e^{(\xi_1 \ldots \xi_n \xi_{n-1} \ldots \xi_1)} e^{\xi_1 \ldots \xi_n} \theta \theta \cdot \theta \ldots \theta \cdot \theta \]

\[ = \det M \]
If we compare:
\[ \int \cdots \int e^{-\theta^T M \theta} = \frac{(2\pi)^n}{\det M} \quad \text{conventional} \]
\[ \int \cdots \int e^{-\theta^T M \theta} = \det M \quad \text{Gaussian} \]
If we have a source:
\[ Z(y_1, y) = \int \cdots \int e^{-\theta^T M \theta + \eta^T \Phi + \bar{\Phi} \eta} \]
\[ = e^{\eta^T \overline{M}^{-1} \eta} \int \cdots \int e^{-\bar{\eta}^T \bar{M}^{-1} \eta} \]
\[ = e^{\eta^T \overline{M}^{-1} \eta} \det M \]

Finally we can calculate the Green's function for the free
Dirac field:
\[ Z[J, \gamma] = \int \cdots \int e^{i \int d^4x \left( \gamma^\mu \bar{\psi} \partial_\mu \psi + \text{other terms} \right)} \]
where \( \gamma^\mu, \gamma^0, \gamma^1, \gamma^2 \) are all Gaussian covariant numbers, and
\[ \bar{\psi} = \gamma^0 \psi \quad \Phi = \gamma^\mu \partial_\mu \phi. \]
Writing the spinor matrices:
\[ i \delta [\gamma, \bar{\gamma}] = \int d^4x d^4y \overline{\psi}_a(x) \left( i \delta^{(4)}(x-y) - i \bar{\psi}_a(x-y) \right) \psi_b(y) \]
\[ M_{ab}(x, y) \]

we have:
\[ Z[J, \gamma] = \int \cdots \int e^{i \int d^4x \psi_1(x) \overline{\psi}_a(x) M^{-1}(x, y) \psi_b(y) \delta^{(4)}(x-y)} \]

Again we take a Fermion-tensor:
\[ M_{ab} \left( \rho \eta \right) = - i \frac{1}{(2\pi)^n} \delta^{(n+1)}(\rho - \eta) \left( \phi \Psi \bar{\Psi} \right)^n \]
so:
\[ M^{-1} M_{ab} \left( \rho \eta \right) = - i \frac{1}{(2\pi)^n} \delta^{(n)}(\rho - \eta) \]
and hence:
\[ Z[J, \gamma] = \int \cdots \int e^{-i \int d^4x \overline{\psi}_a(x) M^{-1}(x, y) \psi_b(y) \delta^{(4)}(x-y) \psi_b(y) \delta^{(4)}(x-y)} \]
\[ S_F(x-y) = \int \frac{d^4y}{(2\pi)^4} \frac{i}{\not{p} + \not{w} + \not{r}} e^{-i\rho \cdot (x-y)} \]

Again we have:

\[
\langle \Omega| T \psi(x) \bar{\psi}(y) |\Omega\rangle = \frac{(i)^4}{(2\pi)^4} \frac{\delta^4(\not{p} - \not{q})}{\delta^4(\not{y} - \not{0})} \int \frac{d^4p}{(2\pi)^4} \frac{1}{\not{p} - \not{w} + \not{r}} \]

\[
= \frac{(i)^4}{(2\pi)^4} \frac{\delta^4(\not{p} - \not{q})}{\delta^4(\not{y} - \not{0})} S_F(x-y)
\]

and more generally we get Wick's theorem for fermionic fields (the
\textit{Gaußian} numbers keep track of the \textit{amps}). As before one can
view all the \textit{Gaußian} algebra formalism as purely formal —
we are really just keeping track of the free-field
combinatorics.

1.6 \textit{EM propagator}

The only free theory we have seen is electromagnetism:

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

Now the field is known and we should be able to follow the scalar
field dynamics of the path integral:

\[ \frac{i}{2} [\lambda \psi] = \int dA_\mu \ e^{i S_{\lambda\psi}} \]

writing:

\[ S = \int dx \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) \]

\[ = \frac{i}{4} \int dx \ A_\mu(x) \left( \gamma^\mu \not{D}^\nu - D^\mu \gamma^\nu \right) A_\nu(x) \]

\[ = \frac{1}{4} \int d^4p \ \tilde{A}_\mu^*(p) \left( -p^\mu \gamma^\nu + p^\nu \gamma^\mu \right) \tilde{A}_\nu(p) \]

so we have \textit{Gaußian} numbers we have:

\[ \tilde{M}_{\mu\nu}(p, q) = i \left( 2\pi \right)^4 \delta^{4}(p-q) \left( p^\mu \gamma_{\nu} - p^\nu \gamma_{\mu} \right) \]

where:

\[ 2 \left[ \lambda \psi \right] = \int dA_\mu \ e^{i S_{\lambda\psi}} \]

\[ = \frac{i}{4} \int dx \ \sum \left( \partial_\mu A_\nu \right) \tilde{A}_\mu^*(p) \tilde{M}^\nu(p, q) A_\nu(q) \]

\[ + \int dx \ \sum \left( \partial_\nu A_\mu \right) \tilde{A}_\mu(p) \tilde{M}^\nu(p, q) A_\nu(q) \]
However, we note:
\[
\tilde{\Lambda}^{\mu}(p, q) \rho \nu = i (2\pi)^{4} \delta^{(4)}(p-q) (p^{\mu} p^{\nu} - p^{\rho} p^{\nu} \rho^{\mu}) = 0
\]
so \(\tilde{\Lambda}^{\mu}(p, q)\) is a tensor which has a free expansion, and hence

\[
\tilde{\Lambda}^{\mu}(p, q) \text{ is singular and cannot be inverted}
\]

This is really just a consequence of gauge invariance
\[
A_{\mu}(x) \rightarrow A_{\mu}(x) = A_{\mu}(x) + \frac{1}{2} \partial_{\mu} \lambda(x)
\]

is a symmetry of the action. This implies that
\[
\int d^{4}q \tilde{\Lambda}^{\mu}(p, q) \left( - \frac{i}{2} \phi_{\nu} \tilde{\sigma}_{\mu \nu}(q) \right) = 0
\]
as indeed we see above. \(- \frac{i}{2} \phi_{\nu} \tilde{\sigma}_{\mu \nu}(q)\) is Hermitian and hence \(\tilde{\Lambda}^{\mu}(p, q)\) cannot be inverted.

The solutions to gauge fix and have path integrals give a very elegant way to treat the problem, first introduced by Faddeev and Popov (1967). Consider some gauge fixing condition
\[
G(A_{\mu}) = 0
\]
For example for Lorentz gauge we take \(G(A_{\mu}) = \partial_{\mu} A_{\nu} \). In the space of \(A_{\mu}(x)\) we can consider the flows under
\[
A_{\mu}^{(\omega)} = A_{\mu} + \frac{1}{2} \partial_{\mu} \lambda
\]
Now consider a vector function \( \sigma(x) \), \( \delta \) \( \sigma(x) \), \( \sigma(x) \)

\[
\delta \sigma(x)
\]
then:
\[
\int d\sigma_{1} \ldots d\sigma_{N} \det \left( \frac{\partial \delta \sigma_{i}}{\partial \sigma_{j}} \right) \delta^{\omega}(\delta \sigma(x))
\]
\[
= \int d\sigma_{1} \ldots d\sigma_{N} \delta^{\omega}(\delta \sigma(x)) = 1
\]
Thus for the path integral with the flows \( A^{(w)}_\mu \) we can write:

\[
1 = \int D\alpha(x) \det \left( \frac{\delta \mathcal{G}(A^{(w)}_\mu)}{\delta \alpha} \right) \delta(\mathcal{G}(A^{(w)}_\mu))
\]

Now we can insert this in the path integral:

\[
\tilde{Z}[\lambda] = \int DA_\mu(x) D\alpha(x) \det \left( \frac{\delta \mathcal{G}(A^{(w)}_\mu)}{\delta \alpha} \right) \delta(\mathcal{G}(A^{(w)}_\mu)) \exp\left(i\mathcal{A}^{\nu}_\mathcal{M} + i\int dx d^2x \alpha^{\nu}_\mathcal{M} \right)
\]

Let choose a particular \( \mathcal{G}(A^{(w)}_\mu) \):

\[
\mathcal{G}(A^{(w)}_\mu) = \mathcal{M}_\mu \alpha^{\nu}_\mathcal{M} - \omega(x)
\]

then:

\[
\mathcal{G}(A^{(w)}_\mu) = \mathcal{M}_\mu \alpha^{\nu}_\mathcal{M} - \frac{i}{2} \mathcal{A}^{\nu}_\mathcal{M} - \omega
\]

and we have the equality:

\[
\frac{\delta \mathcal{G}(A^{(w)}_\mu)}{\delta \alpha(x)} = \frac{\delta \mathcal{G}(A^{(w)}_\mu)}{\delta \alpha(y)} \delta^2(x-y)
\]

Hence we can change integration variables from \( A_\mu(x) \) to \( \tilde{A}_\mu(x) = A^{(w)}_\mu + \alpha_\mu(x) \) and since the action is invariant under this change we have:

\[
\tilde{Z}[\lambda] = \det \left( \frac{i}{2} \mathcal{A}^{\nu}_\mathcal{M} \right) \int DA_\mu(x) D\tilde{A}_\mu(x) \delta(\mathcal{G}(\tilde{A}_\mu)) \exp\left(i\mathcal{A}^{\nu}_\mathcal{M} + i\int dx d^2x \alpha^{\nu}_\mathcal{M} \right)
\]

(Note we have assumed \( \int dx d^2x A_\mu(x) \alpha^{\nu}_\mathcal{M} = \int dx d^2x \tilde{A}_\mu(x) \tilde{\alpha}^{\nu}_\mathcal{M} \) which is true provided \( \partial^\nu \alpha = 0 \). But since notation depends on \( \alpha(x) \) we have:

\[
\tilde{Z}[\lambda] = \det \left( \frac{i}{2} \mathcal{A}^{\nu}_\mathcal{M} \right) \int DA_\mu(x) \delta(\mathcal{A}^{\nu}_\mathcal{M} - \omega) \exp\left(i\tilde{\mathcal{A}}^{\nu}_\mathcal{M} + i\int dx d^2x \tilde{\alpha}^{\nu}_\mathcal{M} \right)
\]

This relation is true for any function \( \omega(x) \); the final trick is to interpret over \( \omega(x) \) into a Gamma:

\[
\text{add } \text{ like } \int D\omega(x) \exp\left(-i\int dx d^2x \omega(x) \mathcal{A}^{\nu}_\mathcal{M} \right)
\]

which puts these normalizations by our such \( N(\mathcal{S}) \)
hence define:

$$Z[\lambda_\mu] = \frac{Z[\lambda_\nu]}{\sqrt{\det(\tilde{G})}} / N^{12}$$

we have (remember Am for $\tilde{A}_m$)

$$Z[\lambda_\mu] = \int DA_\mu e^{iS(A) + i\int dx \tilde{A}^m A_m (x)}$$

where

$$S(A) = \frac{1}{e} \int d^4 x \left( -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_m A_n)^2 \right)$$

\[= \frac{1}{e} \int d^4 x \ A_m (x) \left[ -\partial^2 \partial^0 - d^2 (1 - \frac{1}{3}) \right] A_n (x) \]

so, now we have

$$\tilde{M}^{\mu\nu}(p,q) = e^{(2\pi)^4 \delta^{(4)}(p-q) (p^2 q^2 - (1-\frac{1}{3}) p^2 q^2)}$$

and that can be worked going:

$$Z[\lambda_\mu] = e^{-\frac{1}{2} \int dx \partial y \ D_{\mu\nu}(x-y) D^{\mu\nu}(y)}$$

where

$$D_{\mu\nu}(x-y) = \int \frac{d^4 p}{(2\pi)^4} - e^{i(p_{\mu} x - (1-\frac{1}{3}) p_{\mu} y)} \frac{1}{p^2 + \epsilon}$$

we see that $S = 1$ gives the same as we have seen before ("Feynman game"). Another possibility can come in convenient change in $S = 0$ ("Landau game").

by carrying this has given us a good (unitary) theory - but the new aspect of $S_g$ is not quite invariant - (we fixed a gauge) and we would like to argue that $Z[\tilde{A}_m]$ give the we correct $S$-matrix elements we (say) QED when we project onto physical vacuum (or vacuum polarized).
Let's note a couple of things. First remember that the difference between $\hat{\tau}(x)$ and $\tau(x)$ was

$$\hat{\tau}(x) = N(x) \det(\xi \xi^T) \cdot \int \mathcal{D}(x) \cdot \frac{1}{2} \tau(x)$$

Functional integral over gauge transformations.

We see that we have factored out and infinite volume of the gauge transformations. These were the "redundant" part of the original path integral $\int \mathcal{D}A(x)$. Also note that we could remove the "det(ξξT)" factor because it was independent of $A_\mu(x)$. When we came to non-Abelian gauge theories this will no longer be true.

We can now write down the full partition function for QCD

$$\frac{1}{Z}[\bar{\psi}, \psi, \bar{A}, A] = \int \mathcal{D}(x) \mathcal{D}A(x) e^{iS[A(x), \psi(x), \bar{\psi}(x)]}$$

where

$$S[A] = \int \mathcal{D}x \left[ i \bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x) + \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2 \xi} \xi(x) \right]$$

We are now using $\bar{\psi}, \psi$ for the fermionic source. The Fadeev-Popov procedure ensures that

any correlation function of gauge invariant operators will be independent of the choice of $\xi$.

Similarly, because we can go through the same steps:

$$\int \mathcal{D}(x) \mathcal{D}A(x) e^{iS[A(x), \psi(x), \bar{\psi}(x)]}$$

provided we choose variables $A_\mu \rightarrow \tilde{A}_\mu = A_\mu + \frac{2i}{\xi} \partial \bar{\psi} \psi$ to be gauge invariant.

(we may change variables $A_\mu \rightarrow \tilde{A}_\mu = A_\mu + \frac{2i}{\xi} \partial \bar{\psi} \psi$ to the gauge invariant objects under the operators $\partial \bar{\psi} \psi$ invariant (so they are gauge invariant!). However correlation functions of gauge dependent operators $\langle \bar{\psi} \psi \rangle \neq 0$ will be dependent on the choice of $\xi$.}
We see that the correlation functions $\mathbb{E}[\psi(\mathbf{r}_{1})\psi(\mathbf{r}_{2})]$ are well-defined and:

1. If $\mathbb{E}[\psi(\mathbf{r}_{1})\psi(\mathbf{r}_{2})]$ are gauge-invariant then the correlation functions are trace at $\mathbb{E}=0$.

2. For some non-gauge-invariant quanta (e.g., $\psi \Delta \Delta \psi$ where $\Delta$ is not present), $\mathbb{E}[\psi(\mathbf{r}_{1})\psi(\mathbf{r}_{2})]$ give an extremum at $\mathbb{E}=0$ so that true correlation functions are defined (but depend on the gauge choice and $\mathbb{E}$).

What about $S$-wave elements? Few photon states we know that only some polarizations are physical. As written we have, for example:

\[ |p, \pi, \pi \rangle = \sqrt{2} \frac{e^{i\phi}}{\sqrt{26}} a_{\pi, p}^{\dagger} |\mathbf{r} \rangle \]

with all polarizations, we then physically we want to only:

\[ A_{\mu}^{\pi} (x) \xrightarrow{\mu \rightarrow \omega} \sqrt{2} \int d^{3}q \frac{1}{(2\pi)^{3}} \frac{1}{\sqrt{26}} \left( \epsilon_{\mu}^{\pi} (q) a_{\pi, p}^{\dagger} e^{-iqx} + e^{+\mu} (p) a_{\pi, p} e^{irq} \right) \]

and we have photon states:

\[ |p, r, i, n \rangle = \sqrt{26} \frac{1}{\sqrt{26}} (a_{r, p}^{\dagger})^{+} |\mathbf{r} \rangle \quad r=0, 1, 2, \ldots \]

Recall the physical states are plus:

\[ |p, r=1, 2, \ldots \rangle \text{ transverse polarization.} \]

In the full Hilbert space we can regard this as a projector:

\[ P^{\text{phys}} = P \quad P^{2} = P. \]

We have the unitary component in the $S$-wave. For the $\mathbf{T}$ theory $\mathbf{U} = \mathbf{G}$ (let us call this $S_{\mathbf{pp}}$):

\[ S_{\mathbf{pp}}^{\dagger} S_{\mathbf{pp}} = 1 \]

The physical $S$-wave is:

\[ S = P^{\ast} S_{\mathbf{pp}} P \]

and there is no guarantee that it is unitary. For example as a simple
Example: Consider a 2D Hilbert space with
\[
S_{ef} = \left( \begin{array}{cc} a & -b^* \\ b^* & a^* \end{array} \right) \quad \text{and} \quad S_{ef}^* = S_{ef}^{-1}
\]

and take
\[
P = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)
\]

so \( P \overline{S} P^T = S_{ef} \). Let \( P = \left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right) \), then \( S_{ef}^* = S_{ef} \).

However, in QFT, the Ward-Takahashi identities actually imply \( S^* = S^{-1} \).

We get a sensible S-matrix. We will come back to this again when we consider non-Abelian theories.

Finally note that we can derive the Ward-Takahashi identities from the path integral. Consider making a change of variable
\[
\psi(x) \rightarrow \varphi'(x) = e^{i(x \cdot a)} \psi(x)
\]
\[
\overline{\psi}(x) \rightarrow \overline{\varphi}'(x) = e^{-i(x \cdot a)} \overline{\psi}(x)
\]
then \( \partial \varphi = \partial \psi \) and \( \overline{\partial} \overline{\varphi} = \partial \overline{\psi} \), but the actual s is not crucial.

Infinitesimally we get:
\[
\delta \psi = -\partial \overline{\psi} \quad \overline{\psi} = e^{i(x \cdot a)} \overline{\psi}
\]
and further comes
\[
\delta (\overline{\psi} \psi) = \overline{\delta \psi} \psi + \overline{\psi} \delta \psi = \partial \overline{\psi} \psi + \overline{\psi} \partial \psi
\]
Hence making these changes at the path integral gives:
\[
\mathcal{O} = \int \mathcal{D} \overline{\psi} \mathcal{D} \psi \mathcal{D} \mathbf{A} \mathbf{m} e^{i \overline{\psi} \mathbf{A} \mathbf{m} \psi + \overline{\psi} \partial \psi + \overline{\psi} \cdot \partial \psi}
\]

\[
= \int \mathcal{D} \overline{x} \mathcal{D} \overline{x} \mathcal{D} \mathbf{A} \mathbf{m} \mathcal{D} \mathbf{m} e^{i \overline{x} \cdot \partial \overline{x} + \overline{x} \mathbf{m} \overline{x} + \overline{x} \cdot \partial \overline{x}}
\]

\[
= e^{-i \int \overline{x} x (x) \partial \mathbf{m} + \overline{x} \mathbf{m} x (x) + \overline{x} \cdot \partial \mathbf{m} \overline{x}}
\]

\[
\partial_x \left( \int \overline{x} x (x) \partial \mathbf{m} \mathcal{D} \mathbf{m} e^{i \overline{x} \cdot \partial \overline{x} + \overline{x} \cdot \partial \overline{x}} \right)
\]

\[
= \frac{\partial \partial_x \overline{x} x (x) \partial \mathbf{m}}{\partial \mathbf{m} + \partial \overline{x} \cdot \partial \mathbf{m} \overline{x}}
\]

\[
= +i \mathbf{m} \int \overline{x} x (x) \partial \mathbf{m} \mathcal{D} \mathbf{m} e^{i \overline{x} \cdot \partial \overline{x} + \overline{x} \cdot \partial \overline{x}}
\]

\[
\overline{x} x (x) \mathbf{m} \overline{x} \overline{\mathbf{m}} x (x)
\]

\[
\mathbf{m} \mathcal{D} \mathbf{m} e^{i \overline{x} \cdot \partial \overline{x} + \overline{x} \cdot \partial \overline{x}}
\]

\[
\overline{x} x (x) \mathbf{m} \overline{x} \overline{\mathbf{m}} x (x)
\]

\[
\mathbf{m} \mathcal{D} \mathbf{m} e^{i \overline{x} \cdot \partial \overline{x} + \overline{x} \cdot \partial \overline{x}}
\]

\[
\overline{x} x (x) \mathbf{m} \overline{x} \overline{\mathbf{m}} x (x)
\]

\[
\mathbf{m} \mathcal{D} \mathbf{m} e^{i \overline{x} \cdot \partial \overline{x} + \overline{x} \cdot \partial \overline{x}}
\]
This relation generalizes the WT relation. For example if we act with:

\[ \left( \frac{(-i)^2 \frac{\delta}{\delta \psi(x_2)} \frac{\delta}{\delta \bar{\psi}(x_1)} }{2 \cos \theta} \right) \]

given the simplest non-WT identity:

\[ \frac{\delta}{\delta \psi(x)} \psi(x) \bar{\psi}(x) \Psi(x) \]

\[ = 2 \delta^{(0)}(x-x_2) \delta^{(0)}(x-x_1) \left( \delta^{(0)}(x-x_2) - \delta^{(0)}(x-x_1) \right) \]

The terms on the RHS are sometimes called "contact terms".

Note that it was key in this derivation that we integrated by parts outside the path integral. The problem is that terms like

\[ \int D\Phi \delta \Phi \delta \Phi \Psi(x) \]

is not well defined because derivatives of fields usually have to be continuous, not necessarily differentiable. (Simultaneous arguments can be used to show that correlation functions satisfying the equations of motion up to contact terms.)

2.7. Perturbation theory and the effective action

The expansion of correlation functions in terms of Feynman diagrams follows very naturally from the path integral formalism.

Consider:

\[ L = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{i}{4} \lambda \phi^4 = L_0 - \frac{i}{4} \lambda \phi^4 \]

Then:

\[ \langle \mathcal{O}[\Phi(x)] - \mathcal{O}(\Phi_{\text{ext}}) \rangle = \frac{\int D\Phi(x) \phi(x) \phi(x) e^{iS_0}}{\int D\Phi(x) e^{iS_0}} \]

\[ = \frac{1}{2 \cos \theta} \int D\Phi(x) \phi(x) \phi(x) \phi(x) e^{iS_0 - \frac{i}{4} \lambda \int dx \phi^4} \]

\[ = \frac{1}{2 \cos \theta} \int D\Phi(x) \phi(x) \phi(x) \phi(x) \sum_{k=0}^{\infty} \left( \frac{-i}{4} \lambda \int dx \phi^4 \right)^k e^{iS_0} \]
Since $\mathcal{M}e^{i\phi}$ gives the free-field correlation function we see that

$$\langle \phi_1\phi_2\phi_3\phi_4 \rangle = \frac{\langle 0 | T(\phi_1(\ldots)\phi_4) | e^{-i\phi_1|0\rangle} \rangle}{\langle 0 | T e^{-i\phi_1|0\rangle} \rangle}$$

as we derived in the operator formalism. Using Wick's theorem the
part that yields the graph, just as before, the Feynman rules:

- propagator: $\frac{1}{x-y} = D_{\phi}(x-y)$
- vertex: $\phi(x)$
- symmetry factor.

The combinatorics work as in the example:

$$\langle \phi_1\phi_2\phi_3\phi_4 \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{4}\right)^k L$$

so:

- $\frac{1}{k!}$: cancels permutations symmetry of vertices in diagram
- $\frac{1}{4^k}$: cancels permutations symmetry of lines into vertices.

Thus:

- generally each type of diagram contributes with weight one
- occasionally there is some redundancy no symmetry factor.

Recall that we then have:

$$\langle \phi_1\phi_2\phi_3\phi_4 \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{4}\right)^k L$$

More generally we need to choose an appropriate permutation
weighing for each type of integral:

$$ce = \frac{1}{3!} \cdot k\phi_2 \cdot \frac{1}{4!} \cdot \mu \phi_4 \cdot \frac{1}{6!} \cdot \phi_1\phi_2\phi_4$$

Thus we can bend off Feynman rules from the path integral.

Recall we have, from $1+i\phi_0$ factor:

$$\langle \phi_1\phi_2\phi_3\phi_4 \rangle = \sum \text{all Feynman diagrams with no bubbles} \rangle$$
Note that previously (and in Polonyi & Schroderer) these were called "connected" diagrams. But for convenience it is actually more standard to reserve first term for a different notation - so lets call them "no bubble" diagrams from now on.

To see this cancellation we can write each in the parameterized with the generating function for some general scalar theory:

\[ Z[J] = \int D\phi(x) e^{\frac{iS[\phi]}{\hbar}} = \int D\phi D\bar{\phi} \prod_x \delta(\phi(x)\bar{\phi}(x)) e^{\frac{iS[\phi]}{\hbar}}. \]

For example for \( \phi^4 \):

\[ 2 \times \text{quadratic term} = \frac{1}{2} + \frac{1}{2} \phi^4 + \frac{\infty}{\phi^4} + \cdots \]

where the same term is given by

\[ J = (\text{Sinh x}) \]

(Note that the \( \bar{J} \) cancels the parameter's symmetry at the same time.)

In consequence, because each diagram only appears once we can always write:

\[ 2 \times \text{quadratic term} = \left( \frac{1}{2} \phi^4 + \frac{\infty}{\phi^4} + \cdots \right)(1 + \infty + \cdots) = (\text{no bubbles}) \times (\text{bubbles}) \]

and that extends to all powers in \( J - \phi \) term.

\[ \frac{Z[J]}{Z[0]} = \leq (\text{no bubble diagram}) \]

\[ = 0 + \frac{1}{2} \phi^4 + \frac{1}{2} \phi^4 + \frac{1}{2} \phi^4 + \cdots \]

since \( Z[0] = \leq (\text{bubbles}) \).

We can make 2 further simplifications: Consider for example the 4-point function in \( \phi^4 \):

\[ \text{(Diagram)} \]

\[ \leftarrow \phi \rightarrow \]
We can expand:

\[
\begin{align*}
\sum_{j} & = J_{j} \times J_{j} + \frac{3}{2} J_{j} \times J_{j} + \frac{3}{2} J_{j} \times J_{j} + 4 J_{j} \times J_{j} \\
& \quad + \frac{1}{2} \sum_{j} J_{j} \times J_{j} + \frac{1}{2} \left( \frac{1}{2} \sum_{j} J_{j} \times J_{j} \right)^{2}
\end{align*}
\]

or equivalently

\[
\text{quartic term in } Z[J] = \frac{1}{4} \sum_{j} J_{j} \times J_{j} + \frac{1}{2} \left( \frac{1}{2} \sum_{j} J_{j} \times J_{j} \right)^{2}
\]

where

\[
\sum_{j} = \sum_{j} + \frac{1}{2} \left( \sum_{j} J_{j} \times J_{j} \right)^{2}
\]

The factor \( \frac{1}{2} \) comes from the generalization symmetry

\[
\begin{align*}
\sum_{j} & = \sum_{j} \times \sum_{j} \times \sum_{j} \times \sum_{j} \\
& \text{has } 4! \text{ free fermions, i.e. } 4 \text{ spins}
\end{align*}
\]

\[
\begin{align*}
\sum_{j} & = \sum_{j} \times \sum_{j} \times \sum_{j} \times \sum_{j} \\
& \text{has } 3 \cdot 2 \cdot 4 \text{ free bosons or particles and fermionic } 3 \text{-pairs}
\end{align*}
\]

More generally, one can show (except by induction),

\[
Z[J] = \sum_{k=0}^{\infty} \frac{1}{k!} x \left( \text{sum of connected diagrams} \right)^{k}
\]

\[
= \exp \left( \text{sum of connected diagrams} \right)
\]

\[
= e^{-i E[J]}
\]

where connected means there is a path between any 2 points on diagram

\[
G_{\mu}(x_{1}, \ldots, x_{n}) = \left< \phi_{\mu}(x_{n}) \phi_{\mu}(0) \right> \text{ connected}
\]

\[
= \left< \prod_{i=1}^{n} \phi_{\alpha}(x_{i}) \phi_{\beta}(x_{i}) \right> \sum_{C} \text{ connected}
\]

\[
= e^{\frac{1}{2} \sum_{C} \sum_{\alpha, \beta} E[C]}
\]

where the generating functional

\[
-\int \mathcal{D} x \mathcal{D} \phi \left< \phi_{\mu}(x_{n}) \phi_{\mu}(0) \right> = \ln Z[J]
\]
However we can go further: in the expansion:

\[\Gamma = x + 3xx + 4xx + \ldots\]

the diagoon $x^2$ is really just $x$ with a correction on the legs. Thus we should be able to decouple:

\[\Gamma = G_2(x_1, x_2, y_1, y_2) + G_2(x_3, y_3) + \ldots\]

or in terms of correlation functions:

\[G_4(x_1, x_2, y_1, y_2) = G_2(x_1, x_2) G_2(y_1, y_2) G_2(x_3, y_3) G_2(x_4, y_4)\]

Thus gives a sort of “tree-level” expansion of $G_4$. In eq:

\[\Gamma = x + 3xx + \ldots\]

\[\Gamma_3 \text{ vanishes.}\]

there are no $\Delta p$ propagators on external legs (more than are in the expansion of $G_2$). Such expansions exist in all $G_n$ and we define:

\[\Gamma_n(y_1, \ldots, y_n) : \text{vertex function or } 1\text{-point irreducible (1PI) correlation function.}\]

The term 1PI refers to:

- a diagram is 1PI one-particle reducible if it decouples into 2 pieces when a suitable line is cut.

so for example:

\[\Gamma = G_2(x_1, x_2, y_1, y_2) + G_2(x_3, y_3) + \ldots\]
as before we define a generating functional called the "effective action"

\[ i\Gamma[\phi(x)] = \frac{2}{\hbar} \sum \frac{1}{\sqrt{2\pi\hbar}} \int (dx_1...dx_n) \phi(x_1) ... \phi(x_n) \Gamma_n(x_1,...,x_n) \]

so that

\[ \Gamma_n(x_1,...,x_n) = i \sum \frac{\delta \Gamma[\phi]}{\delta \phi(x)} \bigg|_{\phi=\phi_n} \Gamma[\phi] \]

Note we have changed our conventions from the functional form \( \Gamma[\phi(0)] \) to \( \phi(x) \).
The reason for this is that \( \Gamma[\phi] \) is related to \( E[\phi] \) via a

Legendre transform:

\[ \Gamma[\phi] = -E[\phi] + \int dx \phi(x) \phi(x) \]

where:

\[ \frac{\delta E[\phi]}{\delta \phi(x)} = \phi(x) \]

(degree \( \phi \) as functional of \( \phi \),

vice versa).

or equivalently:

\[ -E[\phi] = \Gamma[\phi] + \int dx \phi(x) \phi(x) \]

\[ \frac{\delta \Gamma[\phi]}{\delta \phi(x)} = -\phi(x) \]

since for the first equation:

\[ \frac{\delta \Gamma[\phi]}{\delta \phi(x)} = -\int dy \frac{\delta \phi(y)}{\delta \phi(x)} \frac{\delta \phi(y)}{\delta \phi(x)} \frac{\delta E[\phi]}{\delta \phi(y)} \phi(x) = -\phi(x) \]

hence:

\[ 2 \Gamma[\phi] = -\int \delta \Gamma[\phi]/\delta \phi(x) \delta \phi(x) \]

we find:

\[ \frac{\delta \Gamma[\phi]}{\delta \phi(x)} = -\phi(x) \]

\[ \frac{\delta \Gamma[\phi]}{\delta \phi(y)} = \left( \frac{\delta \Gamma[\phi]}{\delta \phi(x)} \right)^{-1} \]

so setting \( \delta \phi(x) = 0 \):

\[ \phi(x,y) = \Gamma[\phi(x,y)]^{-1} \]

By similar use of the chain rule, remember \( \Gamma \) is like a matrix:
and for matrices \(\delta M^{-1} = -M^{-1}\delta M \cdot M^{-1}\).

\[
\frac{\delta^3 E}{\delta J(x_1) \delta J(x_2) \delta J(x_3)} = \int \delta y_1 \cdot \delta y_2 \cdot \delta y_3 \left( \frac{\delta^2}{\delta \phi(x_1) \delta \phi(y_1)} \right)^{-1} \frac{\delta^2}{\delta \phi(x_2) \delta \phi(y_2)} \left( \frac{\delta^2}{\delta \phi(y_1) \delta \phi(y_2)} \right)^{-1} \left( \frac{\delta^2}{\delta \phi(x_3) \delta \phi(y_3)} \right)^{-1} \left( \frac{\delta^2}{\delta \phi(x_2) \delta \phi(y_2)} \right)
\]

Using the chain rule and

\[
\frac{\delta \phi(x_3)}{\delta J(x_3)} = \frac{\delta^2}{\delta J(x_3) \delta \phi(y_3)}
\]

we have:

\[
\frac{\delta^3 E}{\delta J(x_1) \delta J(x_2) \delta J(x_3)} = \int \delta y_1 \cdot \delta y_2 \cdot \delta y_3 \left( \frac{\delta^2}{\delta \phi(x_1) \delta \phi(y_1)} \right)^{-1} \left( \frac{\delta^2}{\delta \phi(x_2) \delta \phi(y_2)} \right)^{-1} \left( \frac{\delta^2}{\delta \phi(x_3) \delta \phi(y_3)} \right)^{-1} \frac{\delta^2}{\delta \phi(x_2) \delta \phi(y_2)}
\]

so setting \(J = 0\):

\[
G_3(x_1, y_1, x_2) = \int \delta y_1 \cdot \delta y_2 \cdot \delta y_3 \cdot G_2(x_1, y_1) \cdot G_1(x_2, y_2) \cdot G_3(y_3, y_3)
\]

as required. Again using some inductive arguments one can show that this Legendre transform gives the correct result for all acts.

Note that the term "effective action" is motivated the following way. It is given an expansion of \(Z[J]\) in terms of tree diagrams. To see this, consider the classical limit of \(Z[J]\) - the path integral will be dominated by the classical solution. By stationary phase,

\[
Z[J] \approx Z[0] e^{iS[J] + i\text{const} \cdot S[J](x)}
\]

where \(\phi_0\) is a solution of the classical equation of motion:

\[
\frac{\delta S}{\delta \phi(x)} = -S(x)
\]

If we consider \(\delta \phi\) the equation reads:
We have:

\[(\delta^2 + m^2) \Phi_c(x) = J - \frac{i}{3!} A \Phi_c^3\]

We can solve this perturbatively in \( \lambda_c \) using the Feynman propagator:

\[(\delta^2 + m^2) D_F(x-y) = -i \delta^{(4)}(x-y)\]

Hereafter, diagonalizing this gives an expansion in tree diagrams:

Writing, a perturbative expansion:

\[\Phi_c(x) = \Phi_c^{(0)}(x) + \Phi_c^{(1)}(x) + \Phi_c^{(2)}(x) + \ldots\]

we have:

1. Zeroth order:

\[\Phi_c^{(0)}(x) = \frac{i}{\delta!} J \int d^4 y \ D_F(x-y) \ J(y)\]

2. First order:

\[\Phi_c^{(1)}(x) = -\frac{i}{\delta!} \lambda_c \Phi_c^{(0)}(x) \frac{1}{\delta!} J \int d^4 y \ D_F(x-y) \ \frac{i}{\delta!} \lambda_c \Phi_c^{(0)}(y) \]

\[= \frac{1}{\delta!} \lambda_c \int d^4 y_1 d^4 y_2 d^4 y_3 d^4 y_4 \ D_F(x-y_1) \ D_F(y-y_2) \ J(y_1) \ J(y_3) d^4 y_2 \]

3. Second order:

\[\Phi_c^{(2)}(x) = \frac{i}{\delta!} \lambda_c \left( \Phi_c^{(0)}(x) \right)^2 \frac{1}{\delta!} J \int d^4 y \ D_F(x-y) \]

\[\Phi_c^{(2)}(x) = \frac{i}{\delta!} \lambda_c \left( \Phi_c^{(0)}(x) \right)^2 \frac{1}{\delta!} J \int d^4 y \ D_F(x-y) \]

et al. This means that, substituting the solution for \( \Phi_c(x) \) back into \( S[\Phi_c] + \int d^4 x \ J(x) \Phi_c(x) \), gives us the classical limit:

\[4 \Phi_c \ln \left( \frac{e^{\Phi_c}}{2 \pi i} \right) = i S[\Phi_c] + \int d^4 x \ J(x) \Phi_c(x)\]

\[= \frac{1}{\delta!} \lambda_c \left( \Phi_c^{(0)}(x) \right)^2 \frac{1}{\delta!} J \int d^4 y \ D_F(x-y) \]

and we have a tree expansion.

The point about the effective action \( \Gamma[\Phi_c] \) is that we have the
some kind of tree expansion except we have included all the quantum (loop) corrections to the propagator and the vertices
\[ \Gamma_2(x_1, y_2) \]
\[ \times \Gamma_4(x_1, x_2, x_3, x_4) \]
and the loops generate higher-order interaction vertices, e.g., 6pt vertex
\[ \text{Thus } \Gamma \text{ is the "loop corrected" classical action.} \]
\[ -i \delta \text{[} J \text{] } = \sum_{J} \Gamma_J + \sum_{J J J} \text{ etc.} \]
\[ \text{two is for } \lambda \phi^4 \text{, where } \Gamma_{\text{planar}} = 0 \]

where we sum over tree diagrams.

A few further comments. First note that the tree expansion argument above assumed that when \( \lambda = 0 \), we have \( \phi^4 = 0 \), m other words there are no tadpole contributions and for symmetry breaking one needs to slightly modify the \( \phi^4 \) term to get the effective action in these cases.

As we mentioned, \( \Gamma_4 (x_1, \ldots, x_4) \) are calculated using the usual Feynman rules but without propagators or the external leg. Usually overall momentum factor conservation factors are also usually dropped. Thus for example, in momentum space:
\[ \Gamma_4 \approx (-\lambda)^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p + k)^2 - m^2} \frac{1}{(p_k - k')^2 - m^2} \]
[\text{symmetry factor: with no } \delta^4(p_k - k')]
Also, although strictly $A_2(x,y) = G_2(x,y)$, it is usual to define a slightly different vertex in this case:

\[ \boxed{ 1 \Gamma = \mathcal{O} + \mathcal{O} + \mathcal{O} + \ldots } \]

re doesn't include $\mathcal{O}$ sum $\mathcal{O} + \mathcal{O} + \mathcal{O} + \mathcal{O} + \mathcal{O} + \ldots$ is one-vertex reducible, he then has

\[ \boxed{ 1 \Gamma = \mathcal{O} + \mathcal{O} + \mathcal{O} + \mathcal{O} + \mathcal{O} + \ldots } \]

so

\[ G_2(p, q) = \mathcal{G}_2(p, q) \]

\[ = \frac{(2\pi)^4 \delta^{(4)}(p+q)}{p^2-m^2} \sum_{k=0}^{\infty} \left( \frac{i \pi}{p^2-m^2} \Gamma_2(p) \right)^k \]

\[ = \frac{(2\pi)^4 \delta^{(4)}(p+q)}{p^2-m^2} \left( 1 - \frac{i \pi}{p^2-m^2} \Gamma_2(p) \right)^{-1} \]

\[ = \frac{(2\pi)^4 \delta^{(4)}(p+q)}{p^2-m^2-i\pi \Gamma_2(p)} \]

Finally we can define the effective potential by setting $\Phi^2 = A \Phi$ a current

\[ \mathcal{W} = \mathcal{I} \rightangle \left[ \Phi = \Phi \right] = - \int d^4x \nabla_\mu V_{\text{Eff}}(A) = - \langle \text{Volume} \rangle \cdot V_{\text{Eff}}(A) \]

going in the Feynman heuristic to space and time:

\[ V_{\text{Eff}}(A) = \sum_{k=0}^{\infty} \frac{1}{k!} \int d^4x_1 \ldots d^4x_n \Phi_{\nu}(x_1) \cdots \Phi_{\nu}(x_n) \Gamma_n(x_1, \ldots, x_n) \]

we can get:

\[ V_{\text{Eff}}(A) = i \sum_{k=2}^{\infty} \frac{1}{k!} A^k \Phi_{\nu}(0, 0, \ldots, 0) \]

(we assume $\Gamma_0 = 0$ → no tadpoles and ignore the overall constant normalizatiuon. If we make it one-loop for $\lambda > 0$ we get:

\[ -i \text{V_{Eff}(A)} = \mathcal{A} \mathcal{A} + \mathcal{A} \mathcal{A} + \mathcal{A} \mathcal{A} + \ldots \]

at a given order we have: if $p$ is the momentum in the loop tachan

\[ \mathcal{A} \mathcal{A} = \frac{1}{2k} \frac{1}{2\pi} \int d^4p \left( -i \lambda \right)^k \left( \frac{i \pi}{p^2-m^2+i\epsilon} \right)^k A_{\text{eff}} \text{ with } \text{mass zero momentum in external legs} \]
\[ \text{Hence:} \]
\[ \text{Ver}(A) = -i \sum_{k=1}^{\infty} \int_{4\pi} \frac{d^4 p}{(2\pi)^4} s \left( \frac{1}{2} \right) \left( \frac{4\lambda A^2}{p^2 - m^2 + i\varepsilon} \right)^k \]
\[ = -\frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} \ln \left( 1 - \frac{4\lambda A^2}{p^2 - m^2 + i\varepsilon} \right) \]
\[ = -\frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} \ln \left( \frac{p^2 - m^2 - \frac{4\lambda A^2}{p^2 - m^2 + i\varepsilon}}{p^2 - m^2 + i\varepsilon} \right) \]

where the first two terms in the series are divergent. Note that we can also do the calculation by expanding \( \Phi = \Phi_c + \Delta \Phi \) in the path integral (where \( \Phi_c \) is the classical solution to \( S_S(\Phi_0(x)) = -\Delta(x) \)) and expanding to quadratic order in \( \Delta \Phi \) and doing the Gaussian integral.

### 2.9 Quantizing non-Abelian gauge theories

Having discussed \( \Phi \) for interacting theories perturbatively, let's come back to the question of quantizing gauge theories, but now for the non-Abelian case. We will focus on \( \text{SU}(n) \) as our example but essentially all gauge theories follow the same procedure, which will essentially be the same Faddeev-Popov trick we saw for \( \text{O(} \) or \( \text{SU}(n) \) with a fermion field \( \Psi \) in the fundamental (determining) representation:

\[ Z = -\frac{i}{2} \text{tr} \left( F_{\mu\nu} F^{\mu\nu} + i \Psi (i\not\partial - m_0) \not\Psi \right) \]

where:

\[ D_{\mu} \Psi = \partial_{\mu} \Psi + i g_0 A_{\mu} \Psi \]

\[ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + i g_0 [A_{\mu}, A_{\nu}] \]

The gauge field \( A_{\mu} \) is static, \( A_{\mu} \) takes values in the
Lie algebra of $G = SU(n)$. Thus in this case:

- $A_m$ is a traceless, Hermitian $n \times n$ matrix.
- $\psi$ transforms in the $n$-dimensional representation of $SU(n)$

Thus if we introduce matrix indices: $A^a_m$

\[
A^a_m F^{mn}_b = D_m A^a_n - D_n A^a_m + ig (A^a_m c A^c_n - A^a_n c A^c_m)
\]

\[
(D_m \psi)_a = \partial_m \psi^a + ig \sigma^a_b A^b_m \psi^b
\]

The gauge symmetry is:

\[
\psi(x) \rightarrow \psi'(x) = U(x) \psi(x)
\]

\[
A^a_m(x) \rightarrow A^a'_m(x) = U(x) A^a_m(x) U^{-1}(x) - \frac{i}{g} \partial_m U(x) \partial U(x)^{-1}
\]

We can introduce a basis for the Lie algebra: \{ $t^a$ \} such that

\[
A^a_m = A^a_m t^a \quad a = 1, 2, \ldots, n^2 - 1
\]

and the matrices satisfy:

\[
[t^a, t^b] = i f^{abc} t^c
\]

\[
[t_a, t_b] = i f^{abc} t^c
\]

\[
\text{structure constants}
\]

It is conventional to fix the normalization:

\[
\gamma^{ab} = \frac{1}{2} \delta^{ab}
\]

Contracting indices with $\delta_{ab}$ we then have:

\[
F^{ab} = F^{ab}_m t^a
\]

\[
\mathcal{L} = - \frac{1}{4} F^{ab}_m F_{ab} + \frac{1}{2} \psi (\partial + ig A^a_m t^a) +
\]

where:

\[
F^{ab}_m = D_m A^a_n - D_n A^a_m - g f^{abc} A^c_m A^b_n
\]

Note that the quadratic term in $F_{ab}$ means that even two-particle when we ignore $\psi$, we still have an interacting theory.

As for QED, consider the same correlating function generating function, (square of $F_{ab}$)

\[
\frac{1}{2} \langle [\psi \bar{\psi}] \rangle = \int \mathcal{D} A_m \ e^{i S[A] + i \int \bar{\psi} \gamma^m \partial_m \psi - m \bar{\psi} \gamma^0 \psi}
\]

for some set of currents $J_m = \partial_m \bar{\psi} \gamma^0 \psi$

\[
\mathcal{S}[A] = - \frac{1}{4} \int \mathcal{D} A_m \ e^{i F_{ab} \gamma^{ab} + i \bar{\psi} \gamma^0 \partial_m \psi - m \bar{\psi} \gamma^0 \psi}
\]
as before we can rewrite

$$S[A] = \frac{1}{2} \int d^4x \ A_{\mu}^a \left( g_{\mu\nu} \partial^\nu A^a - \gamma^\nu \partial_\nu A^a \right) + \text{cubic} + \text{quartic},$$

and we see that the quadratic operator cannot be inverted,

precisely because of the gauge symmetry. There is a redundancy
in the $A_\mu$ path integral due to the various gauge equivalent
fields $A^a_\mu$.

we can get around this problem just as before. Consider
some gauge fixing functional $G[A_\mu](x)$. For example:

$$G[A_\mu](x) = \delta^4 A_\mu^a(x)$$

If we define the transformed field

$$\hat{A}_\mu^a = U A_\mu U^{-1} - \frac{i}{g_0} U \delta_\mu \ U^{-1}$$

then we have the identity: for any function $\omega(x)$

$$1 = \int \mathcal{D}U \cdot \delta(G[A_\mu](x) - \omega(x)) \frac{\delta G[A_\mu](x)}{\delta U(x)}$$

To define $\mathcal{D}U$, recall that on any Lie group there is a (right-invariant)
Haar measure $\mu_{\text{Haar}}(g)$, with the property that for
a fixed $g_0 \in G$, $\mu_{\text{Haar}}(g) = \delta_{\mu}^{(g)}$ (invariant under the
action of the Lie group on itself). Thus we can define $\mathcal{D}U$
as the limit of the lattice

$$\prod_{x} d\mu(U(x)) \to \mathcal{D}U(x)$$

with the property that for a fixed function $\omega(x)$:

$$\mathcal{D}(U U_0) = \mathcal{D}U$$

Then,

$$\Delta[A^a] = \det \left( \delta G[A^a] \right)$$

we have:

$$\Sigma_{[A^a]} = \int \mathcal{D}U \Delta[A^a] \delta(G[A^a] - \omega) \Delta[A^a] e^{iS[A^a] + \text{source}}$$

Up to the source terms, source terms (as for $\mathcal{D}U$), we can change
variables for $A_\mu^a \to \hat{A}_\mu^a$ and since $\delta A^a_\mu = \partial_\mu \hat{A}^a_\mu$, $\delta A^a_\mu = \delta (\partial_\mu \hat{A}^a_\mu)$.
\[ Z[\Sigma] = \left( \int D\Sigma \right) \cdot \int DA^\mu \delta (G[A^\mu]) \Delta(A^\mu) \mathcal{E}^{\text{S}[A^\mu] + \text{source}} \]

Finally choose \( N(s) \) such that \( \int_0^{\pi/4} \cos^{-2} \frac{\theta}{2} d\theta = \frac{1}{2} \). We have, reLabelling again \( A^\mu \rightarrow A \)

\[ Z[\Sigma] = N(s) \int D\Sigma \cdot \int D\Omega \, \delta \left[ G[A^\mu] - \omega(A) \right] \Delta(A^\mu) e^{i S[A^\mu] + \text{source}} \times e^{-i \int d\Sigma \, \cos^{-2} \frac{\theta}{2} d\theta} \]

so that finally:

\[ Z[\Sigma] = \frac{Z[\Sigma]}{N(s) \int D\Omega} \]

where:

\[ S_A^\prime = \int d\Sigma \left[ - \frac{1}{2} \, F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} \, \delta \left( G[A^\mu] \right) \frac{1}{2} \, \Gamma \left( G[A^\mu] \right) \right] \]

The new ingredient is \( \Delta(A^\mu) = \det \left( \delta A^\mu / \delta U(y) \right) \). If we note:

\[ i\alpha = \delta U \cdot U^{-1} \quad \text{so} \quad \delta U = i \alpha \cdot U^{-1} \quad \delta U^\dagger = U^{-1}(-i\alpha) \]

then, quasi:

\[ \delta A_{\mu}^\nu = i \alpha \cdot U_{\mu}^{\nu} - U_{\mu}^{\nu} \cdot i\alpha - \frac{1}{g_0} \, F_{\mu \nu} (U^{-1} \cdot i\alpha) \]

\[ \delta A_{\mu}^\mu = i \alpha \cdot U_{\mu}^{\mu} - U_{\mu}^{\mu} \cdot i\alpha = \frac{1}{g_0} \, \delta \left( \alpha \right. \left. , A_{\mu}^\mu \right) U_{\mu}^{\mu} \cdot i\alpha \]

\[ = - \frac{1}{g_0} \left( \partial_{\mu} \alpha \cdot i g_0 \left( A_{\mu}^\mu, \alpha \right) \right) = - \frac{1}{g_0} \, D_{\mu}^\mu \alpha \]

Now let's choose:

\[ G[A^\mu](x) = \delta^\mu A_{\mu}(x) \]

hence:

\[ \delta G[A^\mu](x) = \delta^\mu - \frac{1}{g_0} \, D_{\mu}^\mu \delta \left( x(y) \right) \]

so:

\[ \delta \left[ G[A^\mu] \right] = - \frac{1}{g_0} \, \delta \left( x(y) \right) D_{\mu}^\mu \cdot U^{-1}(y) \]

The determinant \( \Delta \) is in both \( x \) and \( y \) indices and as a functional
operator. Since $\det U = 1$, dropping an overall $g_0$ factor (which only gives a constant), we have:

$$\Delta [A^\mu] = \det (i\partial^\mu D^\mu)$$

But we can include this in the path integral by using matrix Airy-type fields $c(x) = c^a(x)$ \(a = 1, \ldots, m = 4\).

$$\det (i\partial^\mu D^\mu) = \int \delta c^a \delta c^b e^{-i\int d^4x \frac{1}{2m} c^a D^\mu D^\mu c^b}$$

$$= \int \delta c^a \delta c^b e^{-i\int d^4x \frac{1}{2m} (\partial^\mu D^\mu)}$$

Note $c^a, c^b$ are Faddeev-Popov ghost fields.

* spin-0 but anti-commuting! violate spin-statistics then.

* they evade the theorem because have a Hilbert space

with negative norm states $\langle +14 \rangle < 0$.

Putting everything together:

$$Z[\Gamma] = \int \mathcal{DA}_\mu \mathcal{DC} \delta \mathcal{D} e^{-iS_3(\frac{\partial}{\partial A^\mu}) + \frac{1}{2m^2} (\partial^\mu A^\mu)^2 + \frac{1}{2m} A^\mu D^\mu c}$$

where:

$$S_3(A^\mu, c) = \int d^4x \left( -\frac{i}{2} F^\mu_\nu F^\nu_\mu + \frac{1}{4} \frac{1}{2m} (\partial^\mu A^\mu)^2 + \frac{1}{2m} A^\mu D^\mu c \right)$$

If we expand using $E^\mu$:

$$Z = \frac{1}{2} \Omega(A^\mu, c^a) (E^\mu A^\mu - \partial^\mu A^\mu) - \frac{1}{2} \delta_5 (E^\mu A^\mu) (E^\mu A^\mu) + \frac{1}{2} g_0 f_{abc} (E^\mu A^\mu - \partial^\mu A^\mu) A^\nu A^\nu + \frac{1}{2} g_0 f_{abc} f_{de} A^\nu A^\nu A^\mu A^\nu$$

we find:

$$Z = \frac{1}{2} \Omega(A^\mu, c^a) (E^\mu A^\mu - \partial^\mu A^\mu) - \frac{1}{2} \delta_5 (E^\mu A^\mu) (E^\mu A^\mu) + \frac{1}{2} g_0 f_{abc} (E^\mu A^\mu - \partial^\mu A^\mu) A^\nu A^\nu + \frac{1}{2} g_0 f_{abc} f_{de} A^\nu A^\nu A^\mu A^\nu$$

$$+ g f_{abc} (\partial^\mu c^a) A^\mu c^a$$
If we include the Fermions (labeled by $i$) and $\bar{\psi}$ - bar of $\psi$;

\[ \psi = \bar{\psi} (i\gamma^\mu \partial_\mu - g (\bar{\psi} \gamma^\mu \gamma^5 \psi) A_\mu) + \bar{\psi} \]

We can then read off the Feynman rules. The free part of the lagrangian is:

\[ L_0 = -\frac{i}{2} \partial_\mu A^\mu - \partial_\nu A^\nu \right) (\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2g^2} (\partial^\mu A^\nu - \partial^\nu A^\mu) (\partial^\alpha A^\beta - \partial^\beta A^\alpha)
\]

\[ + (\partial^\mu A^\nu) (\partial_\mu A^\nu) + \bar{\psi} (\gamma^\mu \partial_\mu - m) \psi \]

That gives the propagator:

\[ \frac{\delta^{ab} D_{\mu\nu}(k)}{k^2 - m^2} = \frac{i\delta^{ab} (\gamma^\mu - i\gamma^5 k^\mu k^\nu)}{k^2 - m^2 i\epsilon} \]

For the interaction:

\[ g_0 \bar{\psi} \gamma^\mu (k_1 - k_2) \gamma^\nu (k_3 - k_4) \gamma^\alpha (k_4 - k_3) \gamma^\beta (k_3 - k_4) \gamma^\sigma \]

Which can be:

\[ \Delta S = g_0 \int \text{Feynman integral} \left( D_{\mu\nu} A^\mu - \partial_\nu A^\mu \right) \tilde{A}^{\alpha\beta} A^{\alpha\beta} \text{d}x \]

Taking the Fourier transform:

\[ \Delta S = \frac{1}{2g_0} \int \text{Feynman integral} \left( k_1^\mu g_{\mu\nu} - k_1^\mu g_{\mu\nu} \right) \tilde{A}^{\alpha\beta} (k_1) \tilde{A}^{\alpha\beta} (k_2) \text{d}x \]

And then symmetrize on $A (k_1 A(k_2) A(k_3))$ with $\frac{1}{3!}$ and $i$. 

\[ \text{and symmetrize on } A (k_1 A(k_2) A(k_3)) \text{ with } \frac{1}{3!} \text{ and } i. \]
\[
\Delta S = \int d^4x \left( - \frac{i g_0}{2} \text{faceface } A_\mu^a A^\nu_b A_\rho^c A^\sigma_d \eta^{ab} \eta^{cd} \right)
\]

after including \( i \frac{g_0}{2} \text{faceface } A_\mu^a A^\nu_b A_\rho^c A^\sigma_d \eta^{ab} \eta^{cd} \) and symmetrizing over \( A_\mu \) we have:

\[
\Delta S = - \frac{i g_0}{2} \left[ \text{faceface } (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) \\
+ \text{faceface } (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) \\
+ \text{faceface } (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) \right]
\]

For the ghost-gauge vertices:

\[
\Delta S = \int d^4x \left( - \frac{i g_0}{2} \text{faceface } (\partial \mu \partial \nu - \partial \mu \partial \nu) \right) A^\mu B^\nu C^a
\]

so

\[
\Delta S = \int d^4x \left( - \frac{i g_0}{2} \text{faceface } (\partial \mu \partial \nu - \partial \mu \partial \nu) \right) A^\mu B^\nu C^a
\]

\[
= \int d^4x \left( - \frac{i g_0}{2} \text{faceface } \eta^{\mu\nu} \eta^{\rho\sigma} \right) A^\mu B^\nu C^a
\]

Finally we have, for the fermions:

\[
\Delta S = \int d^4x \left( - \frac{i g_0}{2} \text{faceface } (\partial \mu \partial \nu - \partial \mu \partial \nu) \right) A^\mu \overline{\psi} \gamma^\beta \psi
\]

so

\[
\Delta S = \int d^4x \left( - \frac{i g_0}{2} \text{faceface } (\partial \mu \partial \nu - \partial \mu \partial \nu) \right) A^\mu \overline{\psi} \gamma^\beta \psi
\]

With these Feynman diagrams one can compute correlation functions in QCD, and other gauge theories.

Note finally that although we didn't need to, we could have included ghost fields for A\(^\mu\).
we had a $\det(D^{-1})$ factor that was independent of $A_\mu$ and hence we regarded it as contributing an overall constant. However we could have rewritten it using ghosts:

$$ Z_{\phi} = \int d^4A_\mu \det\left( \frac{i}{2} \nabla^2 \right) e^{i S_\phi[A]} + \text{sum}$$

$$ = \int d^4A_\mu D^2 A_\mu e^{i S_\phi[A]} + \text{sum}$$

where:

$$ S_\phi[A, \bar{c}, c] = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \bar{c} \gamma_\mu \gamma_5 A^\mu c - \bar{c} \gamma_\mu \gamma_5 c \right)$$

Since there is no coupling between $c^* \& c$ and the other fields, they will never contribute to any Feynman diagrams.

2.4 Ghosts, BRST symmetry and unitarity

What is the physical interpretation of the ghosts? When we gauge fix we get a propagator (e.g in Feynman gauge)

$$ M_{\mu\nu} = \frac{-i \partial_\mu \partial_\nu}{k^2 + i\epsilon}$$

The propagator's behavior physical and unphysical degrees of freedom of the gauge field. - it includes the time like & longitudinal polarizations. In general, one cannot worry that this will give the wrong answers in loop diagrams, for example:

$$ + $$

have non-physical modes in internal propagators.

However there are cancelled by the ghost contribution:

$$ -ve \text{ sign since antisymmetric loop.}$$
One can already see evidence of this cancellations at the level of the path integral. Consider the limit $g_0 \to 0$ in Feynman gauge in $d$-dimensions:

$$ S_3 \left[ A, c^*, c \right] \to \int d^4x \left( \frac{1}{2} A_\mu^a \left( \eta^{\mu\nu} \partial^2 \right) A^a - c^a \left( \partial^2 \right) c^a \right) $$

so doing the gaussian integrals:

$$ Z[c] = \int D A_\mu \, D c^* \, D c \, e^{i S_3} $$

$$ = \text{const.} \left( \det \partial^2 \right)^{-\frac{d}{2}} \left( \det \partial^2 \right)^{-\frac{d}{2}} \left( \text{dim. of gauge} \right)^{-\frac{d}{2}} $$

$$ = \text{const.} \left( \det \partial^2 \right)^{-\frac{d}{2} - \frac{d}{2} - \frac{d}{2}} $$

so the gauge is "canceled" because of the two non-physical degrees of freedom at the gauge field.

The clearest way to see this cancellation extends to the full theory is via the appearance of a new symmetry. First we need to rewrite the action using a Lagrange multiplier field.

Let $B_a$ be a new (commuting) field, then:

$$ \int d^4x \left( \frac{i}{2} \delta_a B^a + B_a \partial_\mu A^a \right) $$

$$ = \text{const.} \, e^{-i \int d^4x \frac{i}{2} \left( \partial A^a \right) \left( \partial A^a \right)} $$

by completing the square and doing the integral. Hence to for $\mathbf{ACO}$ we can rewrite:

$$ Z \left[ \text{sine} \right] = \int D A_\mu \, D F^2 \, D c^* \, D c \, D B_a \, e^{i S_3 \left[ A, B, c^*, c \right] + \text{sine}} $$

where

$$ S_3 \left[ A, B, c^*, c \right] $$

$$ = \int d^4x \left( - \frac{1}{2} \partial F^2 + \frac{1}{2} \left( \partial \Omega - \mathbf{A} \right) \mathbf{A} + 2 \nabla A^a \, D c + \frac{1}{8} \delta^2 B^a + \mathbf{A} \mathbf{B} \right) $$

This action has a new symmetry called BRST (Becchi, Rouet, Stora and Tyutin).
Let $\epsilon$ be an infinitesimal constant parameter.

The infinitesimal BRS transformations are:

$$\delta A^a = \epsilon (2\partial_a + g_0 F^{\mu} \lambda^a_{\mu})$$

$$\delta t = i g_0 e c^a t_a \cdot \psi$$

$$\delta c = -i g_0 e f_{abc} c^b c^c$$

$$\delta c^a = \epsilon B^a$$

$$\delta B^a = 0$$

or in matrix form:

$$\delta A_{\mu} = \epsilon \partial_{\mu}$$

$$\delta \psi = i g_0 e c \cdot \psi$$

$$\delta c = i g_0 e c^2$$

$$\delta B = 0$$

where $c^2 = c^a c^b t_{ab} = \epsilon e (t_{ab} - t_{ba}) = \epsilon i f_{abc} c^b c^c$.

To check the invariance we note that the $A_{\mu}$ and $\psi$ transformations are infinitesimal, just conventional gauge transformations: $\chi = ec$,

$$\delta A_{\mu} = \delta \partial_{\mu} (ec)$$

$$\delta \psi = i g_0 (ec) \cdot \psi$$

hence $\delta F^a_{\mu
u}$ and $\delta \psi [\psi - m] \psi$ are invariant. For the other terms:

$$\delta (D_{\mu} c) = \delta (D_{\mu} c - i g_0 [\lambda_{\mu}, c])$$

$$= D_{\mu} \delta c - i g_0 [\delta \lambda_{\mu}, c]$$

$$= i g_0 e (D_{\mu} c^2 + c (D_{\mu} c)) - i g_0 [e \partial_{\mu} c, c]$$

$$= i g_0 e ([D_{\mu} c + c (D_{\mu} c)] - i g_0 e (D_{\mu} c^2) + i g_0 c \cdot e (D_{\mu} c)$$

$$= 0$$

so $\delta B = 0$

$$\delta S_{\mu} = \int d^4 x \left( 2 \epsilon \partial^a (\delta c^a) D_{\mu} c + 2 \epsilon \psi B_{\mu} \delta (\delta A^a) \right)$$

$$= \int d^4 x \left( -2 i \epsilon B \partial^a D_{\mu} c + 2 i e \partial^a B_{\mu} D_{\mu} c \right)$$

and hence we have a symmetry.

We can use the notation used at some previous Ga:

$$\delta e A_{\mu} = \epsilon g \cdot A_{\mu}$$

$$\delta \psi = \epsilon g \cdot \psi$$

etc.
an important property is then that

- $\phi^2 = 0 ;;$ BRST symmetry is nil-nilpotent.

For example:

$$\delta \phi : e^\phi \cdot A^\mu = \delta e (e^\phi A^\mu) = 0 \quad \text{from previous calculations}$$

$$\delta \phi : e^\phi \cdot c = \delta e (i g_0 e^\phi c) = -g_0^2 [e (e^\phi c) c + e^\phi c c]$$

$$= -g_0^2 e e^\phi (c^2 - c) = 0$$

One then finds that $\Delta$ gives a natural way to define the physical states.

Let us go to the canonical picture. There will be a Noether current for the BRST symmetry and a corresponding current charge operator $\hat{A}$ that will generate the symmetry; so as operator:

$$\delta \hat{A}^\mu = [\hat{E}, \hat{A}^\mu] \quad \delta \hat{c} = [\hat{E}, \hat{c}]$$

Let $\mathbb{H}$ be the full Hilbert space. We can then define 2 types of subspace:

- $\mathbb{H}_{\text{phys}} = \{ \phi : \mathbb{H} : \hat{A}_1 \phi = 0 \}$

- $\mathbb{H}_{\text{exact}} = \{ \phi : \mathbb{H} : \hat{A}_1 \phi = 0 \} \text{ for some } |x| \gg e \mathbb{H}; 3$

then the physical states are elements of $\mathbb{H}_{\text{phys}}$ modulo states in $\mathbb{H}_{\text{exact}}$.

$\mathbb{H}_{\text{phys}} = \mathbb{H}_{\text{full}} / \mathbb{H}_{\text{exact}}$ where $|\phi\rangle = |\hat{A}_1 \phi\rangle$.

that if we identify states as being the same if they differ by an exact state.

To see how this works, consider the asymptotic states where we effectively have the free theory and so can take $g_0 \to 0$. If we ignore the

particles for now, we have a general one-particle state:

$$| \psi, \pi, \pi^\pm, \beta > = (e^{i \beta} \hat{A}^+ + e^{i \beta^\dagger} \hat{\pi}^+ + e^{i \beta^\dagger} \hat{\pi}^+ + e^{i \beta^\dagger} \hat{\pi}^+) |\psi\rangle$$

That is, we have two possible states (up to factors):

- $|\psi\rangle$
- $|\psi^+\rangle$

Recall that the $B$ equals a metric set (up to factors).

$$B = \delta \pi \hat{A}^+$$

hence $\hat{B}^+ = k \hat{\pi} \hat{A}^+$
with \( g_0 = 0 \) the BRST transformations imply, up to factors:
\[
[Q, \hat{\phi}^\mu] = \kappa^\mu \hat{\phi}^\mu \quad \text{and} \quad \kappa^\mu \hat{\phi}^\mu + \kappa^\mu \hat{\phi}^\mu = 0
\]

Thus, given \( \langle \phi | \phi \rangle = 0 \), we have
\[
Q \mid \phi \rangle = (\kappa^\mu \hat{\phi}^\mu + \kappa^\mu \hat{\phi}^\mu) \mid \phi \rangle = 0
\]

Hence for this to be physical \( Q \mid \phi \rangle = 0 \) so
\[
\kappa^\mu \hat{\phi}^\mu = 0 \quad \kappa^\mu = 0
\]

so
\[
\mid \psi \rangle_{phys} = \mid \phi \rangle (\kappa^\mu \hat{\phi}^\mu + \kappa^\mu \hat{\phi}^\mu) \mid \phi \rangle
\]

But we need to mod out by exact states:
\[
\mid \chi \rangle_{exact} = Q \mid \phi \rangle = (\kappa^\mu \hat{\phi}^\mu + \kappa^\mu \hat{\phi}^\mu) \mid \phi \rangle
\]

Thus we can choose \( \kappa^\mu \) to shift
\[
(\kappa^\mu + \kappa^\mu) + \kappa^\mu \hat{\phi}^\mu = \kappa^\mu + \kappa^\mu
\]
\[
\kappa^\mu \hat{\phi}^\mu = \kappa
\]

Thus we can set \( \kappa = 0 \) and the freedom \( \kappa^\mu + \kappa^\mu \)
just removes the longitudinal polarization, leaving the 2 physical polarization states.

One can also use this structure to prove that the projected S-matrix \( S = P \Sigma P \) is unitary. First we note that \( \hat{Q} \) is a symmetry, hence it commutes with the Hamiltonian \( [\hat{Q}, \hat{H}] = 0 \) and hence since the evolution \( \hat{H} \) the S-matrix is controlled by \( \hat{H} \) we have \( [\hat{H}, \hat{S} \Sigma P \hat{H}] = 0 \).

Let \( \mid \psi \rangle \) be a physical state; that is \( \mid \psi \rangle \in \text{Hilbert} \) then
\[
\hat{Q} \Sigma P \mid \psi \rangle = \Sigma P \hat{Q} \mid \psi \rangle = 0
\]

so \( \Sigma P \mid \psi \rangle \in \text{Hilbert} \). Hence for any two \( \mid \psi \rangle, \mid \psi \rangle \in \text{Hilbert} \)
\[
\langle \psi | \Sigma P \mid \phi \rangle = \frac{1}{\sqrt{2}} \langle \phi | \Sigma P \mid \phi \rangle
\]
where the \( |\psi\rangle\) are a complete set of states for \( \text{H}\). Now recall that
\( \text{H}\psi = \text{H}\psi_1 \text{H}\psi_2 \text{H}\). Exact states have a strange property. Let \( |\psi\rangle = \hat{Q}|\rho\rangle \in \text{H}\) and \( |\chi\rangle \in \text{H}\) then
\[\langle \chi | 1\rangle = \langle \rho | \hat{Q} | 1\rangle = 0\]
(and \(\langle \chi | x\rangle = \langle \rho | \hat{Q}^\dagger | x\rangle = 0\)). Hence
since \( \hat{S} \in \text{Sp} | 1\rangle \in \text{H}\),
\[\langle \chi | \hat{S}^\dagger \hat{S} | 1\rangle = 0 \quad \text{for all } |\chi\rangle \in \text{H}\text{exact}\]
and hence:
\[\langle \psi' | \hat{S}^\dagger \hat{S} | 1\rangle = 2 \sum_{|\chi\rangle \neq \text{H}\text{exact}} \langle \psi' | \hat{S}^\dagger \hat{S} | \chi\rangle \langle \chi | \hat{S}^\dagger \hat{S} | 1\rangle\]
\[= \langle \psi' | \hat{S}^\dagger \hat{S} | 1\rangle\]
where \( \hat{S} = \hat{R}^\dagger \hat{P} \hat{S}^\dagger \hat{P} \). Since \( \hat{S}^\dagger \hat{S} = I_4 \) we thus have
\( \hat{S}^\dagger \hat{S} = I_4 \).

This statement is equivalent to saying that the ghosts cancel the propagator of non-physical polarizaton states in loop diagrams.

Remarkably this same BRST symmetry would have appeared whatever gauge fixing condition we had considered, and physical states would always be elements of the BRST "cohomology" \( \text{H}\text{phys} = \text{H}\text{exact} \).

In fact in some ways the BRST structure is more fundamental than the Fadeev-Popov procedure. Quenching a theory with gauge symmetries really corresponds to finding a theory with an appropriate BRST symmetry and then quantizing that theory. It may or may not be possible to derive this theory via Fadeev-Popov procedure.)
Finally we note that we can again derive Ward-Takahashi
identities for the BRS symmetry, whereas we did not do so here.
### III Renormalization

#### 3.1 Ultra-violet divergences and cut-offs in $\chi E^4$

Let's look at some one-loop diagrams in $\chi E^4$. Consider first the two-point vertex:

$$\Pi_2(p) = \frac{k}{p} = -\frac{i\lambda_0}{(2\pi)^4} \int \frac{d^4k}{E^2-m^2+i\varepsilon} = \frac{i\lambda_0}{(2\pi)^4} \int d^4k$$

We need to evaluate

$$I_1(k) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\varepsilon}$$

which clearly diverges at large $k$.

To see this explicitly we first need to rotate the $k_0$ integral. Let

$$k_0 = \sqrt{E^2 + m^2}$$

we have poles:

$$\int \frac{dk_0}{k_0^2 - 2\omega \varepsilon + i\varepsilon}$$

so writing $k^0 = k_0$

$$\int_{-\infty}^{\infty} \frac{dk_0}{k_0^2 - 2\omega \varepsilon + i\varepsilon} = i \int_{-\infty}^{\infty} \frac{dk_E}{k_E^2 - 2\omega \varepsilon + i\varepsilon} = -i \int_{-\infty}^{\infty} \frac{dk_E}{k_E^2 + 2\omega \varepsilon + i\varepsilon}$$

So:

$$I_1(k) = -i \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2}$$

Using spherical polar coordinates:

$$I_1(m_0) = \frac{i}{(2\pi)^4} \int dS_2 \int_0^\infty \frac{k^3 dk}{k^2 + m_0^2}$$

unit 3-sphere:

$$= \frac{i}{(2\pi)^4} 2\pi^2 \int_0^\infty \frac{k^3 dk}{k^2 + m_0^2} = -\frac{i}{8\pi^2} \int_0^\infty \frac{d^3k}{k^2 + m_0^2}$$
If we introduce a cutoff and set \( u = k^2 \):

\[
I_1(u_0) = -\frac{i}{16\pi^2} \int_0^\infty \frac{dw}{u + u_0} = -\frac{i}{16\pi^2} \int_0^1 \frac{du}{1 - \frac{u_0}{u + u_0}} \]

\[
= -\frac{i}{16\pi^2} \left( \lambda^2 - m_0^2 \ln \frac{\lambda^2 + m_0^2}{m_0^2} \right) \]

\[
= -\frac{i}{16\pi^2} \left( \lambda^2 - 2m_0^2 \ln \left( \lambda/m_0 \right) + O(m_0^4/\lambda^2) \right) \]

so we have a quadratic and logarithmic divergence. What does this mean? Recall that the 2-point function is given by

\[
\langle \Phi(x) \Phi(y) \rangle = \frac{i}{(2\pi)^4 \delta^{(4)}(x-y)} \frac{i}{p^2 - m_0^2 - i\epsilon} \]

and to one-loop order

\[
c\hat{\Pi}^2(p) = \frac{1}{2\lambda_0} I_1(u_0) + O(\lambda_0^2) \]

\[
= \delta^{(4)} \left( \lambda^2 - m_0^2 \ln \frac{\lambda^2 + m_0^2}{m_0^2} \right) + O(\lambda_0^2) \]

Recall the physical particles will have a mass given by the position of pole in the propagator. (see Kallen-Lehman representation).

So: the physical mass is given by:

\[
m_{\text{phys}}^2 = m_0^2 + \frac{\lambda_0}{32\pi^2} \left( \lambda^2 - m_0^2 \ln \frac{\lambda^2 + m_0^2}{m_0^2} \right) + O(\lambda_0^2) \]

We see that the physical mass depends on the large energy physics.

How do we interpret this? Physically, we should think of our \( \phi^4 \) theory as an effective theory valid at low energies. At high energies, there may be new states or new particles in the theory which could contribute in loop diagrams. Since we don't know what this high energy content is, we simply cut off the integrations at some high energy scale \( \Lambda \) - the idea is that all the unknown high energy physics can be encoded in the cutoff.
New UV physics

Effective $\Lambda^4

\Lambda \gg m_{\text{phys}}$

This is very similar to situations in condensed matter. All ferromagnets
near the phase transition can be described by an effective
magnetization field $M(S)$ without knowing the details of the
high-energy structure (lattice of atoms, spins, etc.).

Note that we already see a "fine-tuning" problem. Since we
have some measured mass $m_{\text{phys}} < \Lambda$, we can then
deduce what the bare mass in the Lagrangian was:

$$m_0^2 + \frac{\lambda_0}{32\pi^2} \Lambda^2 \ll \Lambda^2$$

or:

$$\frac{m_0^2}{\Lambda^2} + \frac{\lambda_0}{32\pi^2} \ll 1$$

Thus we need to choose the bare mass to be very close to $\lambda_0\Lambda^2/32\pi^2$
so that the difference is small.

We can also look at the coupling constant. The four-point
vertex is given by:

$$\tilde{\Gamma}_4(p_1, p_2, p_3, p_4) = \times + 3 \otimes \times + 6(\lambda_0^2)$$

where:

$$\Gamma = \frac{(-\lambda_0)^2}{2} \int \left[ \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m_0^2} \frac{i}{(q + p_1 + p_2)^2 - m_0^2} \right]$$

$$= \frac{1}{2} (-\lambda)^2 I_2(p_1 + p_2, m_0)$$

(there are 2 more diagrams with $p_1+p_3$ & $p_1+p_4$.)
If we consider a scattering process \( q \bar{q} \rightarrow q \bar{q} \), the amplitude is given by the

\[ i M_{q \bar{q} \rightarrow q \bar{q}} = \bar{\Pi}_4 (-p_1, -p_2, p_3, p_4) |_{p_i^2 = m_i^2} \]

Thus the overall magnitude of \( \bar{\Pi}_4 \) tells us about the

strength of the coupling:

\[ \bar{\Pi}_4 = -\lambda_0^4 + \frac{1}{2}(\lambda_0^2)^2 f(s, t, u) + \ldots \]

where \( s = (p_1 + p_2)^2 \), \( t = (p_1 - p_3)^2 \), \( u = (p_1 - p_4)^2 \) are the Mandelstam

variables (the Lorentz invariant co-ordinates). We later do

the \( I_2 \) integrals to find the explicit \( s, t, u \) dependence but

for now let's measure the strength of the interaction by

defining: (as in the effective potential)

\[ -i \lambda_{\text{phys}} = \bar{\Pi}_4 (0, 0, 0, 0) \]

\[ = -\lambda_0^4 + \frac{1}{2}(\lambda_0^2)^2 I_2(0, m_0) + \ldots \]

(one can of course use other conventions \( e.g. \bar{\Pi}_2 \rightarrow \bar{\Pi}_2 \). We have

\[ I_2(0, m_0) = \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 - m_0^2 + i\epsilon)^2} = i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 + m_0^2)^2} \]

\[ = \frac{d}{dm_0^2} \bar{\Gamma}_1 (\text{phys} m_0) \]

\[ = \frac{i}{8\pi^2} \left[ \log (\lambda/m_0) - \frac{1}{2} \right] + \mathcal{O}(m_0^2/\lambda^4) \]

hence:

\[ \lambda_{\text{phys}} = \lambda_0 \left[ 1 - \frac{3\lambda_0^2}{16\pi^2} \log (\lambda/m_0) + \mathcal{O}(\lambda_0^3) \right] \]

Again we see that the physical coupling depends on the unknown

CU physics through the cutoff \( \lambda \). - but there is no fine-tuning

problem because we have only a logarithmic dependence on \( \lambda/m_0 \).

However, we do see a question about how to organize the

perturbative expansion, we have been thinking of it as an
expansion in the bare coupling $\lambda_0$, but it might be more natural to view it as an expansion in the physical coupling $\lambda_{\text{phys}}$ - in particular the $\ln \left( \frac{\mu}{\Lambda} \right)$ term means there is no reason one is small when the other is small. This is the issue of "counter-terms" and we will come back to it later.

### 9.2 Power counting and renormalisability

We have seen that the structure of $\Pi_F$ and $\Pi_\text{phys}$ depends on the cut at scale $\Lambda$, but this dependence corresponds to a renormalisation of the mass and coupling constants. The question remains though whether there are further divergences in other diagrams.

Consider for instance the connected diagram:

\[ \gamma_6 \approx 2 \lambda_0 \left( \frac{q q q q}{q q q q} \right) \]

\[
= \begin{array}{c}
\begin{array}{c}
\text{phys dependence}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{new divergence?}
\end{array}
\end{array}
\]

The first diagram will depend on $\lambda_{\text{phys}}^2$ at the cut at $\Lambda$ via $\lambda_{\text{phys}}^2$ but in principle, we could get a new divergence from $\gamma_6$ in the second diagram. We need to quantify the degree of divergence to other vertex diagrams.

Let's move in $d$-dimensions, and consider a diagram with:

- $E = \#$ of external legs.
- $L = \#$ of loops.
- $V = \#$ of vertices.
- $I = \#$ of internal legs.

$\gamma_6$ containing $\lambda_0$. (not necessarily $\lambda_0^2$)
If we expand the vertex function with the cutoff:
\[
\tilde{\Pi}_e(\varphi_1, \ldots, \varphi_e) = \sum_{\nu} \lambda^\nu \lambda^0 a_\nu (\varphi_1, \ldots, \varphi_e)
\]
where the function \( a_\nu \) are dimensionless then:

\[
D_\nu = \text{superficial degree of divergence} \text{ of diagram with } V \text{ vertices}
\]
so that if \( D_\nu > 0 \) then the diagram is potentially divergent. By dimensional analysis we have:

\[
D_\nu = \left[ \tilde{\Pi}_e \right] - \nu \left[ \lambda \right]
\]

To calculate \( \left[ \tilde{\Pi}_e \right] \) consider the connected diagram:

\[
\left. \left< \mathcal{G}_e(t_1, \ldots, t_e) \right> = \left< \mathcal{L}_1 T \varphi_1(0^+) \varphi_e(k_e^-) \right> \right|_{\text{can be different fields } \varphi_i}
\]

so

\[
\left[ \mathcal{G}_e \right] = - \frac{\epsilon}{\epsilon - 1} \left[ \varphi_i \right]
\]

If we take a Fermi function:

\[
\left[ \tilde{\mathcal{G}}_e(\varphi_1, \ldots, \varphi_e) \right] = \left[ \mathcal{G}_e \right] - d E = \frac{\epsilon}{\epsilon - 1} \left[ \varphi_i \right] - d
\]

\[
\left[ \tilde{\mathcal{G}}_e \right] = \left[ \tilde{\Pi}_e \right] \left[ \delta^{(4)}(k_1 + \ldots + k_e) \right]
\]

where \( \delta_i(k) \) is external leg propagator. To calculate \( \left[ \tilde{\mathcal{G}}_e \right] \) we cannot use the same expression for the \( 2 \varphi_1 \) case:

\[
\left< \mathcal{L}_1 T \varphi_1(k_1) \varphi_1(k_2) \right> = \left[ \mathcal{G}_{2,2}(k_1, k_2) \right]
\]

\[
= 2 \left[ \varphi_1 \right] - 2d
\]

\[
= \left[ \delta^{(4)}(k_1 + k_2) \right] + \left[ \delta_c(k_1) \right]
\]

so

\[
\left[ \delta_c \right] = 2 \left[ \varphi_1 \right] - 2d - \left[ \delta^{(4)}(k_1 + k_2) \right] = 2 \left[ \varphi_1 \right] - d
\]

hence:

\[
\left[ \tilde{\Pi}_e \right] = \frac{\epsilon}{\epsilon - 1} \left[ \varphi_i \right] - d - \frac{\epsilon}{\epsilon - 1} \left[ \delta_c \right] = \left[ \delta^{(4)}(k_1 + \ldots + k_e) \right] - \left[ \delta^{(4)}(k_1 + \ldots + k_e) \right] = - \frac{\epsilon}{\epsilon - 1} \left[ \varphi_i \right] + d.
\]
Hence
\[ D_V = -\frac{\xi}{\lambda^d} \left[ \phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \phi_7 \phi_8 \right] + d - V \lambda^d \]

We see that there are 3 possibilities:

+ super-renormalizable : \[ \lambda^d > 0 \]
  only a finite number of divergent diagrams.

+ renormalizable : \[ \lambda^d = 0 \]
  infinite number of divergent diagrams but
  only finite number of divergent vertices. (eg d=2, 3, 4, 5, \ldots)

+ non-renormalizable : \[ \lambda^d < 0 \]
  infinite number of divergent vertices
  (eg gravity: \[ \lambda^d = -2 \], Fermi theory \[ \lambda^d = -2 \]).

The point is that for renormalizable theories the dependence on \( \lambda^d \) can be incorporated into a finite number of renormalization to physical parameters (masses, coupling constants) in the theory. For a non-renormalizable theory, we have to introduce new \textit{bare} coupling constants to cancel the \( \lambda^d \)-dependent terms.

For example: in d-dimensional \[ \lambda^4 \phi^4 \] model for \( \lambda^4 \phi^4 \) we have
\[ \lambda^d = \frac{4-d}{4} \quad \lambda^4 = 4-d \]
so for \( d = 4 \) the theory is:

\[ d < 4 : \text{super-renormalizable} \]
\[ d = 4 : \text{renormalizable} \]
\[ d > 4 : \text{non-renormalizable} \]

For example: for \( d > 4 \):
\[ D_V = -6(d-4) + d + V(d-4) = -2(d-5) + V(d-4) \]
diverges if V is large enough: \( V > \frac{d-5}{2d-3} \)
to remove the a priori \( λ \)-dependence we need to include a new interaction term in the basic Lagrangian:

\[
\Delta \mathcal{L} = -\frac{1}{6} \mu_0 \phi^6
\]

and set \( \mu_0 \) \( \lambda \)-dependent pieces in \( \tilde{\Phi} \):

\[
\tilde{\Phi}_\lambda = -i \mu_0 + (\lambda \text{-dependent part}) + \ldots
\]

We then have to measure the six-point coupling and \( \mu_0 \) may, or may not be measured. (might be zero, might not.) But this procedure continues for all \( \tilde{\Phi}_\lambda \)

as a number of new couplings, set by measurement

has little predictive power!

Finally note that for \( d = 4 \):

\[
\begin{align*}
\mathcal{D} \nu &= -2 + 4 = \lambda^2 \\
\mathcal{D} \nu &= -4 + 4 = 0 \quad \ln(\lambda/m_0)
\end{align*}
\]

For \( \tilde{\Phi}_2 \) we can expand in powers of \( p^2 \) to get subleading terms:

\[
\frac{i \tilde{\Phi}_2(p)}{\alpha(\lambda) + \beta(\lambda) p^2 + \mathcal{O}(p^4)} \quad \alpha(\lambda) \sim \lambda^2 \quad \beta(\lambda) \sim \ln(\lambda/m_0)
\]

The \( \beta(\lambda) \) terms give a \( \alpha \) \( \lambda \)-wave function renormalization:

\[
\tilde{\mathcal{G}}_2(p, \eta) = (2\pi)^d \delta^{(d)}(p - \eta) \, \mathcal{D}(p)
\]

where, using Källen-Lehman

\[
\mathcal{D}(y) = \frac{i}{p^2 - m_0^2 - \mathcal{O}(\tilde{\Phi}_2) + i\epsilon} = \frac{i \mathcal{Z}}{p^2 - m_0^2 + i\epsilon}
\]

so:

\[
\mathcal{Z}^{-1}(p^2 - m_0^2 + i\epsilon) = p^2 - m_0^2 - \alpha(\lambda) - \beta(\lambda) p^2 + i\epsilon
\]

\[
= \left(1 - \frac{\lambda}{m_0^2 + \alpha(\lambda) + \beta(\lambda) p^2 + i\epsilon}ight)^{-1}
\]

\[
= \left(1 - \beta(\lambda)^{-1}\right) \left( (m_0^2 + \alpha(\lambda)) - \alpha(\lambda) \right) + i\epsilon
\]

so

\[
\mathcal{Z}(\lambda) = \left(1 - \beta(\lambda)^{-1}\right) \quad m^2 = \mathcal{Z}(\lambda)(m_0^2 + \alpha(\lambda))
\]
It is a peculiarity of $\exp x$ that $\exp x = 1$ to one-loop order - this is not for instance the case in QED.

Finally let us note that the actual degree of divergence can be more than $\Delta x$ if a sub-diagram diverges. For example in $d=4$

$$\int dq_1 dq_2 \frac{1}{q_1^2 q_2^2} = \ln(N/m^2) \Lambda^2 \not= \ln(N/m^2)$$

However this is really just a weak renormalizable in an internal propagator. The full argument that renormalizable theories actually only have a finite number of renormalizable parameters requires some complicated diagrammatic arguments.

* BPHZ (Bogolubov, Parasiuk, Hepp & Zimmermann) theorem: in a renormalizable theory $[\Lambda] = 0$ all $\Lambda$-dependent terms (divergences) can be removed by a finite number of renormalization, at a finite number of parameters.

3.3. Wilsonian action and the renormalization group

We have argued that for renormalizable physical theories, $\Delta x$ depend on a finite number of physically determined parameters that are functions of the bare parameter and the cutoff scale $\Lambda$. In this sense, all the unknown high energy physics can be incorporated in a single scale $\Lambda$. To see better how thin lines, and also in what cases we can actually have a complete theory, even at high energies, we switch to the Wilsonian picture.

(No, we diverge here a little from the discussion in Peskin and Schroeder. We follow instead notes of David Skennerton (Part III) and which are based on Tim Hollowood (0909.0859). )
The change of picture is to define the quantum theory as a path integral with modes only up to momentum $\Lambda_0$. Dropping the current terms for simplicity, consider the regularized path probability function:

$$\mathcal{Z}_{\Lambda_0} (g_{i0}) = \int_{\Lambda_0} \mathcal{D}\phi \ e^{iS_{\Lambda_0} [\phi]}$$

where (in d-dimensions)

$$S_{\Lambda_0} [\phi] = \int d^d x \left( \frac{1}{2} \partial^2 \phi \partial^2 \phi^* + \frac{1}{2} \Lambda_0^{d-4} g_{i0} O_i \right)$$

Here $\Lambda_0$ is some fixed cutoff scale used to define the theory.

- $\int_{\Lambda_0} \mathcal{D}\phi$ means the path integral over smooth functions with momentum $p^2 \leq \Lambda_0^2$.
- $g_{i0}$ set of dimensionless coupling constants.
- $O_i$ all possible Lorentz-invariant operators that are monomials in $\phi$.

Here:

$$O_i = \phi^2, \phi^3, \ldots, \phi \left( \partial^2 \phi \right) \ldots, \phi^{d+1} \phi^* \quad \text{etc.}$$

and

$$d_i \equiv \text{dim of dimension of } O_i$$

Also if $[\phi] = 1d - 1$, then for $O_i$:

- if $O_i = \phi^2$, $d_i = d-2$.
- if $O_i = \phi (\partial^2 \phi)^2$, $d_i = 3 \left( \frac{1}{2} d - 1 \right) + 2 = \frac{3}{2} d - 3 \quad \text{etc.}$

Note that the cutoff means that, if we "write outside" the Euclidean path integral everywhere so $p^2 \leq \Lambda_0^2$ everywhere it is well-defined — we do not need to worry about measures and if the measure gives a sensible path integral. Our comment is that the function can be arbitrary: we consider functions of the form

$$\phi(x) = \int \frac{d^d p}{(2\pi)^d} \Phi(p) e^{-ip\cdot x} \quad \text{where } \Phi(p) = 0 \text{ if } p^2 > \Lambda_0^2$$
Note that two is the same as putting on a lattice. For simplicity consider a dimension:

$$\varphi(x) = \sum_n \varphi_n \delta(x + 2\pi n b)$$

has the momenta

$$\varphi(p) = \int dx \varphi(x) e^{ipx} = \sum_n \varphi_n e^{-2\pi i p b}$$

so \(\varphi(p + 2\pi b) = \varphi(p)\) so effectively \(0 \leq p \leq 2\pi b\) so

$$\Lambda = \frac{2\pi b}{\varphi(x)}$$

where \(b\) is the lattice spacing.

We can use the ladder operator splitting \(\varphi(x)\) into high and low energy modes and "integrate out" the high energy modes. Consider \(\Lambda < \Lambda_0\):

$$\varphi(x) = \varphi_0(x) + \chi(x)$$

$$= \int \frac{dp}{(2\pi)^d} \varphi(p) e^{-ipx} + \int \frac{d\nu}{(2\pi)^d} \varphi(\nu) e^{-i\nu x}$$

then:

$$\delta \varphi(x) = \delta \varphi_0(x) \delta \chi(x)$$

and we can also consider during the \(\delta \chi(x)\) path integral:

$$\mathcal{Z}_{\Lambda_0} (\{q_0\}) = \int \mathcal{D} \varphi_0 \left[ \mathcal{D} \chi \ e^{i \mathcal{S}_{\Lambda_0} [\varphi_0 + \chi]} \right]$$

$$= \int \mathcal{D} \varphi_0 \ e^{i \mathcal{S}_{\Lambda_0} [\varphi_0]} \text{ path integral}$$

where we get a new effective Wilsonian effective action

$$\mathcal{S}_{\Lambda} [\varphi'] = -i \log \left( \int \mathcal{D} \chi \ e^{i \mathcal{S}_{\Lambda_0} [\varphi_0 + \chi]} \right)$$

(don't confuse this with the effective action \(\mathcal{S}_{\varphi'/x}\)) because we can again write \(\mathcal{S}_{\Lambda} [\varphi']\) in terms of a set of operators:

$$\mathcal{S}_{\Lambda} [\varphi'] = \int d^d x \left( \frac{i}{2} \varphi'(x) \partial^2 \varphi(x) - \frac{i}{2} \Lambda^d d^d \varphi(x) ^2 \varphi_{\nu}(x) \varphi_{\nu}(x) \right)$$

where \(\varphi(x)\) are the wavefunction \(\varphi(x)\) and couplings \(\varphi_{\nu}(x)\) as a
\[ \text{funct} \phi \Lambda \vdash \text{(Note that } \mu \text{ is the number of fields in the operator } \Phi) \text{. Of course the } \Lambda \gamma \text{, by definition, the vacuum function calculated for } S \gamma (\pi) \text{ is the same as that for } \Lambda \gamma \phi \vdash S \gamma \phi : \]

\[ Z \Lambda \phi (\gamma i) = \int_{\Lambda \phi} \mathcal{D} \phi \ e^{iS \Lambda \phi (\pi)} = \int_{\Lambda \phi} \mathcal{D} \phi \ e^{iS \phi (\pi)} \]

\[ = Z \Lambda (\gamma i (\Lambda), 2 \gamma \phi (\Lambda)) \]

Hence we get the Callan-Symanzik equation:

\[ \frac{\Lambda \partial Z \Lambda}{\partial \Lambda} = \Lambda \phi \gamma \]

\[ = \left( \Lambda \partial Z \Lambda \frac{\partial Z \Lambda}{\partial \phi \gamma} \partial \phi \gamma, + \Lambda \partial \phi \gamma \Phi \frac{\partial Z \Lambda}{\partial \phi \gamma} \partial \phi \gamma, - \Lambda \partial \phi \gamma \Phi \frac{\partial Z \Lambda}{\partial \phi \gamma} \right) = 0 \]

Conveniently one defines

\[ \Lambda \partial Z \Lambda = \beta \]

"beta function"

\[ -\frac{1}{2} \Lambda \partial \phi \gamma = \gamma \phi \]

"anomalous dimension of } \phi \}

We see that the "anomaly" sum with scale } \Lambda \text{, how they grow is controlled by an infinite set of differential equations. (Note also that had we had more terms in } \Lambda \text{ we would have many anomalous dimensions.)

We can also derive Callan-Symanzik equations for correlation functions:

Provided } \mu \text{ are operator separated by distances larger than } 1/\Lambda \text{ we have:

\[ \langle \Omega | T \phi (x_1) \cdots \phi (x_n) | \Omega \rangle = \frac{1}{Z \Lambda \phi} \int_{\Lambda \phi} \mathcal{D} \phi \ \phi (x_1) \cdots \phi (x_n) e^{iS \Lambda \phi (\pi)} \]

\[ = \frac{1}{Z \Lambda \phi} \int_{\Lambda \phi} \mathcal{D} \phi \ \phi (x_1) \cdots \phi (x_n) e^{iS \Lambda \phi (\pi)} \]

Because of the } \phi \text{ factor } \phi \text{ is not canonically noncanonical in } S \Lambda \phi \phi \text{. We can define a new field
Can numerically normalised field: \( \tilde{\phi} = \frac{1}{\mathbb{Z}_N} \phi \) and define the continuum function:

\[
\mathbb{E} \mathbb{E}_{(x_1, \ldots, x_n; g_i)} = \frac{1}{Z_N} \int \mathcal{D} \tilde{\phi} \mathcal{D} \phi \phi(x_1) \cdots \phi(x_n) e^{-S_N[\phi]} e^{\beta N g_i} \mathcal{D} \phi \mathcal{D} \tilde{\phi}
\]

(note that \( S_N[g] = S_N[Z_N^{1/N} \tilde{\phi}] \) is independent of \( Z_N \)).

Hence:

\[
\langle \mathbb{E} \mathbb{E}_{(x_1, \ldots, x_n; g_i)} \rangle = Z_N^{-1/2} \mathbb{E} \mathbb{E}_{(x_1, \ldots, x_n; g_i)}
\]

are \( \Lambda \) and since there are independent of \( \Lambda \) we have:

\[
(\frac{\partial}{\partial \Lambda} + \beta \frac{\partial}{\partial g_i} + \Lambda \gamma_0) \mathbb{E} \mathbb{E}_{(x_1, \ldots, x_n; g_i)} = 0
\]

(since \( \gamma_0 = -\frac{1}{\Lambda} \frac{\partial \ln Z_N}{\partial \Lambda} \)).

We can consider an \( \infty \)-dimensional space \( M \) of couplings \( \beta \)-domain, in \( \beta \)-space the set \( \{ g_i \} \) defines a point in \( M \). The Callan-Symanzik equations then define a flow on \( M \) called "renormalized flow of couplings".

\[
M
\]

\[
\{ g_i(\Lambda) \}
\]

among different flows as \( \Lambda \) decreases.

Note that there are special points in the flow. (Not not really a guy?)

- Critical point at \( \beta g_i \) flow: \( \beta_0 g_i(\beta_0) = 0 \)
- so we are at a point \( g_i = g_i^* \) where the theory is independent of \( \Lambda \)

A very simple example is:

- Gausean critical point: \( g_i^k = 0 \)
- so we can only have a kinetic term: \( \tilde{S}_N = \int \mathcal{D}\phi \left( \frac{1}{2 \Lambda} \phi \phi \right) \)
- Free theory: No interactions.
However, there can also be non-trivial fixed points in 4D critical points, where we have an interesting theory independent of scale: i.e.

- Critical points can be strongly  \( \bar{\lambda} \)  - coupled and hence hard to find.

The condition \( \delta i \phi \bar{\lambda} \) \( \bar{\lambda} \)  = 0 is one equation for each coupling and so generally we expect critical points are isolated points in \( \bar{\lambda} \) (though don’t have to be).

There scale invariant theories are very special. For example consider varying the metric: by a scale transformation:

\[
g_{\mu \nu} = \eta_{\mu \nu} \Rightarrow g'_{\mu \nu} = \bar{\lambda}^2 g_{\mu \nu} \Rightarrow x' = \bar{\lambda} x
\]

we have:

\[
\delta g_{\mu \nu} \frac{\delta}{\delta g_{\mu \nu}} \ln \mathcal{Z} (g_{\mu \nu}) = \frac{1}{2} \int d^4 x \bar{\lambda} \delta g_{\mu \nu} \mathcal{T}_{\mu \nu} \xi = \bar{\lambda} S \delta g_{\mu \nu}
\]

\[= \delta \bar{\lambda} \delta g_{\mu \nu} \propto \mathcal{Z} (g_{\mu \nu}) \]

Thus at critical points: for \( \delta g_{\mu \nu} = \bar{\lambda} g_{\mu \nu} \) we have \( \mathcal{Z} (g_{\mu \nu}) \) independent of scale and hence:

\[
\eta_{\mu \nu} \propto \mathcal{Z} (g_{\mu \nu}) \]

is scale invariant.

Similarly the Callan-Symanzik equation for \( \mathcal{Z} \) gives that:

\[
\Lambda \mathcal{Z}^2 \left( \frac{\partial}{\partial \Lambda} \mathcal{Z} \right) + 2 \bar{\lambda} \phi^* \Gamma^2 (\phi, \mu) = 0
\]

so

\[
\Gamma^2 (\phi, \mu) = \Lambda^{-2 \bar{\lambda} \phi^*} f (\Lambda x - y, g^\phi) \]

(where \( \delta \bar{\lambda} \delta g_{\mu \nu} = 0 \) \( \phi^* = \bar{\lambda} \phi (g^\phi) \)). By dimensional analysis:

\[
\left. \Lambda \mathcal{Z} \right| \phi (0) \mathcal{Z} \left| \phi (0) \right| = \Lambda^{4 - 2} f (\Lambda x - y, g^\phi)
\]

hence at the critical point we must have:

\[
f (\Lambda x - y, g^\phi) = \frac{c_\phi (g^\phi)}{(\Lambda x - y)^{2 \phi}} \]

\[
\Rightarrow \delta \phi = \frac{1}{2} (4 - 2) + \phi^* \]

"scaling dimensions"
so that

\[ \Gamma_2^{(d-2)}(x,y) = \frac{\Lambda^{d-2}}{1 \cdot 2 \cdot \cdots \cdot (d-2) \Lambda x y} \frac{c(y^*)}{(x^{-1} y^{-1} \Lambda x y)} = \Lambda^{-2d}\Lambda x y \frac{c(y^*)}{(x^{-1} y^{-1} \Lambda x y)} \]

and we have a long-term test conclusion – no support for

Yukawa factor \( e^{-m_0 x y} \).

In all known examples, scale invariance actually implies centural invariance:

* Theory is invariant under \( SO(d,2) \) "centrality group" which has the Lorentz group as a subgroup, includes scale transformations, but also more ("special centural transformation").

In 2d the centural group is \( SO(2,2) \) in Su, and there is no proof that scale invariance \( \Rightarrow \) centural invariance. This is believed to be true in other dimensions but (as yet!) there is no proof.

What does the flow look like near a critical point? We can expand:

\[ g(x) = g_0(x) + \delta g(x) \text{ then: } \Lambda \frac{\partial \delta g}{\partial x} = B_{ij} \delta g_{ij} + \partial_{ij}(\delta g_{ij}) + O(\delta g^2) \]

We can then diagonalize \( B_{ij} \) with

\[ \sigma_i : \text{eigenvalue of } B_{ij} \text{ with eigenvalue } \Delta_i = \Lambda_d \]

so (just convenient to include \( \Lambda_d \) in definition of eigenvalue)

\[ \Lambda \frac{\partial \sigma_i}{\partial x} = (\Delta_i - \Lambda_d) \sigma_i + O(\sigma^2) \quad \text{or: } \sigma_i(x) = (\Lambda_d / \Lambda_0)^{\Delta_i - \Lambda_d} \sigma_i(0) \]

If these were no constraints from pedagogical need ("equation correction") we expect \( \Delta_i = \Lambda_i \) (classical scaling dimension \( \Theta_i \) operator).

We can then identify 3 types of scaling:

\[ \Lambda_0 : \text{critical point) } \]

\[ \Lambda_+ : \text{upper critical point) } \]

\[ \Lambda_- : \text{lower critical point) } \]

We can then identify 3 types of universality:
we have:

1) relevant coupling: $\Delta i > d$
   
   or decreases as we leave lower $\Lambda$, so any coupling flows back to the fixed point $g^*$

2) relevant coupling: $\Delta i < d$
   
   or increases as we leave lower $\Lambda$, so any coupling flows away from the fixed point $g^*$

3) marginal coupling: $\Delta i = d$
   
   there is no flow to leading order. If we expand to the next order:

   $\Lambda(\Sigma_{\text{marg}}/\partial \Lambda) = C \Sigma_{\text{marg}}^2 \Rightarrow \frac{1}{\Sigma_{\text{marg}}} = A + \log(\Lambda/\mu)$

   if we rewrite the constant $A = -\log(\mu/\Lambda)$, then:

   $\Sigma_{\text{marg}} = \frac{1}{C \log(\mu/\Lambda)}$

   Assuming $\Sigma_{\text{marg}} > 0$ we know have 2 possibilities:

   i) $\Lambda < \mu$, $C > 0$ : "marginally relevant"

   ii) $\Lambda > \mu$, $C < 0$ : "marginally irrelevant"

   If $\Lambda(\Sigma_{\text{marg}}/\partial \Lambda) = 0$ to all orders we have an "exactly marginal gaussian coupling".

Note also that by dimensional analysis: $\beta_0(g(\Lambda))$ has no explicit $\Lambda$ dependence
e: $\beta_0$ only depends on $\Lambda$ through $g_0$. Thus

$\beta_0$ is a vector field on $\mathcal{M} 
\Rightarrow$ flow lines cannot cross

By distinguishing coupling that vary we get a picture of the general

Also note that, classically, we expect to quantum quantum effects

$\Delta i = di >$ dimension of operator $\beta_i$

since only a finite number of charges have $d_i < d$ we expect, even

after quantum correction, that there are only a finite number of

relevant couplings.
We get the following picture near $g_i$:

- Critical surface $C$
- All points flow to $g_i^*$

Relevant couplings, "renormalized trajectory".

There is a very large subspace $C \subset (g, \phi)$ at points in $\mathcal{W}$ that flow to the fixed point $g_i^*$ — these are the flows of the relevant operators. This is called the "critical surface".

There are a small number of relevant operators which generate a trajectory the flows away from $g_i^*$.

The key point is the following:

- Starting at a point near $g_i^*$, these under the flow the relevant operators are quickly suppressed.
- The flow focuses onto the renormalized trajectory for these few relevant operators only.

This is universally:

- Many different theories flow to the same low-energy ($\"mnu-nil\") physics, determined by the relevant couplings.

We say such high energy theories are in the same "universality class". This is the reason we can do low-energy physics without knowing all the high energy details — these are captured by the values of a few couplings.
The RG flow also captures how we can define a QFT independently of the high-energy physics. We reverse the picture:

- Fix a physical theory by giving $g_i(\Lambda)$ at some fixed scale $\Lambda = M$ and see how the cutoff theory $S_{\Lambda_0} [g_i]$ depends as $\Lambda_0 \to \infty$ — "continuum limit."

Thus we flow up in scale:

$\Lambda_0 \to \Lambda_0 \to \Lambda_0 \to \infty$

Company with our earlier cutoff calculations for $\chi^4$: that decreases as $\Lambda_0$:

Calculating with $S_{\Lambda_0} [g_i]$ gives finite answers independent of $\Lambda_0$ and $\Lambda_0$.

- Physical masses, wave functions, $\mu$, $\lambda$, $\phi$, etc. will be finite function of $g_i(M)$ at $\Lambda$ independent of $\Lambda_0$.

Recall the exponentials:

$$\sigma_i(\Lambda) = (\Lambda/M)^{\delta_i^0} \sigma_i(M) \quad \sigma_{i0} = \sigma_i(\Lambda_0)$$

Hence as $\Lambda_0 \to \infty$:

- Relevant coupling $\sigma_{i0}$ diverges.
- Relevant coupling $\sigma_{i0}$ go to zero.

Thus if we stay on the $\chi^4$ renormalization trajectory for the relevant coupling, we can take the limit $\Lambda_0 \to \infty$:

We need to be on the solid trajectory corresponding to relevant couplings only.
As we will see in a moment, $\lambda \phi^4$ is not at this type since the coupling
is nearly marginally irrelevant (same is true for $\lambda \phi^6$), but $\lambda \phi^4$
is since the coupling is marginally relevant

3.4 Local potential approximation and $\lambda \phi^4$

We saw already that there is a Gaussian critical point at $g_c = 0$.

Let's try and calculate the RG flows near this point, using the
approximation where we neglect operators $O_i$ with derivatives
and we also preserve the symmetry $\phi \to -\phi$. Thus:

$$\hat{S}_\lambda[\phi] = \int d^4x \left( \frac{1}{2} \partial^2 \phi \phi^2 + V(\phi) \right)$$

where:

$$V(\phi) = \frac{\lambda}{4!} \phi^4 - \frac{\lambda}{6} \phi^3$$

and we have absorbed $Z_\lambda$ into a rescaling of $\phi$. If we split:

$$\phi(x) = \phi^i(x) + \chi(x)$$

so:

$$Z_\lambda = \int_{[0,\lambda-\delta \lambda]} \mathcal{D} \phi^i \int_{[\lambda-\delta \lambda, \lambda]} \mathcal{D} \chi e^{i \lambda \phi_x^i, \chi}$$

and expand to quadratic order in $\chi$

$$Z_\lambda = \int_{[0,\lambda-\delta \lambda]} \mathcal{D} \phi^i e^{i \lambda \phi_x^i, \chi} \int_{[\lambda-\delta \lambda, \lambda]} \mathcal{D} \chi e^{i \Delta S[\phi^i, \chi]}$$

where:

$$\Delta S[\phi^i, \chi] = \int d^4x \left( \frac{1}{2} \partial^2 \phi \phi^2 + \frac{1}{2} V''(\phi^i) \chi^2 + \Theta(\chi^3) \right)$$

$$= -\frac{1}{2} \int d^4x \left( \partial^2 + V'' \right) \chi^2 + \Theta(\chi^3)$$

we can do the path integral:

$$\int_{[\lambda-\delta \lambda, \lambda]} \mathcal{D} \chi e^{i \Delta S[\phi^i, \chi]} = \frac{\text{const}}{\sqrt{|\text{det} M|}}$$
As for matrices we have \( \ln \det M = \text{tr} \ln M \) so

\[
2 \lambda = \int_{(0, \lambda - \delta \lambda)} d \Phi' e^{-i \mathcal{S}_N[\Phi'] + i \delta \mathcal{S}_N[\Phi']}
\]

where

\[
\delta \mathcal{S}_N[\Phi'] = \frac{1}{i} \text{tr} \ln M
\]

In momentum space, setting \( \Phi' = \text{const} \) (since we only want the correction to the potential) we have:

\[
\tilde{M}(p, q) = -i (p^2 - V'' + i \varepsilon) (2\pi)^d \delta^{(d)}(p - q)
\]

To take the trace we set \( p = q \) and integrate (this is the analogue of summing over the zero \( c = 0 \) indices in \( M^{ij} \)) so

\[
\delta \mathcal{S}_N = \frac{i}{2} \int_{A - \delta A} \frac{d^4 p}{(2\pi)^d} \ln (-i (p^2 - V'' + i \varepsilon)) (2\pi)^d \delta^{(d)}(0)
\]

\[
= \iota \int d^4 x \delta V(p')
\]

where (since the Fourier transform of a constant \( \lambda \) is \( (2\pi)^d \delta^{(d)}(0), \lambda \))

\[
\delta \mathcal{S}_N = \frac{i}{2} \int_{A - \delta A} \frac{d^4 p}{(2\pi)^d} \ln (p^2 + V'')
\]

\[
= \frac{1}{2} \int_{A - \delta A} \frac{d^4 p}{(2\pi)^d} \ln (p^2 + V'')
\]

\[
= \frac{1}{2} \frac{1}{(2\pi)^d} \int d^{d-1} \rho \cdot A^{d-1} \delta A \ln (p^2 + V'')
\]

\[
= \Lambda^{d-1} \delta A \ln (\Lambda^2 + V'')
\]

where

\[
\alpha = \frac{\text{volume}(\mathbb{S}^{d-1})}{2(2\pi)^d} = \frac{1}{(4\pi)^{d/2} \Gamma(1/2)} = \frac{\text{volume}(\mathbb{S}^d)}{\Gamma(1/2)}
\]

where

\[
\Gamma(n) = n! \quad \text{if} \quad n \in \mathbb{N} \cup \{1, 2, \ldots, 3\}
\]

\[
\Gamma(\xi) = 2 \pi^{(\xi-1)/2} \Gamma(1/2 - \xi/2) \Gamma(-\xi) \quad \text{if} \quad \Re(\xi) < 0
\]

\[
\Gamma(1) = 1 \quad \Gamma(1/2) = \sqrt{\pi} \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt
\]

Hence:
we also have for the classical reaction \( S_{\lambda} \): \( S_{\lambda} - \delta \lambda \left[ g_\lambda + \delta g_\lambda \right] = \delta \lambda \left[ g_\lambda + g_\lambda \right] \)

so for the potential \( V \):

\[
V = \frac{\delta \lambda}{\lambda} \left[ (\lambda - k\lambda - 1) \frac{\delta g_\lambda}{\delta \lambda} \right] \frac{\delta \lambda}{\lambda} + \frac{\delta \lambda}{\lambda} \left[ \Lambda \frac{\delta g_\lambda}{\delta \lambda} \right] \frac{\delta \lambda}{\lambda}
\]

so \( \delta g_\lambda = (\lambda - k\lambda - 1) \frac{\delta g_\lambda}{\delta \lambda} \). Putting this together with \( \delta V \) from the path integral with \( \lambda \) gives, neglecting a Taylor expansion at \( \delta V \):

\[
V = \Lambda \frac{\delta g_\lambda}{\delta \lambda} (k\lambda - 1) \frac{\delta g_\lambda}{\delta \lambda} - \Lambda \frac{\delta g_\lambda}{\delta \lambda} \frac{2\lambda}{\delta \lambda} \frac{\delta \lambda}{\lambda} \left[ \Lambda^2 + V'' \right] \bigg|_{\phi = 0}
\]

or term by term:

\[
\Lambda \frac{\delta g_\lambda}{\delta \lambda} = -2g_\lambda - \frac{a g_\lambda}{1 + g_\lambda} \quad \text{and} \quad \Lambda \frac{\delta g_\lambda}{\delta \lambda} = (d - n) g_\lambda - \frac{a g_\lambda}{1 + g_\lambda} + 3a g_\lambda^2 \frac{1 + g_\lambda}{(1 + g_\lambda)^2}
\]

Taking \( \lambda = 4 \): near the Gaussian fixed point we see \( g_\lambda \) is irrelevant for \( 2k \geq 6 \). Keeping only \( g_\lambda \) and \( g_\lambda \) gives:

\[
\Lambda \frac{\delta g_\lambda}{\delta \lambda} = -2g_\lambda - a g_\lambda \quad \text{and} \quad \Lambda \frac{\delta g_\lambda}{\delta \lambda} = 3a g_\lambda^2 \quad \text{with} \quad a = \frac{1}{16\pi^2}
\]

To leading order we have eigenvalues \( \sigma = \left( \frac{g_\lambda}{g_\lambda} \right) \) of \( B = \left( \begin{array}{cc} 2 & -a \\ 0 & 1 \end{array} \right) \):

\[
\sigma_{uu} = \left( \begin{array}{c} \rho \\ 0 \end{array} \right) \quad \text{and} \quad 4 - 4 = 0 \quad \sigma_{uu} = (\Lambda \sigma_{uu}) \quad \sigma_{uu} = (\Lambda \sigma_{uu})^2 \sigma_{uu,0}
\]

\[
\sigma_{wuy} = \left( \begin{array}{c} -\frac{a}{2\pi^2} \sigma_{uu} \\ \lambda \end{array} \right) \quad \text{and} \quad 4 - 4 = 0
\]

Working to second order for the marginal coupling and taking \( g_\lambda > 0 \) for stability:

\[
\lambda = \frac{1}{3a \sigma_{uu}(\mu / \lambda)} = \frac{16\pi^4}{3a \sigma_{uu}(\mu / \lambda)} \quad \text{maximally irrelevant}
\]

Thus we have the mean and quantum couplings:

\[
\mu^2 = g_\lambda \lambda^2 = \lambda^2 \sigma_{uu,0} - \frac{1}{2\pi^2} \Lambda \lambda^2 = \mu_0^2 - \frac{1}{2\pi^2} \Lambda \lambda^2
\]

\[
\lambda = \frac{16\pi^4}{3a \sigma_{uu}(\mu / \lambda)}
\]
We see the same dependence on $\Lambda$, as we saw before in calculating the one-loop diagram. The flow is:

\[ g_1 \rightarrow \text{Gaussian fixed point} \rightarrow \text{marginal relevance} \]

Since $\lambda$ is marginally irrelevant there is no attractor limit (only the free theory with $g_2$ becoming the relevant mass coupling $g_2$ has an attractor limit).

\[ \lambda = \frac{16\pi^2}{3\log(\mu/\Lambda)} \]

"Landau pole" as $\Lambda \rightarrow \mu$

This means that if we want to keep $g_4$ we need to have some "new physics" at a scale $\Lambda = \mu$. However $\lambda$, because large, is really our perturbative calculable breaking down — it is possible that the theory has a strongly coupled fixed point somewhere with large $g_4$ (and maybe $g_6$, etc.). This is called "asymptotic safety" — but there is no evidence for this in $\chi^k$.

8.5 Perturbative nonmaintainable and dimensional regularization

The Wilsonian description gives a very elegant picture of why QFTs exist and at what new theory can be independent of the high energy physics. However it is not always the most useful when doing calculations.

Let's go back to the theory defined with some cutoff $\Lambda$. 
An important point is that given the renormalonic:

\[ \chi_{\text{phys}} = \chi_0 \left[ 1 - \frac{3\lambda_0}{16\pi^2} \ln (\lambda^2/\mu^2) + O(\lambda^2) \right] \]

means that:
- the physical coupling \( \chi_{\text{phys}} \) can be very different
- from the bare coupling \( \chi_0 \)
- we really want to do perturbative theory in \( \chi_{\text{phys}} \)

The solution is to choose the counter terms:

\[
Z = \frac{1}{2} (\partial \varphi_0 \partial^2 \varphi_0) - \frac{1}{2} m_0^2 \varphi_0^2 - \frac{1}{4} \lambda_0 \varphi_0^4
\]

\[
= \frac{1}{2} \left( \partial^2 \varphi_R \right) \left( \partial^2 \varphi_R \right) - \frac{1}{2} m_R^2 \varphi_R^2 - \frac{1}{4} \lambda_R \varphi_R^4
\]

\[+ \frac{1}{2} \delta z \left( \partial \varphi_R \partial^2 \varphi_R \right) - \frac{1}{2} \delta m^2 \varphi_R^2 - \frac{1}{4} \delta \lambda \varphi_R^4 \]

where:
\[ \varphi_R = 2^{-11} \varphi \]

\[ \sqrt{(1 + \delta z)} \varphi_R = \frac{1}{2} \varphi \]

\[ 2m_0^2 = m_R^2 + \delta m^2 \]

\[ 2^2 \lambda_0 = \lambda_R + \delta \lambda \]

and

\[ \delta \text{ do perturbative theory in } \lambda_R \ (\text{not } \lambda_0) \]

This means we need to include new Feynman diagrams:

\[ \frac{1}{p} \rightarrow i (\delta z \cdot p^2 - \delta m^2) \]

\[ \frac{\delta z}{p} \rightarrow -i \delta \lambda \]

and we choose the counter terms to cancel the heavy \( \lambda_0 \)-dependent contributions for loops, so that \( \lambda_R \), \( m_R \) are the physical

couplings & masses. Note this is simply more a rewriting in

order to organize the theory as an expansion in \( \lambda_R \), we will have

\[ \delta m^2 \sim O(\lambda_R) \]

\[ \delta \lambda \sim O(\lambda_R^2) \]

\[ \delta z \]

\[ 0 \]

\[ + \]

\[ \infty \]
However the split into $\lambda_R$ and $\delta \lambda$ etc is not unique. There are many different "renormalization schemes" where we choose by "hunts" (i.e. no independent) parameters. For example, using our prescription that $\lambda_{phys} = \lambda_F (0, 0, 0, 0)$, we have:

$$\lambda_{phys} = \lambda_R \left[ 1 - \frac{3\lambda_R}{16\pi^2} \ln \left( \frac{\Lambda^2}{m^2} \right) \right] + \delta \lambda$$

(as a one-loop perturbation in $\lambda_R$). If we choose

$$\delta \lambda = \frac{3\lambda_R^2}{16\pi^2} \ln \left( \frac{\Lambda^2}{m^2} \right) + C$$

then, given the freedom in $\lambda_R$ choosing $\delta \lambda$, the final part $C$

$$\lambda_{phys} = \lambda_R + C$$

we can think of this as a different choice of defining how we measure the physical coupling. Recall $\lambda_F$ is a function of $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$. Then we can choose a set the freedom in $C$, can be thought of as fixing:

$$\lambda_R = \tilde{\lambda}_F (s_0, t_0, u_0) \quad \text{for some } s_0, t_0, u_0$$

where $\tilde{\lambda}_F (s_0, t_0, u_0) \neq \lambda_F (0, 0, 0, 0) = \lambda_R / \lambda_{phys}$.

Often one chooses a renormalization scheme that depends on some mass scale $M$. (For example, $s_0 = t_0 = u_0 = M^2$ in $\lambda_F$). Of course the final answers - correlation functions - must be independent of scheme, and hence of $M$. Just as for the Wilson renormalization we can formulate this as a differential equation:

$$G_n (x_1, \ldots, x_n, \mu^2 (\mu), \lambda_R (\mu^2), M)$$

$$= \langle \mathcal{O}_1 \mathcal{O}_2 (x_1) \cdots \mathcal{O}_n (x_n) \rangle \mathcal{O}_n$$

$$= 2^{\frac{n}{2}} \langle \mathcal{O}_1 \mathcal{O}_2 (x_1) \cdots \mathcal{O}_n (x_n) \rangle \mathcal{O}_n = \pm M \nu \lambda_{phys} \Gamma_n (x_1, \ldots, x_n)$$

where $\Gamma_n (x_1, \ldots, x_n)$ is independent of $M$ we have:

$$0 = \left( \frac{\partial}{\partial \lambda} + \beta \frac{\partial}{\partial \lambda_R} + \mu \gamma_2 \frac{\partial}{\partial \mu^2} + \nu \gamma_4 \right) \Gamma_n (x_1, \ldots, x_n, \mu^2 (\mu), \lambda_R, M)$$
\[ \beta_\lambda = \frac{M}{\partial M} \partial \lambda \quad \gamma_2 = \frac{M}{\partial \lambda} \partial \gamma^2 \quad \delta q = \frac{1}{2} \frac{M}{\partial M} \delta \lambda \]

These are again called the Callan-Symanzik equations, though here it is the independence of the nonperturbative prescription not the Wilsonian scale \( \Lambda \) that is encoded. Of course one possible prescription is to take: the Wilsonian couplings:

\[ \lambda_R = \delta \lambda(\Lambda) \bigg|_{\Lambda = m} \quad m_R = \Lambda^2 g_2(\Lambda) \bigg|_{\Lambda = m} \]

\[ Z = Z(\Lambda) \bigg|_{\Lambda = m} \]

and then the 2 Callan-Symanzik equations encode the same theory.

We now turn to dimensional regularization. Although the cutoff \( \Lambda \) is conceptually very simple, it is usually impractical to integrate multiloop integrals directly. The technique of dimensional regularization is much more convenient, and compatible with gauge invariance.

Standard technique is dimensional regularization: consider integrals in \( d = 4 - \varepsilon \) dimensions. We have

\[ I_{2n}(m) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - m^2 + i\varepsilon)^n} = (-1)^n \frac{i}{n!} I_{2n}^\varepsilon(m) \]

where:

\[ I_{2n}^\varepsilon(m) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m^2)^n} \]

If we general to \( d \)-dimensions keeping a factor of \( p^4 \), some mass scale \( \mu^4 \) so the dimension of \( I_{2n}^\varepsilon \) remains \( 4 - n \)

\[ I_{2n}^\varepsilon(m, \mu) = \mu^4 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m^2)^n} = \frac{\mu^4}{(2\pi)^4} \frac{\Gamma(\frac{d-2}{2})}{\Gamma(\frac{d}{2})} \int_0^\infty \frac{p^{d-2}dp^2}{(p^2 + m^2)^n} \]

\[ = \frac{\mu^4}{(4\pi)^{d/2} \Gamma(\frac{d}{2})} \int_0^1 x^{n-\frac{d-1}{2}} (1-x)^{\frac{d-1}{2}-1} dx \quad x = \frac{m^2}{p^2 + m^2} \]
but we have the Euler beta function
(not the RG beta function!!)
\[ \int_0^1 x^{\alpha - 1} (1-x)^{\beta - 1} \, dx = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \]

and hence
\[ I_n^\varepsilon (m, |e\varepsilon|) = \frac{\mu^4 m^{4-2n}}{(4\pi)^{\frac{1}{2}} \Gamma(n)} \frac{\Gamma(n-2d) \Gamma(2d)}{\Gamma(n)} = \frac{\mu^4 m^{4-2n} \Gamma(n-2d)}{(4\pi)^{\frac{1}{2}} \Gamma(n)} \]

Going back to the Minkowski metrics and using \( d = q - \varepsilon \)
\[ I_n^\varepsilon (m, |e\varepsilon|) = \frac{(-1)^n}{16\pi^2} \left( \frac{4\pi\mu^2}{m^2} \right)^{\varepsilon/2} \frac{\Gamma(n-2+\frac{1}{2}q)}{\Gamma(n)} \]

For \( q \leq 2 \), \( \Gamma(q) \) has poles at \( q = 0, -1, -2, \ldots \) but no zeros for \( q = 1, 2, 3, \ldots \)

Hence
\[ I_n^\varepsilon (m, |e\varepsilon|) \text{ is finite for } n \geq 3 \]

but expanding in \( \varepsilon \): (recall \( \Gamma(1) = \Gamma(2) = 1 \))
\[ I_2 (m, |e\varepsilon|) = \frac{i}{16\pi^2} \left( 1 + \frac{1}{2} \ln \left( \frac{4\pi\mu^2}{m^2} \right) \right) \Gamma(\frac{1}{2}\varepsilon) + O(\varepsilon^2) \]
\[ I_1 (m, |e\varepsilon|) = \frac{-i\mu^2}{16\pi^2} \left( 1 + \frac{1}{2} \ln \left( \frac{4\pi\mu^2}{m^2} \right) \right) \Gamma(-1+\frac{1}{2}\varepsilon) + O(\varepsilon^2) \]

But we have the expansions:
\[ \Gamma(\frac{1}{2}\varepsilon) = \frac{\varepsilon}{2} \Gamma(1) = \frac{\varepsilon}{2} - \gamma + O(\varepsilon) \]
\[ \Gamma(-1+\frac{1}{2}\varepsilon) = -\left( \frac{\varepsilon}{2} + 1 - \gamma \right) + O(\varepsilon) \]

(Where \( \gamma \) is the Euler-Mascheroni constant \( \gamma \approx 0.577216 \ldots \). Hence)
\[ I_2 (m, |e\varepsilon|) = \frac{i}{16\pi^2} \left( \frac{\varepsilon}{2} + \ln \left( \frac{4\pi\mu^2}{m^2} \right) - \gamma \right) + O(\varepsilon) \]
\[ I_1 (m, |e\varepsilon|) = \frac{i\mu^2}{16\pi^2} \left( \frac{\varepsilon}{2} + \ln \left( \frac{4\pi\mu^2}{m^2} \right) - \gamma + 1 \right) + O(\varepsilon) \]

with the cutoff we had:
\[ I_2 (m, \Lambda^2_0) = \frac{i}{8\pi^2} \left( \ln \left( \frac{\Lambda^2_0}{m^2} \right) - \frac{1}{2} \right) + O(m^4/\Lambda^2_0) \]
\[ I_1 (m, \Lambda^2_0) = \frac{-i\ln \Lambda^2_0}{16\pi^2} + \frac{i\mu^2}{8\pi^2} \ln \left( \frac{\Lambda^2_0}{m^2} \right) + O(m^4/\Lambda^2_0) \]
we see that \(\bar{N}e\) is acting like the cut-off:

\[
\log (\Lambda_0/n) - \frac{1}{2} = \frac{1}{2} + \frac{1}{2} \log (4\pi m^2 n^2) - \frac{1}{2}
\]

but the covariant regularization uses the quadratic divergence

\[\bar{N}e \text{ as is general.}
\]

dimensional regularization only keeps \(\log\) divergences.

We have seen how to calculate \(\text{In}(\Lambda_0/n)\) but more generally in Feynman

diagrams we have integrals of the form

\[
\int \frac{d^4k}{(2\pi)^4} \frac{k^\mu \cdots k^\nu}{(k - m^2)(k + p_1^2 - m^2) \cdots (k + p_n^2 - m^2)}
\]

for example

\[
\begin{align*}
\infty &= -\frac{1}{4} \lambda R \sum_i I_2(p_i + p_2, mw) \\
I_2(q, mw) &= \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k - m^2 + i\epsilon)(k - mw + i\epsilon)(k + q)^2 - mw + i\epsilon)}
\end{align*}
\]

The key "trick" is to use Feynman parameters: i.e. the identity

\[
\frac{1}{A B} = \int_0^1 dx \frac{1}{[x A + (1-x) B]^2}
\]

This can be extended to give an expansion for \(1/(k - mw)\). Choosing

\[
A = (k - q)^2 - mw^2 + i\epsilon \quad B = k^2 - mw^2 + i\epsilon
\]

we get

\[
I_2(q, mw, \epsilon) = \lambda R \int_0^1 dx \frac{1}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta(x) + i\epsilon)^2}
\]

where \(\Delta(x) = -x(1-x)q^2 + mw^2\)
But then we do the finite $I_2 (A \psi, e)$ so

$$I_2 (q, m \psi, e) = \frac{\imath}{16 \pi^2} \int_0^1 \! dx \left[ \frac{2}{\epsilon^2} + \log \frac{4\pi\mu^2}{\Delta(x)} - \gamma \right] + O(\epsilon)$$

$$= \frac{\imath}{16 \pi^2} \left( \frac{2}{\epsilon^2} + \int_0^1 \! dx \log \frac{4\pi\mu^2}{m_x^2 - x(\Delta m)^2} - \gamma \right) + O(\epsilon)$$

and then final results can be done analytically. This allows us to calculate $\hat{\Gamma}_2 (q, \psi, m \psi, e)$ exactly at one-loop. The same kind of techniques extend to all loops.

At one-loop we have:

$$\hat{\Gamma}_2 (p)_{\text{new}} = \frac{\imath \lambda \epsilon m^2}{32 \pi^2} \left( \frac{2}{\epsilon^2} + \log \frac{4\pi\mu^2}{m_x^2} - \gamma + 1 \right) + \imath (p^2 \delta^2 - \delta m^2)$$

$$\hat{\Gamma}_2 (p, \psi, m \psi, e)_{\text{new}} = \chi \chi \chi \chi + \chi$$

$$= -\imath \lambda \left[ 1 - \frac{3 \lambda \epsilon}{32 \pi^2} \left( \frac{2}{\epsilon^2} + \log \frac{4\pi\mu^2}{m_x^2} - \gamma + f(s, \tau, u) \right) \right] - \imath \delta \lambda$$

($f(s, \tau, u)$ cancels momentum dependence of $I_2 (q, m \psi)$). We see that

for dimensional regularization a natural renormalisation scheme is:

* minimal subtraction: cancel $\frac{2}{\epsilon^2}$ under $\delta m^2$

* modified minimal subtraction, $\bar{\text{MS}}$: at scale $M$:

  cancel $\frac{\mu^2}{\epsilon} + \frac{1}{2} \log \frac{4\pi\mu^2}{m_x^2} - \frac{\epsilon}{2} \gamma$ terms

since it always appears in this combination.

Then:

$$\hat{\Gamma}_2 (p) = \frac{\imath \lambda \epsilon m^2}{32 \pi^2} \left( \log \frac{M^2}{m^2} + 1 \right) + O(\lambda \epsilon)$$

$$\hat{\Gamma}_2 (p, \psi, m \psi, e) = -\imath \lambda \left[ 1 - \frac{3 \lambda \epsilon}{32 \pi^2} \left( \log \frac{M^2}{m^2} + f(s, \tau, u) \right) \right] + O(\lambda \epsilon^3)$$
One can use the same technique to calculate $\beta$-function in QCD and QED. Since we can tune the renormalization scale $\Lambda$ as being equivalent to the scale $\Lambda$ in the Wilsonian picture, we can calculate the $\beta$-function using $\frac{\partial^2 \beta}{\partial \ln \Lambda^2}$. In QCD one finds:

$$\beta e^2 = \Lambda \frac{\partial \beta e^2}{\partial \Lambda} = \frac{N_f}{6\pi^2} e^4$$

marginally irrelevant

where $N_f = \#$ of Fermion species. Hence:

$$e^2(\Lambda) = \frac{6\pi^2}{N_f \log (\mu/\Lambda)}$$

Landau pole, no continuum limit.

However for QCD (Gross, Politzer, Wilczek) for $SU(n)$

$$\beta g = \Lambda \frac{\partial \beta g}{\partial \Lambda} = \frac{3g^3}{16\pi^2} \left( -\frac{11}{3} n_f + \frac{2}{3} N_f \right)$$

where $N_f$ is number of Fermion in the fundamental (adjoint) representation.

Hence:

For $N_f < \frac{11}{3} n_f$: marginally relevant

we have:

$$g^2(\Lambda) = \frac{8\pi^2}{\frac{11}{3} n_f - \frac{2}{3} N_f} \frac{1}{\log (\Lambda_{\text{QCD}}/\Lambda_{\text{scale}})}$$

**Asymptotic freedom**: weakly coupled at high energy.

Gaussian tree fixed point

The scale $\Lambda_{\text{QCD}}$ freezes the strength of the coupling. At lowenergies our description in terms of gluons and quarks breaks down. $\Lambda_{\text{QCD}}$ scale of hadron masses. \( \Lambda_{\text{QCD}} \approx 250 \text{ MeV} \)