Preliminaries - eg sec wald chapter 2

Let \( M \) be a manifold. (Key idea is that of a tangent vector - directional derivative)

Let \( \mathcal{F} = \{ f: M \to \mathbb{R}, \text{with } C^\infty \} \)

a) Vector \( V \) at a point \( p \in M \) is a map \( V: \mathcal{F} \to \mathbb{R} \) with \( f \to V(f) \) satisfying:
   i) Linearity \( V(a f + b g) = a V(f) + b V(g) \) for \( a, b \in \mathbb{R} \).
   ii) Leibnitz \( V(f g) = f(p) V(g) + g(p) V(f) \).

In local coordinates we write:

\[
V = V^m \frac{\partial}{\partial x^m} \quad \text{and} \quad V(f) = V^m \frac{\partial f}{\partial x^m}(p)
\]

where \( V^m \) are the components of \( V \).

b) Vector field on \( M \)

Specification of a vector at each point on \( M \) \( V(p) \)

Vector field is smooth if \( V(p)(f) \) is smooth

\( \Rightarrow \) \( V^m(p) \) is smooth

Useful fact: For a vector field \( V \) in local coordinates such that \( V = \frac{\partial}{\partial x^1} \) i.e. \( V = (V^1, 0, 0, \ldots) \).
c) Smooth curve $\gamma : \mathbb{R} \to M$
\[ \gamma (\lambda) \]

A point $p$ on $\gamma$ can specify a vector $V$
\[ V(f) = \frac{d}{d\lambda} \left( f \circ \gamma \right) \bigg|_{\gamma^{-1}(p)} \]

in coords:
\[ V(f) = \frac{\partial f}{\partial x^\mu} \frac{dx^\mu}{d\lambda} \Rightarrow V^\mu = \frac{dx^\mu}{d\lambda} = \dot{x}^\mu \]

d) Given a vector field $V$ can construct the "integral curves" with properties that one and only one curve passes through each pt $p$.
and the tangent vector to curve at $p$ is $V|_p$.

Useful fact: for a vector field $V$ can choose local coordinates so that
\[ V = \frac{\partial}{\partial x^1} \]

Useful fact: for a vector field $V$ in local coords, such that
\[ V = \frac{\partial}{\partial x^1} \quad \text{i.e.} \quad V^1 = 1, \quad 0, \ldots \]
c) Lie derivative $L_V W$.

Let $T$ be a tensor of type $(c,s)$ (in components $T^{\mu_1 \ldots \mu_c}_{\nu_1 \ldots \nu_s}$).

- $L_V T$ is of type $(c,s)$.

**Idea:** take derivative along integral curves of $V$ (see Wald).

**Ex:** $\mathcal{L}_V (g^{\mu \nu}) = g^{\mu \nu} (\partial_\mu V^\nu - \partial_\nu V^\mu)$

Let's see it in action:

i) Acting on functions: $L_V (f) = V^\mu \partial_\mu f$.

ii) Acting on vectors: $L_V W = [V, W]$ where $[V, W](f) = V(W(f)) - W(V(f))$.

**Ex:** $(L_V W)^\mu = V^\nu \partial_\mu W^\nu - W^\nu \partial_\mu V^\nu$.

**Properties**

- $L_V$ commutes with contraction.
- $L_V (S\circ T) = (L_V S) \circ T + S \circ (L_V T)$ (Helmholtz).
  
  eg: $A_{\nu V} = S_{\mu} T^\nu$ then $(L_V A)_{\mu \nu} = (L_V S)_{\mu} T^\nu + S_{\mu} (L_V T)^\nu$.

This is in fact sufficient to define on all tensors.
\[ L^g_{\mu} T^\nu = V^g \partial^\mu T^\nu + T^\nu \partial^\mu V^g \]

\[ L^g_{\mu} A^\nu = V^g \partial^\mu A^\nu + A^\nu \partial^\mu V^g + A^\mu \partial^\nu V^g \]

**Now**, assume we have a metric on the manifold \( M \)

Then, we have a Levi-Civita connection \( \nabla \)

Covariant derivative \( \nabla \)

\[ \nabla^\mu = \partial^\mu + \Gamma^\nu_{\mu} V^\nu \]

**Fact**: In component expressions for \( L^g_{\mu} T \)
we can replace partial derivatives by covariant derivatives.

\[ L^g_{\mu} f = \nabla^\mu f \]

eg. \( (L^g_{\mu} W)^\nu = V^\mu \nabla^\nu W^\nu = W^\nu \nabla^\nu V^\mu \)

Consider

\[ (L^g_{\mu} g)^{\alpha \beta} = V^g \partial^\mu g^{\alpha \beta} + g^{\nu \beta} \partial^\mu V^\nu + g^{\mu \beta} \partial^\nu V^\nu \]

\[ = \partial^\mu V^\nu + \partial^\nu V^\mu \]

\[ = 2 \partial^\mu (V^\nu) \]

**Killing Vector**: \[ L^g_{\mu} g = 0 \] \( \Rightarrow \partial^\mu V^\nu = 0 \)

Also, locally we can choose co-ords \( \xi^i \)

\[ V = \frac{\partial}{\partial \xi^1} \]

In these co-ords \( L^g_{\mu} g = 0 \) \( \Rightarrow \frac{\partial g_{\mu \nu}}{\partial \xi^1} = 0 \)
\[ \overline{A}^{\mu \nu} = \frac{1}{2}(A^{\mu \nu} + A^{\nu \mu}) - \text{symmetrisation} \]
\[ \overline{A^{[\mu \nu]} = \frac{1}{2}(A^{\mu \nu} - A^{\nu \mu}) - \text{antisymmetrisation} \]

\[
\begin{cases}
\text{Suppose } S^{\mu \nu} \text{ is a symmetric tensor i.e. } S^{\mu \nu} = S^{\nu \mu} \\
\text{then } S^{\mu \nu} T^{\mu \nu} = S^{\mu \nu} T^{\nu \mu} \\
\text{Suppose } A^{\mu \nu} \text{ is an antisymmetric tensor i.e} \\
A^{\mu \nu} = -A^{\nu \mu} \\
\text{Then } A^{\mu \nu} T^{\mu \nu} = 0 \text{ for all } T^{\mu \nu}
\end{cases}
\]

\[
\begin{cases}
\text{If } S^{\mu \nu} \text{ is symmetric then } S^{\mu \nu} = S^{\nu \mu} \\
S^{[\mu \nu]} = 0 \\
\text{If } A^{\mu \nu} \text{ is antisymmetric then} \\
A^{\mu \nu} = 0 \\
A^{[\mu \nu]} = A^{\mu \nu}
\end{cases}
\]

\[ \text{If } S^{\mu \nu} \text{ is symmetric & } A^{\mu \nu} \text{ is antisymmetric then } S^{\mu \nu} A^{\mu \nu} = S^{\mu \nu} A^{\mu \nu} \]

* Can define symmetrisation in some indices
* Can
\[ T^{\mu\nu} = \frac{1}{3} \left( T_{\mu\nu} + T_{\nu\mu} + \frac{\Phi}{G} \right) + T_{\mu\nu} + T_{\nu\mu} + T_{\nu\mu} + T_{\nu\mu} \]

\[ T^{\mu\nu} = \frac{1}{3} \left( T_{\mu\nu} - T_{\nu\mu} \right) + T_{\mu\nu} - T_{\nu\mu} + T_{\mu\nu} - T_{\nu\mu} \]
killing vectors "generate isometries" under

\textbf{Units} \quad c = \hbar = 1 \quad mostly

\underline{Geodesic motion of test particles}

spacetime \((M, g)\), \((g\, \text{ Lorentzian})\)

particle of mass \(m\) moving on a curve \(\gamma\)

w. parameter \(\lambda\) \(\text{f}r\text{m} \, A \to B\)

use coordinates \(x^\mu(\lambda)\)

Action

\[ I = m \int_{\gamma_A}^{\gamma_B} d\tau \]

where \(\tau\) is proper time

\[ \int x^\mu(\lambda) = m \int_{\gamma_A}^{\gamma_B} \left\{ - x^\mu \dot{x}^\nu \Gamma_{\nu \mu} (x) \right\} d\tau \quad \text{or} \quad d\tau^2 = - dx^\mu dx^\nu \Gamma_{\mu \nu} \]

Geodesics \(\Leftrightarrow \delta I = 0 \quad \text{for variations } \omega, \delta x(\lambda) = dx(\lambda) \quad \text{with } m \neq 0\)

Convenient to use an alternative action by introducing a new field object - the "einbein" - \(\xi(\lambda) > 0\)

\[ \hat{I}(\xi(\lambda), \eta(\lambda)) = \frac{1}{\xi} \int_{\gamma_A}^{\gamma_B} d\tau \left[ \xi^{-1} x^\mu \dot{x}^\nu \Gamma_{\nu \mu} - m^2 \xi \right] \]

\[ \text{This gives equivalent equations of motion} \]

\(\text{when } m \neq 0\)

Proof

\[ \frac{\partial \hat{I}}{\partial \xi} = 0 \quad \Rightarrow \quad \frac{\delta \hat{I}}{\delta x} = 0 \]

\[ \Rightarrow \quad \xi = \frac{1}{m} \int -x^\mu \dot{x}^\nu \Gamma_{\nu \mu} d\tau = \frac{1}{m} \frac{dx}{d\lambda} \quad \Rightarrow \quad \xi \equiv e \left[ x \right] \]
Also have
\[ \nabla [x, e(x)] = -\nabla [x] \nabla \gamma \Rightarrow \]
\[ \frac{\delta \nabla}{\delta x(\lambda)} = \frac{\delta \gamma}{\delta x(\lambda)} e(\lambda) \]
\[ \frac{\delta \nabla}{\delta x(\lambda)} e(\lambda) = \frac{\delta \gamma}{\delta x(\lambda)} \]
So \[ \frac{\delta \nabla}{\delta x} = 0 \Rightarrow \frac{\delta \gamma}{\delta x} |_{e(0)} = 0 \]
So geodesic \[ \frac{\delta \gamma}{\delta x} |_{e(0)} = 0 \Rightarrow e = e(x) \]

**Exercise:**
\[ \begin{cases} \frac{D}{d\lambda} ^\gamma x^m = x^m + \rho^m_{\lambda k} x^k x^l f^l = (e^{-1} e^* \dot{x}) \dot{x}^m \\ e = \frac{1}{m} \frac{d\gamma}{d\lambda} \end{cases} \]

The freedom in choice of \( \lambda \) is freedom of choice of \( x \)
To get an affinely parameterised geodesic
we choose \[ \dot{e} = 0 \Rightarrow \lambda = a \tau + b \]
\[ + \frac{D}{d\lambda} x^m = 0 \]
Helpful to recall that a vector \( V^m \) is said to lie "parallelly transported along the curve" if with tangent vector \( \frac{dx}{d\lambda} \)
\[ e^\lambda V^m = f V^m \]
\[ \Rightarrow D.A V^m = f V^m \]

Thus, a geodesic has a tangent vector which is is a curve that
Parallely transported along it.

Note: cannot use \( I[x] \) when \( m=0 \), but can still use \( \mathcal{I}[x, e] \), so \( \mathcal{I} \) is more general.

**Summary**

Affinely parametrised geodesics:

\[
\frac{D}{dt} x^i = 0
\]

\[
\int ds^2 = 0 \quad \text{"null" \ (m=0)}
\]

\[
\int ds^2 = -dt^2 \quad \text{\( \pm \) \ (m \neq 0), \ "timelike"}
\]

**Killing vectors & conservation laws**

Let \( K^m \) be a Killing vector.

Consider \( x^m \to x^m + \epsilon K^m \) \( \epsilon \) infinitesimal.

\[
\delta I = \delta [x + \epsilon k, e] - \delta [x, e] \quad \downarrow \text{exercise!}
\]

\[
= 3 \int d\Sigma \ e^{-1} \dot{x}^i \dot{x}^j K_{ij} \quad \downarrow \text{exercise!}
\]

\[
= 3 \int d\Sigma \ e^{-1} \dot{x}^i \dot{x}^j K_{ij} \big|_{g_{ij}}
\]

\[
= 0
\]

Noether's theorem \( \Rightarrow \) a conserved charge \( Q \)

\[
Q = \frac{\partial L}{\partial \dot{x}^m} \bigg|_{x^m}
\]

where \( \frac{\partial L}{\partial \dot{x}^m} = \frac{\partial}{\partial \dot{x}^m} = e^{-1} \dot{x}^i g_{ij} \)
Spherical symmetric collapse $\propto$ Schwarzschild black hole

Consider a ball of pressure free dust that undergoes gravitational collapse. Idealised model of a star.

Recall Birkhoff's Theorem

Spherical symmetry $\Rightarrow$ vacuum Einstein eqns

$\Rightarrow$ outside the star have Schwarzschild

$$ds^2 = -(1-2m)dt^2 + (1-2m)^{-1}dr^2 + r^2 d\Omega^2$$

$$d\Omega = d\theta + sin^2 \theta \, d\phi$$

Killing vectors

$x^i \in \mathfrak{s}^2 \Rightarrow$ obvious

$\exists$ 2 more, $\mathfrak{so}_2(\mathbb{R})$ algebra under $[,]$

so $\mathfrak{so}(3) \subset \mathfrak{X}$

\* It is another killing vector

It do timelike (i.e. $\square K^I \leq 0$) for $r > 2M$

$\Rightarrow$ stationary

In addition $t \rightarrow -t$ a discrete symmetry

$\Rightarrow$ static

Interestingly that spherical symmetry $\Rightarrow$ static.
Outside star → Schwarzschild
Surface of star at $r = R(t)$ is also Schwarzschild by continuity.

A point on the surface follows a radial time-like geodesic

$$\theta = \phi = 0$$

$$\chi^m(t) = (t(t), r(t), \theta_0, \phi_0)$$

$$\tau - \text{proper time}$$

Want $R(t) \equiv r(t(t))$

$$d\tau^2 = - ds^2 \implies 1 = \int \left(1 - \frac{2M}{R} - \left(\frac{2M}{R}\right)^2 R^2 \right) \left(\frac{dt}{d\tau}\right)^2$$

$d\tau$ is a Killing vector

$$\xi = p_m k^m = P_+ = e^{-\varphi} g_{tt} \frac{dt}{d\tau}$$

$$\xi$$ constant

$\varepsilon \equiv - g_{tt} \frac{dt}{d\tau} = \left(1 - \frac{2M}{R}\right) \frac{dt}{d\tau}$ is constant on the geodesic

$$\text{energy} = \text{mass}$$

Substitute, rearrange

$$e^{\varphi} \left(\frac{dt}{d\tau}\right)^2 \left(1 - \frac{2M}{R}\right)^2 \left(\frac{2M}{R} - 1 + \varepsilon^2\right)$$

Take $0 < \varepsilon < 1$ to have gravitationally bound orbits.
\[ \frac{dR}{dt} \]

\[ R = 0 \text{ at } R_{\text{max}}, \text{ particle starts from rest here} \]

\[ R \text{ then decreases as approaches } 2M \text{ at } t \to \infty \]

\[ t_{2M} = \int_{0}^{2M} dt = -e \int_{R_{\text{max}}}^{2M} \frac{dR}{R(\frac{2M}{R} - 1 + e^2)^{1/2}} \to \infty \]

Exercise

According to observer at asymptotic \( \infty \), they see the surface of star approach \( R \to 2M \) and stay there ("frozen star")

What about accelerating

However, for observer on surface of star, want \( R(t) \)

\[ \frac{d}{dt} = \left( \frac{dt}{dR} \right)^{-1} \frac{dR}{dt} = e^{t} (1 - 2M) \]

\[ (\frac{dR}{dt})^2 = (1 - e^2) \left( \frac{R_{\text{max}} - 1}{R} \right) \]

\[ (\frac{dR}{dt})^2 \]
Proper time $\tau$ to reach $R=0$:

$$\tau = \int_0^{\tau_0} dt \bigg|_{R=0} = -\int_{r_{\text{max}}}^{\infty} dR \left(\frac{r^3}{R_{\text{max}}^3} - 1\right)^{-1/2} \left(\frac{R_{\text{max}}}{R}\right)^{3/2}$$

$$= \frac{\pi M}{(\pi^2)^{3/2}}$$

*Remark from Reader*

This is correct but have been a bit rough: coords at $r=2m$ breakdown.

**Introduction**

Introduce new coords: examine spacetime of spherical collapse *and* the geometry of Schwarzschild. *But* at the same time.

**Eddington-Finkelstein coordinates (Not written down by Eddington or Finkelstein but Penrose!)**

Idea: replace $t$ by a coordinate $u$ adapted to outgoing radial null geodesics.

On such a geodesic:

$$d\tau^2 = 0 \Rightarrow dt^2 = (1 - 2M/r)^{-2} dr^2 = dr^2$$

Take $r^*_+ = r + 2m \ln \left(\frac{r-2M}{2M}\right)$

$$2M < r < \infty \quad \text{and} \quad -\infty < r^*_+ < \infty$$

on radial null geodesic:

$$d\tau(r^*_+) = 0$$

$U = t - r^*_+$ lines of $U$ constant $\Rightarrow$ outgoing null geodesic

$V = t + r^*_+$ $U$ "" $\Rightarrow$ ingoing ""
\[
(t, \varphi, \theta, \phi) \rightarrow (\varphi, r, \theta, \phi) \quad \text{Ingoing E-F coards}
\]

\[
ds^2 = -(1 - \frac{2M}{r}) \, dt^2 + 2 \, dt \, dr + r^2 \, (d\theta^2 + \sin^2 \theta \, d\phi^2)
\]

**Note:** Initially metric only defined for \( r > 2M \)

But it can now be analytically continued to \( r > 0 \)

In particular in E-F coards the metric is non-singular at \( r = 2M \):

\[
g_{\mu \nu} \bigg|_{r=2M} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 4M^2 & 1 \\
0 & 0 & 1 & \sin^2 \theta
\end{pmatrix}
\]

\[
\det g \bigg|_{r=2M} = 0.
\]

**Theorem:** The star collapses through \( r = 2M \) down to \( r = 0 \)

Recall \( h = \pm 1 \) for \( r > 2M \)

\[
h = \frac{d}{dt} \frac{d}{dt} \frac{d}{dx^1}
\]

\[
|| \text{Kill} ||^2 = g_{\mu \nu} \text{Kill}^\mu \text{Kill}^\nu = g_{\mu \nu} \text{Kill}^\mu \text{Kill}^\nu = -\left( \frac{1 - \frac{2M}{r}}{r} \right)
\]

\( \text{Kill} \) is timelike \( r > 2M \)

null \( r = 2M \)

Space-like \( r < 2M \)
E-F diagram for Schwartzschild

\[ t^* = \nu - r \]

- Ingoing radial null geodesic: \( \nu = \text{const} \)
- Outgoing \( \nu = t - t^* = \text{const} \)

In EIF coordinates:

\[ t^2 - 2r - 4M \ln \left| \frac{r-2M}{2M} \right| = \text{const} \]

Radial timelike curves have tangent vectors that lie inside the light cone at any point.

For \( r > 2M \) outgoing radial null geodesics have increasing \( r \).

For \( r < 2M \) both ingoing and "outgoing" radial null geodesics have decreasing \( r \).

Notice \( r = 2M \) is a null surface, (even though \( r = \text{const} \))
No radial timelike curves come from $r < 2m$ to a point with $r > 2m$.
In fact, can relax the radial condition and conclude no signal emerges from $r < 2m$ to $r = \infty$: we have a black hole.

E+F diagram for spherical collapse of dust:

![Diagram of spherical collapse with labels: interior of star.]

**Definition:** A vector is **causal** if it is timelike or null.
A curve is **causal** if tangent vector is everywhere causal.

**Definition:** A spacetime is **time orientable** if it admits a time orientation – a causal vector field $\tau^a$.

Another causal vector $x^a$ is future directed if it lies in the same light cone as $\tau^a$, otherwise $x^a$ is past directed otherwise (i.e., $x^a\gamma_0 \gamma^0 \leq 0$).
Schwarzschild: For $r > 2M$, choose $k = \frac{\partial}{\partial t}$ as time orientation.

For $r < 2M$, cannot choose $\frac{\partial}{\partial t}$ since $\partial t$ is space-like.

Can choose $\pm t$, since these are null or $\partial t$ is E-F coordinates ($\mathcal{F}_t = 0$).

We need to choose sign to agree with time orientation when $r > 2M$.

\[
k \cdot \left( -\frac{\partial}{\partial r} \right) = k^n \frac{\partial}{\partial t} \partial t
= k^n \left( -\frac{\partial}{\partial r} \right) \partial t
= -g_{\nu r} \partial r
= -1
\]

Therefore, can choose in E-F coords $-\frac{\partial}{\partial r}$ for $r > 0$ as global time orientation.

\[\begin{align*}
\text{Warning: The vector } \frac{\partial}{\partial r_{\text{sch}}} + \frac{\partial}{\partial r_{\text{EF}}} + 1 \text{ holding differs coordinates constant.}
\end{align*}\]

\[
\frac{\partial^2}{\partial r_{\text{sch}}^2} = \frac{\partial}{\partial r_{\text{sch}}^m} \frac{\partial}{\partial x^m} = \frac{\partial^2}{\partial r_{\text{sch}} \partial v} + \frac{\partial}{\partial r_{\text{EF}}} \frac{\partial}{\partial v}
= \frac{\partial}{\partial r_{\text{sch}}} \frac{\partial}{\partial v} + \frac{\partial}{\partial \partial r_{\text{EF}}} \frac{\partial}{\partial v}
\]

Let $X^\mu(\lambda)$ a future directed causal curve.
Assume $r(\lambda_0) \leq 2m$ then $r(\lambda) \leq 2m \quad \forall \lambda > \lambda_0$.

Proof:
Tangent vector $V^\mu = \frac{dx^\mu}{d\lambda}$

Future directed $\Rightarrow V^\mu \cdot \left( \frac{d\lambda}{d\xi} \right) \leq 0$

$\Leftrightarrow -V^\mu g_{\mu\nu} \frac{d\lambda}{d\xi} \leq 0$

$\Leftrightarrow -V^\mu \leq 0$

$\Rightarrow \frac{ds}{d\lambda} \geq 0$

Also have:

$$V^2 = -\left( 1 + \frac{2M}{r} \right) \left( \frac{dr}{d\lambda} \right)^2 + \frac{2}{r^2} \frac{d\theta}{d\lambda} \frac{dr}{d\lambda} + r^2 \left( \frac{d\phi}{d\lambda} \right)^2 + \sin^2 \theta \frac{d\chi}{d\lambda}$$

$\Rightarrow$

$$-2 \frac{d\omega}{d\lambda} \frac{dr}{d\lambda} = -V^2 + \left( \frac{2M}{r} - 1 \right) \left( \frac{d\omega}{d\lambda} \right)^2 + r^2 \frac{d\phi}{d\lambda} \frac{d\omega}{d\lambda} + \sin^2 \theta \frac{d\chi}{d\lambda} \frac{d\omega}{d\lambda}$$

$\geq 0$ \quad $\forall r \leq 2m$ \quad $\Box$

$\Rightarrow \frac{dr}{d\lambda} \leq 0 \quad \text{for} \quad r \leq 2m$

If $r \leq 2m$, $\frac{dr}{d\lambda} < 0$ \quad (if not, $\Box$ \quad with $\frac{dr}{d\lambda} = 0 \Rightarrow$

$\frac{d\theta}{d\lambda} = \frac{d\phi}{d\lambda} = 0 \quad \text{where} \quad V = 0$

$a contradiction$.

Therefore, if $r(\lambda_0) < 2m \Rightarrow r(\lambda) < 2m \quad \forall \lambda > \lambda_0$.

$r(\lambda_0) = 2m$ case can also be shown (as indicated p.16).
Consider any particle (not nec geodetic) in region $r < 2m$.

Use Schw. coords:

$$\frac{dt^2}{= -\left(\frac{2m}{r}\right)dt^2 + \left(\frac{2m}{r}\right)^2 dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

The proper time to reach $r = 0$ is maximised by

$$\frac{ds}{dt} = \frac{dr}{dt} = \frac{dt}{dt} = 0$$

$\Rightarrow$

$$\frac{dr}{dt} = -\left(\frac{2m}{r}\right)^{1/2} dr$$

$\Rightarrow$

$$\int_0^r \frac{dr}{\left(\frac{2m}{r}\right)^{1/2}} = \int_{2m}^0 \left(\frac{2m}{r}\right)^{1/2} = \pi m$$

$\approx 10^{-5} \left(\frac{M}{M_\odot}\right) \text{ seconds}$.

Note this is achieved for a geodetic:

$$\frac{dr}{dt} = -\left(\frac{2m}{r}\right)^{1/2}$$

$$\frac{d}{dt} \left(\frac{r}{m}\right) = \left(\frac{2m}{r}\right)^{1/2} \frac{dr}{dt} = -\frac{m}{r}$$

$$\vec{r}_{rr} \left(\frac{dr}{dt}\right)^2 = \frac{1}{2} \frac{d}{dt} \left(\frac{1}{r^2 (2m)} \right)^2$$

$$-\frac{d}{dt} \left(\frac{dr}{dt}\right) + \frac{m^2}{r^2} \frac{dx^i}{dy^j} \frac{dx^j}{dy^i} = 0$$

Also note $\varepsilon = \left(1 - \frac{2m}{r}\right) \frac{dt}{dr} = 0 \quad \text{(conserved)}$,}
White Holes

What happens if we use outgoing E-F coordinates instead of ingoing?

\[ u = t - r, \quad -\infty < u < \infty \]

\[ ds^2 = -(1-\frac{2m}{r})du^2 - 2drdu + r^2d\Omega \] (Ex.)

Initially defined for \( r > 2m \)

Analytically continue to all \( r > 0 \)

The \( r < 2m \) region is not the same as the \( r < 2m \) region in ingoing E-F coordinates.

This is revealed in (outgoing) E-F diagram

This is a white hole.

It is the time reverse of the black hole

Note: (Einstein equations are time reversal invariant)
Exploding Star?

starts with a singularity!

Also: black holes are stable: small pertuptions are expected to decay.

The time reverse says that pertuptions of white holes will grow, hence are unstable.

Not physical!

Kruskal-Stekelres coordinates

Have seen there are 2 ways to extend schwarzschild metric past r=2M. To see how they are related we introduce Kruskal-Stekelres coordinates.

start at (T=0,\phi=0,\theta=\pi, r=\infty)