Rotating Black Holes

In gravitational collapse, we expect isolated black holes to settle down to a "time independent" equilibrium state (in reality, it's a fast process). Want to classify all such states. A powerful "Uniqueness Theorem".

Define an asymptotically flat spacetime as stationary if there exists a Killing vector \( K \) that is timelike near \( \infty \).

Can normalise so that \( h^2 - 1 \) at \( \infty \).

Local cords near \( \infty \): \( t, x^i \)

\[ ds^2 = g_{tt}(t) dt^2 + 2g_{tt} dt dx^i + g_{ij}(x^i dx^i dx^j) \]

\[ g_{tt} \to 1 \text{ as } |x^i| \to \infty \]

Stationary \( \iff \) \( g_{ti} \to 0 \) (more precise, global definition)

Define an asymptotically flat spacetime as axisymmetric if \( K \) Killing vector in that is spacelike near \( \infty \) and all orbits of \( K \) are closed (ie. \( S^1 \)).

Can choose local cords so that \( M = 0 \) with \( g_{tt} \leq 0 \) and \( \nabla^t K = 0 \).

Recall Birkhoff: spherical symmetry \( \Rightarrow \) static

Converse is not true (eg consider spacetime outside a cuboid).

But if the only object is a black hole we have...
Theorem Israel 1967

(M, g) static asymptotically flat, vacuum black hole spacetime, regular on and outside an event horizon then (M, g) is (isometric to) Schwarzschild.

Note:
1) Static, vacuum multi black holes don't exist
2) I am Einstein Maxwell generalisation which says solution is R-N or M-P.

Theorem Hawking 1973

(M, g) stationary, non-static, asympt. flat, analytic solution of Einstein-Maxwell that is regular on and outside the event horizon then (M, g) is stationary + axisymmetric.

i.e. "stationary = axisymmetric"

(Analyticity here a bit unsatisfactory....)

Theorem Carter (1971) - Robinson (1973)

(M, g) stationary, axisymmetric, asymptotically flat, vacuum, regular on and outside event horizon then (M, g) is part of two parameter family of Kerr solutions. The parameters are mass M and angular Momentum J

Strongly implies that the final state of rotating gravitational collapse is Kerr; with all other information about the collapse is lost - either by radiation or by falling into the black hole.

[Aside: more options - asympt. AdS space -]
There is an Einstein-Maxwell version which states (4.3) should belong to Kerr-Newman family specified by $a$, $J$, $Q$ and $C$ parameters.

Kerr-Newman 1965

"black holes have no hair"

Boyer-Lindquist coordinates:

$$ds^2 = \frac{-\Delta}{\Sigma} dt^2 - 2a \sin^2 \theta \left( \frac{\Delta}{\Sigma} \right) dr dt + \frac{\Delta}{\Sigma} d\theta^2 + \frac{\Sigma}{\Delta} \sin^2 \theta d\phi^2 + \sum \Omega d\Omega^2$$

$$A = -\frac{1}{\Sigma} \sum Q_i \left( dt - a \sin^2 \theta d\phi \right) + \sum P \cos \theta \left( a dt + \sum \phi \right)$$

where

$$\Sigma = r^2 + a^2 \cos^2 \theta$$

$$\Delta = r^2 - 2Mr + a^2 + \ell^2$$

$$e = \sqrt{a^2 + \ell^2}$$

Note 2) $a = 0 \rightarrow K-N$ solution

3) Killing vectors $k = \partial_t$, $\ell = \partial_\phi$, $\Theta = \partial\theta$ (unnormalized)

5) $t \rightarrow t + \phi \rightarrow - \phi$ (dilatation, inversion)

4) $t \rightarrow t + \phi \rightarrow - \phi$ discrete isometry.

6) See later $a = J/\mu = \frac{\text{angular momentum}}{\text{mass}}$
Kerr solution (1963) \( P = Q = 0 \).

Describes rotating black holes

Approximates spacetime outside rotating star

Describes all rotating black holes (neutral)

Analytic structure

Coord. sing. \( \theta = 0, \pi \) — usual

\[ \Delta = \frac{\Delta}{r - r_+} (r - r_-) \]

\[ r_\pm = M \pm \sqrt{M^2 - a^2} \]

Curvature sing. \( \Sigma = 0 \)

\[ r = 0 \quad \frac{\Delta}{r} \quad \theta = \frac{\pi}{2} \]

Again, 3 cases to consider

1) \( a > M \) — superextremal

- Naked singularity
- Non-physical

2) \( M > a \) — of most interest

Start at \( r > r_+ \) define Kerr coordinates \((\varphi, r, \theta, \chi)\)

Analogous to ingoing E-T:

\[
\begin{align*}
\Delta U &= dt + \frac{r^2 + a^2}{\Delta} d\varphi \\
\Delta d\chi &= d\varphi + \frac{a}{\Delta} dr
\end{align*}
\]

\( \Delta = d\varphi = d\chi \)

\( M = d\varphi = d\chi \quad 0 \leq \chi \leq 2\pi \).
\[ ds^2 = \left( \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} \right) du^2 + 2 dudr - 2a \sin^2 \theta \left( r^2 + a^2 - \alpha \right) \frac{dr^2}{\Sigma} \]

\[- 2a \sin^2 \theta \, d \Omega^2 + \left( \frac{r^2 + a^2}{\Sigma} - \Delta \sin^2 \theta \right) \sin^2 \theta \, d\Omega^2 + \Sigma \, d\theta^2 \]

Smooth at \( r = r_+ \) and at \( r = r_- \). So can be analytically continued to new region \( 0 < r < r_+ \).

We then proceed as for R-N case.

To draw a purpose diagram is difficult since no spherical symmetry.

Consider two interesting slices:

\( \theta = 0 \) (or \( \theta = \pi \)) - the axis of symmetry

or \( \theta = \pi/2 \) - the equatorial plane (where \( \Sigma = 0 \))

Note \( \theta = 0 \) or \( \theta = \pi/2 \) are both "totally geodesic".

i.e. a geodesic initially tangent to it stays tangent.

By introducing Kruskal coards, we find in both cases:

\[ \text{the dotted line refers to the singularity at } r = 0 \]

\[ \text{for } \theta = \pi/2 \text{ case} \]

\[ \text{Case} \]
The ring singularity:

Not at fixed $w, r, \Theta$ we have

\[ ds^2 = \left( \frac{(r^2 + a^2)^2 - \Delta a^4 \sin^2 \Theta}{\Delta} \right) \sin^2 \Theta d\chi^2 \]

as $r \rightarrow 0$

\[ ds^2 \rightarrow \left( \frac{a^4 - a^4 \sin^2 \Theta}{a^4 \cos^2 \Theta} \right) d\chi^2 \]

\[ = a^2 \sin^2 \Theta d\chi \]

At $\Theta = \pi/2$ we see this has proper length $\pi^2 a^2$.

If you travel from any other angle $(\Theta + \pi/2)$ one does not hit the ring singularity but emerges into a new region of spacetime.

Closed time-like curves

There are ubiquitous in GR, e.g. periodically identify time in Minkowski space.

...are almost certainly not physical. (Any other role?)

Kerr has CTC's.

Consider a world line $x$:

Calculate norm of $m = \frac{\partial}{\partial \chi}$ at $\Theta = \pi/2$

\[ m^2 = g_{\chi \chi} = r^2 + a^2 + 2ma \]

If $r < a \Rightarrow (r > \frac{2ma}{r^2 + a^2})$ $m$ is time-like.

But orbits of more closed $\Rightarrow$ CTC's near the ring singularity.
However, this is not a physical because Cauchy horizon at $r = r_c$.

Kerr describes rotating black holes but only approximately that outside a star.

3) External case $M = a$ - not physical.

For a collapsing rotating shell, probably get spacelike?

$n:b$ extend for Kerr-Newman

$\mu^2 = a^2 + \ell^2$
\[ \frac{\mathbf{u}}{3} = k \mathbf{u} + \Omega \, H \mathbf{M} \]
\[ \mathbf{u} = \frac{\alpha}{r^2 + \alpha^2} \]

\[ \mathbf{a} = 1 \]
\[ \mathbf{a}_r = 5 \mathbf{r}_r = \mathbf{a}_\theta = 0 \]
\[ \mathbf{a}_r = \Omega H \]
\[ \mathbf{a}_\theta = 0 \]

\[ \mathbf{z} = 0 \]
so \( r = r_+ \) small

Also in 3-L coads

\[ \mathbf{z} = 2t + \Omega H \, 2 \phi \]

\[ \mathbf{z}^m \partial_m (\phi - \Omega H t) = 0 \implies \phi = \Omega H t + \text{const} \]

Recall \( \phi = \text{const} \) an integral curve of \( \mathbf{z} \)

Hence particle on orbits of \( \mathbf{z} \) relate with angular velocity \( \Omega H \) art to a "stationary" observer (are an orbit of \( \mathbf{z}^m \)). eg someone on

\( \text{in interpreted as angular velocity of blobbob.} \)
**Ergosphere**

In B-L coordinates

\[
K^2 = 9t^2 = -\left(\Delta - \frac{a^2 \sin^2 \theta}{\Sigma^2}\right) = -\left(1 - \frac{2Mr}{r^2 + a^2 \cos^2 \theta}\right)
\]

1. **Timelike for** \[r^2 + a^2 \cos^2 \theta - 2Mr > 0\]

2. \[r > M + \sqrt{M^2 - a^2 \cos^2 \theta}\]

K is spacelike outside \(H^+\) in the ergosphere:

\[r_+ = M + \sqrt{m^2 - a^2 \cos^2 \theta} < r < M + \sqrt{M^2 - a^2 \cos^2 \theta}\]

---

A stationary observer is one with \(\mathbf{v} = \mathbf{0}\) parallel to \(K\). Such observers do not exist in the ergo-region; in this region they must rotate relative to observers at \(\infty\), in the same direction as the black hole.

**Note:** See (ii)

\[
\sqrt{u^i u^i} = \sqrt{g_{ij} u^i u^j} = \frac{g_{tt} u^t u^t}{u^2} = \frac{g_{tt}}{u^2} \quad \text{all positive except for} \quad 9t^4 + U^4 = 0\]

\[\Rightarrow U^4 > 0 \quad \text{for future directed} \]

**Rindler Frame - dragging**
Consider a particle with 4-momentum \( p^\mu = \mu \frac{dx^\mu}{dt} \) approaching a Kerr black hole along a geodesic.

**Conserved Energy of Particle** (according to a stationary observer at \( \infty \))

\[ E = -k \cdot p \quad \text{at a pt } P \]

Suppose it decays into 2 particles with momentum \( p_1^\mu + p_2^\mu \). We have \( P^\mu = p_1^\mu + p_2^\mu \) \( \text{ (use inertial frame at point } P) \)

\[ \text{Therefore } E = E_1 + E_2 \]

Normally \( E_1 > 0 \) \& hence \( E_2 < E \)

However, note that \( E_1 = -p_1 \cdot k \)

\[ \text{ & hence if splitting happens inside the ergo-region we can have } E_1 < 0 \& \text{ hence } E_2 > E \]

To see this:

Choose coords in which \( k = (0, 1, 0, 0) \)

\[ \text{ & prepare decay so that } (p_1^\mu) = (1, y, 0, 0) \text{ with } y \text{ small enough} \]

\[ \implies p_1 \cdot k = y \]

Energy has been extracted from the black hole!

When \( E_1 < 0 \) the particle can show that \( E_1 < 0 \) the particle has fallen into the black hole. (Also if \( E < 0 \) then \( L < 0 \).)
How much energy can be extracted!

particle crossing \( H^+ \) : p. 5 \( \geq 0 \) since \( p^\mu, \gamma^\mu \) both future directed a

\( \Rightarrow E = 2 M L > 0 \)

where \( L = M \cdot p \) so conserved angular m.m.

\( \Rightarrow L \leq E / M \)

(note this implies that \( y E < 0 \) then \( L < 0 \))

After particle with \( E, L \) enters, black hole settles down to Kerr with \( S M = E \)

\( 8 J = L \)

\( 8 J = \frac{SM}{\eta_M} = \frac{2M[M^2 + \sqrt{M^4 - J^2}]}{J} \frac{\eta_M}{8} \left[ \frac{a}{J} \right] \)

Recall \( a, J \geq 0 \).

\( \Rightarrow S \eta_M > 0 \)

where \( \eta_M = \left[ \frac{1}{8} \left( M^2 + \sqrt{M^4 - J^2} \right) \right]^{1/2} \)

"irreducible mass"

\( E_{\text{av}} \)

\( A = 4 \pi M^2 \eta_M \)

Hence \( 8 J > 0 \) case of 2nd law!

\( \Rightarrow M^2 = M_{\text{ir}}^2 + \frac{J^2}{4 M_{\text{ir}}^2} > M_{\text{ir}}^2 \)

thus cannot reduce mass below initial value of \( M_{\text{ir}} \).

Hence suppose \( M_0, J_0 \rightarrow M_{\text{ir}} (M_0, J_0) \) to start \((M_{\text{ir}}, \eta_E)\) to end.

The smallest \( M \) can be? \( M_{\text{ir}}, M \geq M^2 (0) > M^2 (0) \)

Hence smallest \( J \Rightarrow M^2 = M_{\text{ir}} (0) = M^2 (0) \)

\( \Rightarrow J = 0 \Rightarrow \eta_E = E / M \) \& decays

14. \( \{M^2 = M_{\text{ir}} (0), \eta_E \}

\( \eta = \text{Efficiency} = \frac{M^2 - M_{\text{ir}}^2}{M_0^2} = 1 - \frac{M_{\text{ir}} (0)}{M_0^2} \approx 1 - \left( \frac{\eta}{2} \right)^{1/2} \)

a. \( M_0 = 9 \text{ or } M_0 \Rightarrow \eta = \frac{M_0 - M_{\text{ir}}}{M_0} \approx 29\% \).
Black Holes - Jerome Gauntlett

Notes on differential forms

Let $M$ be an $n$-dimensional manifold. A $p$-form is a tensor of type $(0,p)$ that is totally anti-symmetric:

$$A_{\mu_1 \ldots \mu_p} = A_{[\mu_1 \ldots \mu_n]}$$  \hspace{1cm} (1)

A 0-form is simply a function and a 1-form is a co-vector. We must have $p \leq n$.

**Wedge product:** Consider a $p$-form, $A$, and a $q$-form, $B$. We define the $(p+q)$ form $A \wedge B$ via

$$(A \wedge B)_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_q} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \ldots \mu_p} B_{\nu_1 \ldots \nu_q]}$$  \hspace{1cm} (2)

We immediately have

$$A \wedge B = (-1)^{pq} B \wedge A$$  \hspace{1cm} (3)

and a corollary is $A \wedge A = 0$ if $p$ is odd.

In a given set of coordinates we can consider a basis for one-forms $dx^\mu$. Using the wedge product we can obtain a basis for $p$-forms and we can write

$$A = \frac{1}{p!} A_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$  \hspace{1cm} (4)

**Exterior derivative:** This is a derivative operation that takes a $p$-form, $A$, to a $p+1$ form, $dA$ whose components are

$$(dA)_{\nu_1 \ldots \nu_{p+1}} = (p+1) \partial_{\nu_1} A_{\nu_2 \ldots \nu_{p+1}}$$ \hspace{1cm} (5)

The factor $(p+1)$ corresponds to the fact that we can think of $d$ as the operation $dx^p \wedge \partial_\nu$ in the sense:

$$dA = \frac{1}{p!} \partial_\nu A_{\mu_1 \ldots \mu_p} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$$= \frac{1}{p!} \partial_\nu A_{\mu_1 \ldots \mu_p} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$$= \frac{1}{(p+1)!} [(p+1) \partial_\nu A_{\mu_1 \ldots \mu_p}] dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_{p+1}}$$  \hspace{1cm} (6)

If $A$ is a $p$-form and $B$ is a $q$-form we have the Leibniz rule:

$$d(A \wedge B) = (dA) \wedge B + (-1)^p A \wedge (dB)$$  \hspace{1cm} (7)

We also have the important property that

$$d^2 = 0$$  \hspace{1cm} (8)

We have not yet assumed that we have a metric defined on $M$. Let us now do so, with components $g_{\mu \nu}$. This gives a unique Levi-Civita covariant derivative $\nabla$. It is useful to note that we can write

$$(dA)_{\nu_1 \ldots \nu_{p+1}} = (p+1) \nabla_{[\nu_1} A_{\nu_2 \ldots \nu_{p+1}]}$$  \hspace{1cm} (9)
where the right hand side is usual integration. One can show that this is a coordinate independent definition. Note that this definition did not require a metric.

Suppose now we have a metric and hence a volume form \( \epsilon \). We can then define the volume of \( M \) to be \( \text{Vol}(M) = \int_M \epsilon \) (which might be infinite). We can also use \( \epsilon \) to define the integral of a function \( f \) on \( M \):

\[
\int_M f \equiv \int_M f \epsilon = \int d^n x \sqrt{|g|} f
\]

and the latter expression should be familiar.

**Stokes Theorem**: Let \( M \) be an oriented \( n \)-dimensional manifold with boundary \( \partial M \) then for an \( (n-1) \)-dimensional form \( A \) we have

\[
\int_M dA = \int_{\partial M} A
\]

Notice that this theorem does not require a metric.

**Gauss Law or Divergence Theorem**: Let \( M \) be an \( n \)-dimensional manifold with boundary \( \partial M \), metric \( g_{\mu \nu} \) and volume form \( \epsilon \). Let \( V \) be a one-form, then \( \ast V \) is an \( (n-1) \)-form and Stokes Theorem says

\[
\int_M dA = \int_{\partial M} A = \int_{\partial M} \ast V
\]

We now want to reexpress the left and right hand sides. From (16) we have

\[
\ast d \ast V = \pm (-1)^{n-1} (\text{div} V) 1
\]

\[
\Rightarrow \pm d \ast V = \pm (-1)^{n-1} (\nabla^\mu V_\mu) \ast 1
\]

\[
\Rightarrow d \ast V = (-1)^{n-1} (\nabla^\mu V_\mu) \epsilon
\]

where \( 1 \) is the trivial 0-form (function) which is 1 everywhere and as noted above \( \ast 1 = \epsilon \). The left hand side of (20) is thus

\[
\int_M dA = (-1)^{n-1} \int_M d^n x \sqrt{|g|} \nabla_\mu V^\mu
\]

We now consider the right hand side of (20). We first assume that \( \partial M \) is specified by an outward pointing normal vector \( n^\mu \), with \( n^a = n^\mu n_\mu g_{\mu \nu} = \mp 1 \) depending on whether the normal is time-like or spacelike. The induced metric on \( \partial M \) is then given by \( h_{\mu \nu} = g_{\mu \nu} \pm n_\mu n_\nu \) (note that \( h_{\mu \nu} n^\nu = 0 \)). This induced metric \( h \) can be used to define a volume \( (n-1) \)-form, \( \tilde{\epsilon} \) on \( \partial M \). We then calculate

\[
\ast V = \frac{1}{(n-1)!} \epsilon_{\mu_1 \cdots \mu_n} V^{\mu_1} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}
\]

\[
= (-1)^{n-1} \frac{1}{(n-1)!} \epsilon_{\mu_1 \cdots \mu_n} V^{\mu_1} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}
\]

\[
= (-1)^{n-1} \frac{1}{(n-1)!} (n^\nu V_\nu) \tilde{\epsilon}_{\alpha_1 \cdots \alpha_{n-1}} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{n-1}}
\]

\[
= (-1)^{n-1} (n^\nu V_\nu) \tilde{\epsilon}
\]

\[(23)\]
$\text{Diff forms} \quad M$ - $n$-dimensional

- **P-form** - totally antisymmetric $(0,p)$ tensor
  \[ \kappa A_{\nu_1 \ldots \nu_p} = A[\nu_1 \ldots \nu_p] \]

\[
\begin{align*}
\{ & \text{0-form} - \text{function} \\
\{ & \text{1-form} - \text{1-form vector} \, \omega \\
\text{Wedge product} & \quad \text{A-form} \, A, \quad \text{q-form} \, \beta \quad A \wedge \beta \quad \text{(n+1) form} \\
(A \wedge \beta)_{\nu_1 \ldots \nu_{p+q}} &= (p+q)! \frac{A_{\nu_1 \ldots \nu_p} \beta_{\nu_{p+1} \ldots \nu_q}}{p! q!} \\
\end{align*}
\]

- **Example**
  \[ A \wedge \beta = (p+q)! \, \beta \wedge A \]

- **Corollary**
  \[ A \wedge A = 0 \quad \text{if} \quad p \text{ odd} \]

Convenient to write components as
\[ A = \frac{1}{p!} \, A_{\nu_1 \ldots \nu_p} \, dx^{\nu_1} \ldots dx^{\nu_p} \]

**Exterior derivative**
\[
\begin{align*}
dA &= \frac{1}{p!} \, \partial_{\nu_1} A_{\nu_1 \ldots \nu_p} \, dx^{\nu_1} \wedge dx^{\nu_2} \wedge \ldots \wedge dx^{\nu_p} \\
&= \frac{1}{p!} \, \partial_\nu A_{\nu_1 \ldots \nu_p} \, dx^\nu \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_p} \\
&= \left( \frac{p+1}{p!} \right) \left( \frac{1}{(p+1)!} \right) \, \partial_\nu A_{\nu_1 \ldots \nu_{p+1}} \, dx^\nu \wedge \ldots \wedge dx^{\nu_{p+1}} \\
\end{align*}
\]

\[ (A \wedge \beta)_{\nu_1 \ldots \nu_{p+1}} = \left( \frac{p+1}{p!} \right) \, \partial_\nu A_{\nu_1 \ldots \nu_{p+1}} \, dx^\nu \wedge \ldots \wedge dx^{\nu_{p+1}} \]
Facts:
\[ d^2 = 0 \]

\[ d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta) \]

If we have a metric \( g \) and hence Levi-Civita \( \nabla \)
\[
(\Phi A)_v \ldots V_{\nu + m} = (p + 1) \nabla [v_1, v_2, \ldots, A v_{p+1}]
\]

Also, define volume form in \( n \) dimensions

\[
E_{\mu_1 \ldots \mu_n} = \frac{1}{n!} \varepsilon_{\mu_1 \ldots \mu_n}
\]

\[ \uparrow \text{tensor} \]

\[ \leftrightarrow \text{even} \quad \text{perm} \quad \text{odd perm} \]

\[ \uparrow = 1, \quad \downarrow = -1 \]

Example:

\[
E_{\iota_1 \ldots \iota_n} = g_{\mu_1 \nu_1} \ldots g^{\mu_n \nu_n} E_{\mu_1 \ldots \mu_n}
\]

\[ = \det g^{-1} \varepsilon_{\mu_1 \ldots \mu_n} \]

\[ = \det g^{-1} \varepsilon_{\mu_1 \ldots \mu_n} \]

\[ = \pm \frac{1}{n!} \varepsilon_{\mu_1 \ldots \mu_n} \]

\[ \rightarrow \text{Riemannian} \]

\[ \rightarrow \text{Lagrangian} \]

\[
E_{\iota_1 \ldots \iota_{n-p}, v_{p+1} \ldots v_n} = \pm p! (n-p)! \delta_{\iota_1 \ldots \iota_{p}}^{v_1 \ldots v_p}
\]

\[ \nabla_{\rho} E_{\mu_1 \ldots \mu_n} = 0. \]
**Def:** Hodge dual of p-form $A$ is the $(n-p)$-form $\star A$

with

$$\star A = \frac{1}{p!} \varepsilon_{\nu_1...\nu_{n-p}} A_{\mu_1...\mu_p} \ A^{\nu_1...\nu_p}$$

**Ex:**

$$\star \star A = \pm (-1)^p (n-p) A$$

**Ex:**

$$(\star d \star A)_{\mu_1...\mu_{p-1}} = \pm (-1)^p (n-p) \nabla^\nu A_{\mu_1...\mu_p} \ A^{\nu_1...\nu_p}$$

**Integration**

**Integral of n-form A in n-dimensions M**

$$\int_M A = \int dx^1 dx^2 ... dx^n A(x) = A_1 dx^1 ... dx^n$$

**Ex:**

Given

$$A = A_1 ... n \frac{dx^1 ... dx^n \ dy^{n-1} ... dy^1}{dy^n}$$

$$\Rightarrow \int_M A = \int \det \frac{\partial x^i}{\partial y^j} A_1 ... n = \int dx^1 A_1 ... n$$

If we have metric $g$ then volume form $\mathcal{E}$

- Define Volume on M as $\int_M \mathcal{E}$

To define integral of a function $f$ on M as $\int_M f \mathcal{E}$

(Note: $\int_M f = \int \mathcal{E} \ f(x)$)
**Stokes Theorem**

Let $N$ be an oriented $n$-dimensional manifold with boundary $\partial N$ and a $n-1$-form $\omega$, then

$$\int_N d\omega = \int_{\partial N} \omega$$

---

**Gauss-Law (Divergence Theorem)**

Manifold $M$ is $n$-dimensional, boundary $\partial M$ metric $g$, $\omega$ is $1$-form, $A = \ast \omega$ is $n-1$-form

$$dA = d \ast \omega$$

Stokes: $\int_M dA = \int_{\partial M} A = \int_{\partial M} \ast \omega$

Note: $\ast (d \ast \omega) = \pm (\ast)_{n-1} V^\nu V^\nu$

$$\Rightarrow (\ast d \ast \omega) = \pm (\ast)_{n-1} (\nabla \cdot \omega) \ast \delta^{\nu \nu}$$

$$\Rightarrow d \ast \omega = (\ast)_{n-1} (\nabla \cdot \omega) \delta^{\nu \nu}$$

$$\Rightarrow \int_M dA = (\ast)_{n-1} \int_M \delta^{\nu \nu} \left[ \frac{1}{(\ast)_{n-1}} \right] (\nabla \cdot \omega)$$

Next: Assume $\partial M$ specified by outward pointing normal $\partial M$. The induced metric on $\partial M$ is

$$h_{\mu \nu} = g_{\mu \nu} \pm N_{\mu} N_{\nu}$$

for $N^2 = -1$

$$\ast \omega = E_{1 \ldots m} V^\nu V^\mu \ldots dx^{m-1} \frac{1}{(n-1)!}$$

$$= (-1)^{n-1} \sum E_{1 \ldots m} V^\nu V^\mu \ldots dx^{m-1} \frac{1}{(n-1)!}$$

$$= (-1)^{n-1} (V^\nu V^\mu) E_{1 \ldots m} \ldots dx^{m-1} \frac{1}{(n-1)!}$$
Hence

\[ \int_m d^n \sqrt{1 - V} = \int_m d^n \sqrt{1 - \phi^2} \]

\[ dA = 0 \quad \text{"closed"} \]

\[ A = dB \quad \text{"exact"} \]

\[ \text{"exact"} \Rightarrow A \text{ closed} \]

\[ \text{Not! Converse.} \]

\( \vec{A} = -\cos \theta \, d\phi \)

\( \nabla \times \vec{A} = 0 \quad \text{\&} \quad \nabla \cdot \vec{A} = 0 \)

\( \begin{align*}
    \vec{F}_{uv} &= \partial_u A_v - \partial_v A_u \\
    \vec{F}_{ri} &= \partial_r \vec{A} \\
    \vec{F}_{tr} &= -\vec{F}_{ri} \quad \Rightarrow \\
    \vec{F}_{\phi} &= \sin \theta \\
    \vec{F}_{\theta} &= 0
\end{align*} \)

\( \vec{F} = \frac{1}{2} F_\phi d\phi + dA = \sin \theta \, d\phi \, d\theta \, d\phi \)
Energy & Angular Momentum: Kevor integrals.

Maxwell's eqs.

\[ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \quad \Rightarrow \quad \nabla \cdot J = 0 \\
\nabla \times E = \frac{\mu_0}{c^2} \nabla \times B = 0 \quad \text{Maxwell's equations} \\
\Rightarrow \quad \nabla \times B = 0 \quad \Rightarrow \quad \nabla \times j = 0 \quad \text{continuity eqn.} \\
\frac{dF}{dt} = 0 \quad \Rightarrow \quad F = dA + \alpha = 0 \\
\]

Take spacelike surface \( S \) and define the electric charge in some region \( B \) to be

\[ Q_B = \int_B \rho dV \]

As usual this electric charge is time independent if \( j \) vanishes on the boundaries.

\[ \sum = \int d\tau + \int_B^S j = \int_{B_2}^S j - \int_{B_1}^S j + \int_{S'} j_d = 0 \]

\[ \Rightarrow \quad \int_{B_1}^S j = \int_{B_2}^S j \quad \Rightarrow \quad \int_{B_1}^S j \times F = \int_{B_2}^S j \times F \]

New Maxwell's equations: 

\[ \Rightarrow \quad \Phi_B = -\frac{1}{4\pi} \int_B d\tau \times F = -\frac{1}{4\pi} \int_{S_B} \times F \]
Consider $A \cdot F \neq 0$ at $s/t = t'$

\[ E_b \text{ has induced metric } h_{\nu \rho} \]

Electric charge: $Q = \frac{1}{4\pi} \int_{S_\infty} \mathbf{E} \cdot \mathbf{d} \mathbf{A}$

Magnetic charge: $P = \frac{1}{4\pi} \int_{S_\infty} \mathbf{B} \cdot \mathbf{d} \mathbf{A}$

Note $A_t = \frac{Q}{r}$, $A_\perp = 0$(cosot etc.)

Show this agrees with $\mathbf{E} \cdot \mathbf{B}$.

Note $\mathbf{E} \cdot \mathbf{B}$ to show that $J = \mathbf{0}$, eg take $\mathbf{t}$ const curl $\mathbf{A}$.

\[ \Theta \]

\[ \]

\[ \]

\[ \]

\[ \]

Before discussing k"aher integrals, let us first consider conserved shear tensor

\[ \partial_{\mu} T^{\mu\nu} = 0 \]

$T^{\mu\nu}$ could be matter or RHS of Einstein eqns.

If $s/t$ time was stationary

3. Killing vector $K^\mu$ (timelike at $\infty$)

Then $\partial_{\mu} J^\mu = T^{\mu\nu} K_\nu$

\[ \partial_{\mu} J^\mu = -\partial_{\mu} T^{\mu\nu} K_\nu - T^{\mu\nu} \partial_{\mu} K_\nu = 0 \]

and can redefine $K^\mu$. 
then can define \( E(\Sigma) = -\int_\Sigma \mathbf{A} \cdot \mathbf{J} \)

d.e.g. \( E_2 - E_1 \) by \( \mathbf{J} \) vanishes on edge of cylinder.

We cannot express this \( \mathbf{J} \) as \( d \mathbf{w}_2 \) and write \( E(\Sigma) \) as an integral on \( \partial \Sigma \).

Physical reason: grav field has energy too.

Kemer integrals

Killing Vector \( K^\mu \): \( \partial_\mu (K^\mu) = 0 \Rightarrow \)

\[ \Rightarrow \partial_\mu K^\nu = \partial_\mu K^\mu = \frac{1}{2} (\mathbf{g}(K))_{\mu\nu} \]

Also

\[ \partial_\nu (\partial_\mu K^\nu) = R_{\mu\nu} K^\nu \] \hspace{1cm} (prob sheet 1)

\[ \Rightarrow \nabla_\nu (d_k K)_{\mu\nu} \]

\[ \Rightarrow \nabla_\nu (\gamma \mathbf{k})_{\mu\nu} = R_{\mu\nu} K^\nu \]

\[ \Rightarrow (d_k \mathbf{k})_{\mu\nu} = -2 R_{\mu\nu} K^\nu \]

\[ \Rightarrow -2 \left( \int \mathbf{T}_{\mu\nu} - \frac{1}{2} \mathbf{g} \mathbf{T} \right) K^\nu \]

\[ = -8 \pi \hat{J}_\mu \]

Here \( \hat{J}_\mu = -2 \left( \mathbf{T}_{\mu\nu} - \frac{1}{2} \mathbf{g} \mathbf{T} \right) K^\nu \)

\[ \Rightarrow \mathbf{d} \times \mathbf{d} \mathbf{k} = 8 \pi \mathbf{J} \]

Here \( T_{\mu\nu} \) in LHS of Einst eq's.

\[ \Rightarrow \mathbf{d} \times \mathbf{d} \mathbf{J} = 0 \]
Schwarzschild

\[ k = \delta_\tau \Rightarrow k^\tau = 1 \]
\[ k_\mu = \delta^\nu_\mu = \delta^\nu t \]

\[ \Rightarrow \hat{k} = \left(1 - \frac{2M}{r}\right) dt \]

\[ (\hat{k})_t = \frac{-2M}{r} \quad (\hat{k})_r = \frac{-2M}{r^2} \]

\[ (\hat{k})^\nu = \frac{1}{2} \hat{k}^{\alpha \beta} (\hat{k})_{\nu \alpha \beta} \]

\[ = \sqrt{-\hat{g}} \quad \hat{g}^{\nu \alpha} (\hat{k})_{\nu \alpha} \]

\[ (T \hat{k})_t = \frac{\Theta t + \phi}{\sqrt{-\hat{g}}} \quad (T \hat{k})_t \quad (\hat{m}_4 / r^2) \]

\[ \Rightarrow (T \hat{k}) = \frac{2M \sin \theta \cos \phi}{r^2} \]

\[ \Rightarrow \quad M_{\text{grav}} = \frac{1}{8\pi} \int_{S_{\infty}} T \hat{k} \]

\[ = \frac{1}{8\pi} \int_{S_{\infty}} -2M \sin \theta \cos \phi d\Omega d\phi \]

\[ = \frac{M}{4\pi} \int_{S_{\infty}} \sin \theta d\theta d\phi \]

\[ = M / \]
Define Komar integral \( \Phi_k(z) = \frac{c}{8\pi} \int \frac{\mathbf{x} \cdot d\mathbf{k}}{\mathbf{x} \cdot \mathbf{k}} \) for convenient constant \( c \).

Defn.: A. Flat spacetime, stationary, define the Komar mass (the total energy of the spacetime) to be

\[
M_{\text{Komar}} = -\frac{1}{8\pi} \int_{\mathbf{x}}^{\mathbf{x}'} \mathbf{x} \cdot d\mathbf{k}
\]

Defn. A. flat, axisymmetric, define Komar angular momentum as

\[
J_{\text{Komar}} = \frac{1}{16\pi} \int_{\mathbf{x}}^{\mathbf{x}'} \mathbf{x} \times d\mathbf{m}
\]

\[\text{Ex:}\]

\[
M_{\text{Komar}} = M, \quad J_{\text{Komar}} = J \quad \text{for Kerr} \quad m = \frac{\mathbf{z}}{r} + \frac{\mathbf{d}\phi}{r^2}
\]

Comment:

For asymptotic flat spacetimes, w/o K. vectors, can still define total energy using the fact that \( J \) is an asymptotic Killing vector \( dt \) as \( r \to \infty \) (in A.F. coords) and this is equivalent to "ADM energy".
Energy Conditions

In relativity we only want to consider "physical" matter which means $T_{\mu\nu}$ should satisfy certain conditions.

Consider an observer with 4-velocity $U^\mu$ would measure $E - m c^2$ current $\mathbf{J} = -T^\mu_{\ \nu} U^\nu$. We would expect physical matter not to move faster than $c$, so we want $J^2 < 0$.

**Dominant Energy Condition**: $-T^\mu_{\ \nu} U^\nu$ is a future directed causal vector (or zero) for all future directed timelike vectors $U^\mu$.

For massless scalar $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial_\phi)^2$

$$J^\nu = -T^\mu_{\ \nu} U^\nu = - (U^\mu \partial_\mu \phi) \partial_\nu \phi + \frac{1}{2} \partial_\nu (\partial_\phi)^2$$

$V$ timelike: $J^2 = \frac{1}{4} V^2 \left[ (\partial_\phi U)^2 \right] \leq 0 \Rightarrow J^\nu$ causal or 0.

Also $V \cdot J = - (V_\mu \partial_\mu \phi)^2 + \frac{1}{2} V^2 (\partial_\phi)^2$

$$\leq 0$$

This vector is orthogonal to $V^\mu$ as spacelike or $0 = \text{norm} \geq 0$.

$\Rightarrow V \cdot J \leq 0$.

$\Rightarrow J^\mu$ future directed (or zero).
Less restrictive condition, (only energy density measure by observers as true)

Weak energy condition: \( T_{\mu \nu} V^\mu V^\nu > 0 \) for any causal vector \( V^\mu \)

A special case is

Null energy condition: \( T_{\mu \nu} V^\mu V^\nu > 0 \) for any null \( V^\mu \)

- Dominant \( \Rightarrow \) weak \( \Rightarrow \) null

Another condition:

Strong energy condition: \( (T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T) V^\mu V^\nu > 0 \) for all causal \( V^\mu \)

The statement Einstein's eqs \( \Rightarrow \) \( R_{\mu \nu} V^\mu V^\nu > 0 \) as "gravity is attractive".

Note:
- Strong \# any others
- Strong needed for some singularity theorems
- Dominant most important physically
- E.g. strong violated by a true cosmological constant.
The ADM mass of an asymptotically flat spacetime satisfying Einstein's equations has $M=0$ if $M=0$ only for Minkowski with $T_{\mu\nu}=0$, provided:

1) The spacetime satisfies dominant energy condition.

2) A non-singular Cauchy surface (otherwise $M<0$ Schwarzschild would be counterexample).

3) Some other technical assumptions.
Hypersurfaces

Let \( S^m \) be smooth fn.
Consider family of hypersurfaces \( S = \text{const} \)

the one-form \( ds \) is normal to \( S \)

\[
T \cdot ds = T^m \frac{\partial S}{\partial x^m} = 0
\]

since \( S = \text{constant} \) on \( S \)

(eg \( T^m = x^m \) then \( T^m \cdot ds = x^m \frac{\partial S}{\partial x^m} = 0 \) on \( S \))

Write \( n^m = \frac{T^m}{\sqrt{-g}} \) for some fn \( f \)

Hypersurfaces (spacelike)
- choose \( f \) such that \( n^2 = +1 \)
- timelike
- null \( n^2 = 0 \)

Spacelike \& timelike cases:

Induced metric \( h^m_n = g^m_n + n^m n_n \)

- Note \( h_{mn} \) is degenerate on full tangent space at \( p \) \& on \( S \)
  but non-degenerate on subspace of vectors tangent to curves in \( S \) (ie. satisfying \( n^m \frac{\partial}{\partial x^m} = 0 \))
- Note \( h^{mn} n^m = h^{mn} n_n = 0 \) so \( h \) is a projection.

eg Minkowski: \( (t, x) \)

\[
\begin{pmatrix}
 1 & 0 \\
 0 & -1
\end{pmatrix}
\]

\[
h_{mn} = \begin{pmatrix}
 1 & 0 \\
 0 & -1
\end{pmatrix}
\]

\[
h_{uv} = \begin{pmatrix}
 1 & 0 \\
 0 & -1
\end{pmatrix}
\]
Aside: The Hamiltonian formulation of GR foliates spacetime with spacelike surfaces $\Sigma$ and uses how an "configuration space" of the one finds extrinsic curvature $\kappa_{\nu} \equiv \nabla_{\nu} N$, where $N$ is

The canonical MB can be built from this

data. Hamiltonian $= NH + NH^\#$ constant $H = 0$

We are particularly interested in the case where one of the $\Sigma$ is null. Will write normal vector as $\nu$ and null hypersurface $N$.

**Null Hypersurfaces**

- With normal $\nu$

  e.g. Schwarzschild in ingoing E-F.

  $S = S(t) = r - 2M$

  $\ell^\nu = f g^{\nu\sigma} \frac{\partial S}{\partial r} + f \frac{\partial}{\partial r} f$\frac{\partial S}{\partial r} + \ldots = f(r-2M)$

  $\ell^\nu = f g^{\nu\sigma} \frac{\partial S}{\partial r} + \ldots = f$

  $\ell = f \left[ \frac{1}{1 - \frac{2M}{r}} \right] dr + dx^a$

  **normal** to $N$

  $\Phi e^{\nu} = g_{\nu\mu} e^{\mu} = f^2 \left( 1 - \frac{2M}{r} \right)$

  $= g_{\nu\mu} \ell^{\mu} + 2\mu \nu e^{\mu}$

  $\Rightarrow r = 2M \quad \ell$ is null ($r > 2M \ell$ is like $r < 2M \ell$ is like $\Phi e^{\nu}$

  $\Phi e^{\nu} \bigg|_{r = 2M} = 0$
A vector \( t \) tangent to \( N \) is \( t = 0 \) but for nice \( N \), since \( \xi^2 = 0 \) we have \( \xi^2 \) is tangent to \( N \)!

\[
\xi^2 = \frac{d^2}{dt^2} \quad \text{for some } x^m / C N
\]

Claim: \( x^m / N \) are geodesic.

\[
P_{\xi} = e^m D_n e^p = e^m D_n (f_{\xi} e^p) = e^m D_n (f_{\xi} e^p)
\]

\[
= (e^m D_n f) e^p + f e^m D_n e^p
\]

\[
= [(e^m D_n f)^2 + f e^m D_n e^p]
\]

\[
= (f_{\xi} e^p)^2 e^2 + \frac{1}{2} \partial_p (e^2)
\]

Now \( \xi^2 = 0 \) on \( N \)

But \( \partial_p \xi^2 \neq 0 \) on \( N \)!

\( \xi^2 \) can be nonzero outside \( N \)

However, \( \xi^2 \) constant on \( N \) \( \Rightarrow \partial_p (e^2) = 0 \)

Thus:

\[
\left. e^m D_n e^p \right|_N = e^p
\]

\( x^m / n \) are geodesics (with tangent \( e^m = f^m \))

We can choose the function \( f \) so that

\[
\left. e^m D_n e^p \right|_N = 0
\]

Defn: Null geodesics \( x^m (n) \), with affine parameter \( t \) for which \( e^m = \frac{dx^m}{dt} \) are normal to null H.S. \( N \) and the "generators of \( N \)"
\[ \text{Kruskal: } U \leq \text{const} \Rightarrow \text{family of null $$t$$-$$s$$} \]

Consider \( N \) to be \( U = 0 \) (horizon)

Normal \( \ell = -\frac{r}{3} \frac{1}{e^{r/2}} \ dV \)

\[ \ell \bigg|_N = -\frac{r}{16m^2} \ dV \]

In this case \( \ell^2 = 0 \), not just on \( N \)

\[ \Rightarrow \quad \Delta \phi = 0 \quad \text{on } N \]

Hence \( \ell^2 \Delta \phi = 0 \) \( \Rightarrow \) \( \phi = \text{constant} \).
Choose \( \ell = -\frac{16m^2}{e} \)

\[ \ell \bigg|_N = \frac{2m^2}{e} \quad \text{is normal to } U = 0 \]

\( \ell \) is affine parameter for the generator of \( N \).

\[ \text{Killing Horizon} \]

Null HS \( N \) is a Killing Horizon of a Killing vector field \( \xi \) on \( N \), \( \xi \) is normal to \( N \)

Since \( \xi^2 = 0 \) on \( N \) \( \Rightarrow \) \( \Delta \phi (\xi^2) \) is normal to \( N \)

Thus \[ \nabla_{\xi} (\xi^2) = 2K \xi^2 \bigg|_N \]

\( K \) is the "surface gravity"

\[ \Rightarrow \quad \xi^2 \nabla_{\xi^2} = K \xi^2 \text{ on } N \]
The norm in absolute do 5T \cdot 5T = 1

Note: k is constant, and w/ is F an N. Then since \( K = c \) and \( k = C \),

The norm in absolute do 5T. Then, \( 3 = c \) and \( k = C \).
Example

Schwarzschild
Use ingoing EF. \((\xi, r, \theta, \phi)\)

\[ \xi = \partial_\nu \quad \text{Killing vector} \]

\[ r = 2m \quad \xi_\nu \quad N \]

\[ \xi_\nu = g_{\nu\mu} \xi^\mu = -\left(1 - \frac{2m}{r}\right) \]

\[ \partial_\mu (\xi^\mu) : \text{only non-zero} \quad \partial_r (\xi^r) = -\frac{2m}{r^2} \]

Next \[ \xi_\mu = g_{\mu\nu} \xi^\nu = g_{\mu\nu} \xi^\nu \]

\[ \begin{cases} \xi_\nu = -1 - \frac{4m}{r} \\ \xi_r = 1 \\ \xi_\theta = \xi_\phi = 0 \end{cases} \]

an \( r = r_+ : \quad \xi_\nu \big|_{r_+} = 1 \quad \text{for} \quad r = r_+ \quad \text{or otherwise} \)

\[ \partial_\mu (\xi^\mu) \big|_{r_+} = -2k \xi_\nu \big|_{r_+} \]

\[ \Rightarrow -\frac{2m}{r^2} = -2k \quad \Rightarrow \quad k = 1 \quad \frac{4m}{r_+} \]

[Can also do Christoffel \[ \xi_\mu \nabla_\nu \xi^\nu \big|_{r_+} = k \xi_\nu \big|_{r_+} \]].

Note \(\xi_\nu = \nabla_\nu \xi^\mu \)

\[ V^\mu = \xi_{\mu r} \quad \xi_\nu V^\nu = \xi_{\nu r} \]

\[ \Rightarrow \quad V^r \neq 0 \quad V_{r+} \quad \quad \Rightarrow \quad V^2 = 0 \quad \text{everywhere} \]
Use k-vector lemma
\[ \mathcal{E}^a \mathcal{E}_a \mathcal{X} = (\mathcal{Q}_a \mathcal{P}_a k) \mathcal{X} \]

\[ \Rightarrow \frac{\mathcal{X}}{\mathcal{P}} \neq 0 \text{ then } \mathcal{Q}_a \mathcal{P}_a k = 0 \]

Actually, sets with \( \mathcal{X} = 0 \) arise as limit points of orbits with \( \mathcal{X} \neq 0 \), so that we have even \( \forall \mathcal{X}_0 \in \mathcal{N} \) when \( \mathcal{X} = 0 \) on \( \mathcal{N} \).

\[ \text{bifurcate} \]

Non-degenerate Killing horizons \( (k \neq 0) \)

Suppose \( k \neq 0 \) on one orbit of \( \mathcal{X} \) in \( \mathcal{N} \), then we show that this only coincides with part of a null generator of \( \mathcal{N} \). We find that we must have what we saw for Kruskal.

Furthermore, can show for bifurcate Killing horizons \( k \neq 0 \) constant on \( \mathcal{N} \).
To prove this, first some general remarks...

Frohlich theorem:

A vector field $\mathbf{V}$ is hypersurface orthogonal if $\nabla_{\mathbf{V}} N = 0$.

Now $\mathbf{V}$ normal to $N$ hence $\nabla_{\mathbf{V}} N = 0$.

Since $\nabla_{\mathbf{V}} \mathbf{V} = \mathbf{V} \nabla_{\mathbf{V}}$,

$\nabla_{\mathbf{V}} \mathbf{V} \cdot N = 0$,

$\nabla_{\mathbf{V}} N = 0$,\n
$\left( \mathbf{V} \cdot \nabla \right) \mathbf{V} = 0$.

$x \mathbf{V} \cdot \nabla$:

$-2 \left( \mathbf{V} \cdot \nabla \right) \mathbf{V} \cdot N = \mathbf{V} \cdot \nabla \left( \mathbf{V} \cdot \nabla N \right) / N$

$\Rightarrow -2 \mathbf{V} \cdot \nabla \frac{\mathbf{V} \cdot N}{N} = \mathbf{V} \cdot \nabla \left( \mathbf{V} \cdot \nabla N \right) / N$

$\Rightarrow \text{for points } \mathbf{V} \neq 0$, $k^2 = -\frac{1}{2} \left( \mathbf{V} \cdot \nabla \right)^2$.

Also true for $\mathbf{V} = 0$ by continuity.

Let take an arbitrary tangent vector to $N$.

$\mathbf{V} \cdot \nabla \left( t \cdot \mathbf{V} \right) \frac{\mathbf{V} \cdot N}{N}$

$\Rightarrow -t \mathbf{V} \cdot \nabla \mathbf{V} \cdot N$

$\Rightarrow -t \mathbf{V} \cdot \nabla \mathbf{V} \cdot N$

Now assume bifurcate $k$ horizon.

$k^2$ constant on each orbit of $\mathbf{V}$. This constant is the same value of $k^2$ at the limit point of $\mathbf{V}$.

So, on $\mathbf{V}^2$ we have from (7) that $k \mathbf{V}^2 = 0$.

Hence constant on $\mathbf{V}^2$ and on $N$. \(\Box\)
\[
\left| \Delta_{\mathbf{k}} \right|^2 = -1
\]

\[
\text{Kruskal } \mathbf{k} = \partial_t \quad \Rightarrow \quad \mathbf{k} = \frac{1}{4M} \left( \nabla \psi - \nabla \psi \frac{\mathbf{r}}{r} \right)
\]

\[
\psi \rightarrow k = f \quad \text{with} \quad \nabla \mathbf{r} \cdot \mathbf{e}_r = 0
\]

\[
\begin{pmatrix}
\psi \\
\mathbf{r} \\
\mathbf{e}_r
\end{pmatrix}
\]

\[
= k_{\mathbf{r}} \mathbf{e}_r
\]

\[
= \frac{k}{r} \mathbf{r}
\]

\[
\Rightarrow \quad \mathbf{k} = k_{\mathbf{r}} \mathbf{e}_r
\]

On \( \psi = 0 \)

\[
k = \frac{\psi}{4M} \quad \text{and} \quad \mathbf{e}_r = \frac{\partial \psi}{\partial r}
\]

\[
f = \frac{\psi}{4M}
\]

\[
\Rightarrow \quad \mathbf{k} = \frac{\psi}{4M} \frac{\partial \psi}{\partial r} \ln \left( \frac{\psi}{4M} \right) = -\frac{1}{4M}
\]

On \( \psi = 0 \)

\[
k = -\frac{1}{4M}
\]

But \( k^2 = \frac{1}{16M^2} \) is constant.

Degenerate horizons \( k = 0 \) -- \( \psi = \text{constant} \) -- No bifurcate \( \Sigma \).