Focus on free, real scalar field $\phi$

Eq of motion

$$\nabla^2 \phi - m^2 \phi = 0$$

$$\begin{cases} 
\text{+ve frequency solutions} \\
\psi_p = \frac{N_p}{\sqrt{p^2 + m^2}} e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} \\
\quad \omega = \sqrt{p^2 + m^2}, \quad \mathbf{k} = \frac{\mathbf{p}}{m}
\end{cases}$$

$$\begin{cases} 
\text{-ve frequency solutions} \\
\psi_p^* = \frac{N_p}{\sqrt{p^2 + m^2}} e^{i\omega t - i\mathbf{k} \cdot \mathbf{x}} \\
\text{plane waves in an inertial frame.}
\end{cases}$$

Note: $\mathbf{k} = \mathbf{p}/m$

$$\begin{align*}
\mathbf{\nabla} \cdot \mathbf{\psi}_p &= -i \omega \mathbf{\psi}_p \\
\mathbf{\nabla} \cdot \mathbf{\psi}_p^* &= +i \omega \mathbf{\psi}_p^*
\end{align*}$$

$$\{ \psi_p, \psi_p^* \} \text{ is a complete basis. Hence we can expand}$$

$$\phi(x) = \int d^3 p \left[ a_p \mathbf{\psi}_p(x) + a_p^* \mathbf{\psi}_p^*(x) \right] \text{ real}$$

Introduce the bracket

$$(f, g) \equiv \int d^3 x \left( \frac{f^*}{\mathbf{k}} \cdot \mathbf{\nabla} + 3 \mathbf{\nabla} f \right) g / \mathbf{k} \quad \text{(by definition)}$$

$$(f, g) = (g, f)^* = - (f^*, f) + (g^*, f) + (f, g^*)$$

Choose $N_p = \frac{1}{\sqrt{2p^0 (2\pi)^3}}$

$$\begin{align*}
(\psi_p, \psi_q) &= \delta^3(\mathbf{p} - \mathbf{q}) \\
(\psi_p, \psi_q^*) &= 0 \\
(\psi_p^*, \psi_q^*) &= -\delta^3(\mathbf{p} - \mathbf{q})
\end{align*}$$

$$(\psi_p^*, \psi_q)$$ 

The def. of

$$- \left( \psi_p^*, \psi_p \right)$$

$$- \left( \psi_p^*, \psi_p^* \right)$$

$$- \left( \psi_p, \psi_q \right)$$

$$- \left( \psi_p, \psi_q^* \right)$$
Quanti\textsuperscript{a}tion

ETCR's
\[
\begin{align*}
[\mathbf{r}(x), \pi(y)] &= i\hbar \delta(x-y) \\
[\mathbf{r}(x), \phi(y)] &= 0 \\
[\pi(x), \pi(y)] &= 0 \\
\pi(x) &= \mathbf{r}(x)
\end{align*}
\]

\[\alpha_p^a \rightarrow \text{ operator with } \alpha_p^a, \alpha_p^a \]
\[(\mathbb{2}) \quad \left[ \alpha_p^a, \alpha_p^b \right] = \hbar^2 \left( \delta^2(x-y) \right) \]
\[
\left[ \alpha_p^a, \alpha_q^a \right] = 0
\]

Vacuum: \( |0\rangle \) defined via \( \alpha_p^a |0\rangle = 0 \)

1. Particle states \( \alpha_p^a |0\rangle \)
2. \( \alpha_p^a \alpha_q^a |0\rangle \) basis for \( \mathcal{H} \)

Number operator
\[ N = \int d^3r \alpha_p^a \alpha_p^a \]

One can show \( |0\rangle \) is Lorentz invariant

1. Suppose we had worked in another inertial frame \( \mathbf{x}' = \mathbf{\Lambda} \mathbf{x} \) then \( |0\rangle' = |0\rangle \).
OFT in curved spacetime

Klein–Gordon eqn  \((\Box \Phi - m^2) \Phi = 0\)

We will assume that the spacetime is globally hyperbolic & \(\Sigma\) a Cauchy surface.

Inner product  
\[ (f, g) = \int_{\Sigma} ds^\mu f^* \overline{g}^\mu \]

is complex.

Then for solutions of Klein–Gordon eqn this is independent of Cauchy surface \(\Sigma\). \(\Sigma'\)

\[ (f, g)_{\Sigma'} = \int_{\Sigma'} ds^\mu f^* \overline{g}^\mu \]

\[ = \int_{\Sigma} ds^\mu f^* \overline{g}^\mu \]

\[ = i \int_{\Sigma} ds^\mu m^2 f^* \overline{g}^\mu \]

\[ = i \int_{\Sigma} D\mu (f^* \overline{g}) \]

\[ = i \int_{\Sigma} [f^* \overline{Dg} - (\overline{Df}^*) g] \]

\[ = f^* m^2 g - (m^2 f^*) g \]

\[ = 0 \]

Fact: consider that we have a highly non-unique basis of solutions such that

\( (\psi, \psi) = 0 \)

\( (\psi^*, \psi^*) = - \delta_j^i \psi_i^* \psi_j + (\psi_i, \psi_j) \infty \)

where \(i,j\) are indices.
We can expand

\[
\langle \psi_i, \phi \rangle = a_i \langle \psi_i, \phi \rangle \quad \text{for all } \phi
\]

Can expand \( \phi = \sum_i q_i \psi_i(x) + a_i^* \psi_i(x) \)

quantize \( \rightarrow \sum_i q_i \psi_i(x) + a_i^* \psi_i(x) \)

with \( \langle a_i, a_j^* \rangle = 0 \quad \text{and} \quad \langle a_i, a_j \rangle = \delta_{ij} \)

Basis for Hilberts are \( a_i/|0\rangle = \psi_i \), \( 10 \rangle, a^*/|0\rangle, \ldots \)

Note: In general is a preferred choice of basis
satisfying \( \phi \) and hence \( A \) a preferred vacuum state!

hence\( A \) the notion of particle-like states is
natural

Consider another basis \( \psi'_i \)

\[
\psi'_i = \sum_j A_{ij} \psi_j + B_{ij} \psi_j^* \quad \psi'_i^* = \sum_j A_{ij}^* \psi_j^* + B_{ij}^* \psi_j
\]

Thus satisfies \( \dagger \) for \( \psi'_i \) provided that

\[
\begin{bmatrix}
A A^* - B B^* = I \\
A^* A - B^* B = 0
\end{bmatrix}
\]

Meeting \( \dagger \)

\[
\psi'_i = \sum_j A'_{ij} \psi'_j + B'_{ij} \psi'_j^*
\]

with \( A' = A^* + B' = -B^* \)

check.
\[ \psi_i' = \sum_j A_{ij} \left( \sum_k A_{jk} \psi_k' + B_{jk} \psi_k'' \right) \\
+ B_{ij} \left( \sum_k A_{jk} \psi_k' + B_{jk} \psi_k'' \right) \\
= \sum_{jk} \left[ (AA')_{ik} \psi_k' + (BB')_{ik} \psi_k'' \right] + \left[ (AB')_{ik} \psi_k'' + (BA')_{ik} \psi_k' \right] \\
= \sum_{ik} \left[ AA^+ + BB^+ \right]_{ik} \psi_k' \\
+ \left[ -ABT + BAT \right]_{ik} \psi_k'' \\
= \psi_i' \] 

Since \( A^+ + B^+ \) must satisfy same conds as \( A^*B \),
we have

\[
\begin{bmatrix}
AA^+ - BB^+ & = 1 \\
A^*B - BTA^* & = 0
\end{bmatrix}
\]

Note directly implied by other conds.

Use invertibility.

\[ \phi = \sum [a_i \psi_i' + a_i^+ \psi_i''] = \sum (\psi_i' + a_i^+ \psi_i') \\
\Rightarrow a_i = \sum_j (A_{ij} a_j + B_{ij} a_j^*) \\
\] 

\[ [a_i^*, a_j^+] = 0 \]

\[ [a_i^*, a_j^+] = \delta_{ij} \Rightarrow AA^+ - BB^+ = 1 \\
AB^* - BAT = 0. \]

The above change of basis, involving \( A^*B \) satisfying conds is called a "Bogoliubov transform."
(Note $\beta = 0$ have $AA^T = AA^+ = 1$ → unitary trans.)

Why doesn't this come up in $\mathbb{M}$ (or $\mathbb{C}^E$?)

For stationary space-times there is a preferred choice of vacuum. Vacuum that utilises the frequency cut to Killing vector $k$.

To see this:

1. $(\delta^\mu_{\nu} - \eta^\mu_{\nu}) L_k \phi = 0$ so $L_k \phi$ solves $\mathcal{L}$-Gordon $\phi$ does

2. $L_k$ is anti-hermitian $\langle f, L_k g \rangle = -\langle L_k f, g \rangle$

$\Rightarrow$ it has imaginary eigenvalues $\mathcal{L}$-eigenbasis

- Eigenfunctions with distinct eigenvalues are orthogonal

Can choose a basis $\{\psi_i\}$

We can choose a basis of positive frequency eigenfunctions with

$$\mathcal{L} \psi_i = -i \omega_i \psi_i \quad \text{with} \quad \omega_i > 0$$

3. Anti-hermitian $\Rightarrow$ can normalise $\langle \psi_i, \psi_j \rangle = \delta_{ij}$

also $\Rightarrow$ distinct eigenvalues orthogonal

$$\langle \psi_i^*, \psi_j \rangle = 0.$$
Particle production in non-stationary spacetime

Globally hyperbolic sandwich spacetime

\[ t > t' \quad \text{M+ stationary} \rightarrow \text{preferred basis} \ U^+(x) \]

\[ t < t' \quad \text{M- stationary} \rightarrow \text{preferred basis} \ U^-(x) \]

\[ \psi(x) \text{ solves Klein-Gordon eqn everywhere in } M \]

\[ \Phi(x) = \sum a^+_i U^+_i(x) + a^-_i U^-_i(x) \quad \text{in } M^+ \]

But we have shown

\[ U^+_i(x) = \sum A^+_{ij} U^-_j + B_{ij} U^+_j \]

\[ a^+_i = \sum A^+_i a^-_j - B^+_i a^+_i \]

Suppose we start with the vacuum state \( 10^- \)

defined by \( a^- 10^- = 0 \quad \forall i \)

What is expected number of particles \( \phi^- \) associated with late time media defined by \( N^- = a^- a^+ ? \)
$$\langle N^2 \rangle_{10^{-7}} = \langle 0 - 1 \ a^+_i a^+_j a_i a_j \rangle_{10^{-7}}$$

$$= \sum_{jk} \langle 0 - 1 \ a^-_k (B^{i k}) (-B^*_i j a^-_j) \rangle_{10^{-7}}$$

$$= \sum_{jk} \langle 0 - 1 \ a^-_k a^-_j \rangle (B^{i k} B^*_i j)$$

$$= \sum_j B_{i j} B^*_{i j} = \text{trace} (B B^*)_{i i}$$

Expected the total # of particles at late times do $\text{trace} B B^*$ which vanishes if $B = 0$. We take $B = 0$.

The time dependent grav. field results in particle production. Emphasis: this within possible because stationary in for past or future.

Comment. One would see $\langle N^2 \rangle$ particles if in a lab stationary wrt to $K$. 
Black holes from grav. collapse are not stationary, so expect particle creation. But the exterior spacetime is stationary at late times 

late time phenomenon determined by collapse? ??

But if a time dilation at $H^+$ is particle creation take as $\mathcal{H}$ only long to escape escape 

late time particle production due to $H^+$ independent of details of collapse. There is such a flux, called it turns out to be thermal. This is
due to Hawking radiation.

Consider massless scalar fields

1) spacetime is globally hyperbolic specifying data on $\mathcal{H}$-sufficient since massless.

2) For $\mathcal{H}$ can construct modes on $g-$ (for past apparent)

$$(\phi, \phi^*)$$

3) It is not a Cauchy surface but $A^+$ exists.

Define modes $$(g, g^*) \phi (h, h^*)$$

Actually I'm same

ambiguity in defining PE frequency on $H^t$. However, the results we are after don't depend on the choice.

We just require $\phi$, $h^*$ form a complete basis.
\( \phi(x) = \sum a_i \phi_i + \text{h.c.} \)
\[ = \sum b_i \phi_i + c_i \phi_i + \text{h.c.} \]

Consider a state in vacuum at early times.
Assume 3 mass particles.
Assume we are in state \( 10 > \) corresponding to vacuum at early times
\[ a_1 \phi_1 = 0 \]

Observer at late time will observe the number of particles in \( i \)th mode:

\[ \langle N_i \rangle = \langle 0 | b_i^+ \phi_i | 0 \rangle = (BB^*)_ii \]

\( \text{Note we can write} \quad g_i = \sum_j A_{ij} f_j + B_{ij} f_j \)

\( \text{Want to calculate} \quad B_{ij} \quad \text{(sure fit complete!)} \)

If we could solve KG exactly \( \phi \) Schwarzschild this would be straightforward, but this is not possible.

Instead consider: the frequency solution to KG at \( 9^+ \) and ask for its form on \( 9^- \).

\( \text{Impose} \quad b.c. \quad \text{on} \quad 9^+ \quad \text{degenerate its form on} \quad 9^- \)
Recall that \( ds^2 = \frac{1}{\chi^2} (dt - \omega dx)^2 - dr_x^2 + r^2 d\Omega^2 \).

And recall \( u = t - r_x \),
\[ v = t + r_x \]

\[ + \quad \text{EX} \quad \phi = e^{-i\omega t} R_{\omega e}(r_x) Y_{lm}(\theta, \phi) \]

The equation is
\[ (\partial_{r_x} + \omega^2) R_{\omega e}(r_x) = V_e R_{\omega e}(r_x) \]

where
\[ V_e(r_x) = \left( \frac{-2M}{r} \right) \left[ \frac{\ell(\ell + 1)}{r^2} + \frac{2M}{r^3} \right] \]

Notice:
\[ A^+ H^+ \lim_{r_x \to -\infty} V_e \to 0 \]
\[ A^+ H^+ \lim_{r_x \to +\infty} V_e \to 0 \]

Have a potential barrier.

\[ V_e \text{ at } r_x \]

\[ -\infty \quad r_x \quad +\infty \]
Near $g^\pm$ solutions are plane waves.

On $g^-$ define

$$f_{\pm}(x) = \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega t} Y_{\pm\mu}$$

Outgoing

$$f_{\pm}(x) = \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega t} Y_{\pm\mu}$$

Ingoing

On $g^+$ define

$$g_{\pm}(x) = \frac{1}{\sqrt{2\pi\omega}} e^{i\omega t} Y_{\pm\mu}$$

Interesting: $g_{\pm} \equiv 9_{\pm} f_{\pm}(x)$ at late times on $g^+$ (to get outgoing mode), will see it is related to $f_{\pm} \equiv f_{\pm}(x)$ on $g^-$.

**Subtlety:** Plane waves such as $g_{\pm}$ are delocalized — they have support everywhere on $g^+$. However, can assemble superposition to construct a local wave packet on $g^+$ localized around some wave number.

Keeping this in mind, we phrase argument in terms of $g_{\pm}$ below for simplicity.

**Key idea**

Need to trace $g_{\pm}$ the solution $f_{\pm}$ back in time in order to express it in terms of the $f_{\pm}$. 
As waves travel inwards from \( 9^+ \), decreasing values of \( r^+ \), it encounters a barrier in \( V_e \).

One part of the wave will be reflected and end up on \( 9^- \) with same \( k \), but with \( 9^- \) gives which gives different \( \phi_0 \) (this leads to a Bogoliubov coefficient \( A \)).

The other part of the wave is transmitted \( g \).

\( \text{This gives rise to a Bogoliubov coefficient } A \text{ which is not the focus.} \)

We then need to expand the transmitted wave in terms of the \( \phi_0 \) frequency modes on \( g^- \) (the \( 0+ \)).

\( 8^- \) a generator of \( 11^+ \) continues it back until it intersects \( g \).

\( \phi^0 \) can always shift to \( \phi^0 = 0 \).
for fixed \( u_0 \), the wave packet is localized around \( u_0 > 0 \) on \( \mathbb{R} \).

Note from

Field is oscillating very rapidly near \( \mathbb{R} \) all the way back to \( \mathbb{R} \). Thus can use geometric optics approx.

Write \( \Phi(x) = A(x) \exp(i S(x)) \) and assume \( i \ll 1 \)

Leading order \( i \ll 1 \)

\[ 0 \ll i \Rightarrow (iS)^2 \ll 0 \]

\[ \Rightarrow \text{surfaces of constant } S \text{ are null hypersurfaces. (hence geodesics)} \]

Consider null congruence containing these hypersurfaces also \( H^+ \) (which is \( S = \infty \))

Let us introduce \( \mathbb{R} \)

\( 2 \) so tangent geodesics \( \mathbb{R} \) introduce connecting null vector with \( L \cdot N = -1 \)

\[ \mathbb{R} \]
Outside matter know we can choose \( \ell^u = (\frac{\partial}{\partial u}) \) as affinely parametrised generator of \( H^+ \)

d hence \( N^u = c \frac{\partial}{\partial v} \) for some (constant \( c > 0 \)).

Hence, outside matter \(-3N^u\) connects to a null geodesic \( g \) with \( u = -c \).

Now, we know, from definition \( \theta = \frac{-1}{\log(-\xi)} \)

\( \Rightarrow \) at late times \( \xi \rightarrow \infty \) outgoing null geodesic with

\( u = \frac{-1}{k} \log(\xi) \)

phase of \( g \sim e^{-i\omega u} + i\omega u = \frac{i\omega \log(\xi)}{k} \).

phase every photon

At \( g \), in \( u, \phi \) coords we have \( \ell^u \sim du \)

\( d \theta = e^{i\omega u} \theta \) for constant \( D > 0 \)

\( \phi \) here

\( \theta = \frac{i\omega \log(-c \xi)}{k} \)
Thus we have

\[ g^T_w(n) = \begin{cases} 1 & w > 0 \\ \frac{iw \log(w)}{K} & w < 0 \end{cases} \]

Now want to decompose it further.

Take Fourier transform

\[ \widehat{g^T_w}(w) = \int_{-\infty}^{\infty} e^{iwn} g^T_w(n) \, dn \]

\[ = \int_{-\infty}^{0} \exp\left( iw n + \frac{iw \log(w)}{K} \right) \, dn \]

**Lemma**

\[ \widehat{g^T_w}(-w) = -\exp(-\pi w/k) \widehat{g^T_w}(w) \quad w > 0. \]

**Inverse Fourier transform**

\[ g^T_w(n) = \int_{-\infty}^{\infty} \frac{dw}{2\pi} e^{-iwn} \widehat{g^T_w}(w) \]

\[ = \int_{0}^{\infty} \frac{dw}{2\pi} e^{-iwn} g^T_w(w) \]

\[ + \int_{-\infty}^{0} \frac{dw}{2\pi} e^{iwn} g^T_w(-w) \]

\[ = \int_{0}^{\infty} \frac{dw}{2\pi} \frac{Nw}{2\pi} f(w) \widehat{g^T_w}(w) \]

\[ + \int_{-\infty}^{0} \frac{dw}{2\pi} \frac{Nw}{2\pi} f(w) \widehat{g^T_w}(-w) \]

\[ N = \text{norming factor} \]
\[ A \omega \omega' = N \omega \omega' \delta^\top \omega (\omega') \quad \omega > 0 \]
\[ B \omega \omega' = N^* \omega \omega' \delta^\top (-\omega') \]

\[ |B \omega \omega'| = \exp\left(-\frac{\pi \omega}{K}\right) / |A \omega \omega'| \quad \text{because of lemma} \]

Normalisation of \( \delta^\top \omega \):

\[ \Pi \omega = T^2 = (\delta^\top \omega, \delta^\top \omega) \]

\[ = \sum_{\omega'} \left( A \omega \omega' f_{\omega'} + B \omega \omega' f_{\omega'}^* \right) \left( A \omega \omega' f_{\omega'} + B \omega \omega' f_{\omega'}^* \right) \]

\[ = \sum_{\omega'} |A \omega \omega'|^2 - |B \omega \omega'|^2 \]

\[ = \left(e^{2\pi \omega / K} - 1\right) \sum_{\omega'} |B \omega \omega'|^2 \]

\[ = \left(e^{2\pi \omega / K} - 1\right) (BB^+)_{\omega \omega} \]

\[ (BB^+)_{\omega \omega} = \frac{\Pi \omega}{e^{2\pi \omega / K} - 1} - \text{can argue \( \Pi \omega \) is the absorption cross section for more few} \]

This is exactly the black body spectrum with \( T_H = K / h \)}
\( g_{\omega} \cdot g_{\omega} = \text{orthogonal} \)

\[ T_{\omega} = (g_{\omega}^T, g_{\omega}) \]

\[ L_{\omega} = (g_{\omega}, g_{\omega}^T) \]

\[ T_{\omega}^2 + L_{\omega}^2 = 1 \]

Now start with mode \( \omega \).

Some gets absorbed by black hole.

Some gets reflection, thus is the same as \( R_{\omega} \).

Hence \( P_{\omega} = T_{\omega}^2 \) is the absorption cross section for the mode \( \omega \) - i.e., the fraction absorbed by the black hole.
Information Paradox

State with matter in pure state. Let it collapse, it forms a black hole & then radiates thermally. It seems after it has evaporated pure → mixed state

Thus is impossible according to unitary evolution in QM.

Originally Hawking argued that QM's would need modification in Q gravity. Most physicists think that there are subtle correlations even in the Hawking radiation that one could in principle reconstruct the initial state.

Contrast with burning a lump of coal: If we studied carefully the final radiation both one would have pure → pure.

This AGKST point of view is hard to implement for black holes as the information going into making the black seems to be lost (uniqueness theorem) & for very long times the radiation is exactly thermal.

(!) not a causal interlude
$T_H = \frac{K}{2\pi}$

Comments

1. Derivation can be generalised to other fields, fermions, bosons, $m = 0, m \neq 0$.

2. Non-spherical collapse

3. Collapse to rotating or charged black holes


$2c. \quad T_H \sim 6 \times 10^{-8} \left( \frac{M_\odot}{M} \right)^{1/3} \text{K}$

- Astrophysical black holes would absorb CMB.
- Micro black holes?! Perhaps formed after Big Bang?

$3c. \quad \frac{dE}{dt} < 0$ / Heats up as it evaporates!!

Thermodynamics: 1st law promoted to thermodynamic law

$\Delta E = T_H dS_{bh} + \mu dJ + \mathcal{E} dQ$

With $S_{bh} = \frac{1}{4} A$

General second law

$S_{bh} + S_{\text{matter outside}} > 0$

Black hole
A riddle

Derivation emphasises role of horizon.

A close closely related phenomenon with observer dependent horizons.

Eg: Riddle - acceleration horizon

World line of uniformly accelerated observer

Also exist in

Eg: de Sitter

Can obtain \( T \) via "Euclideanising" geometry. \( t \rightarrow i \tau \). See page sheet 6.
\[ S_{BH} = k_B \left( \frac{M}{m_0} \right)^2 \]

by comparison \[ S_{Sun} \approx k_B \times 10^{58} \]

Entropy would be much greater if all matter was in black holes, i.e. we live in a low-entropy state.

Comment: This Sun data were the work of other

In statistical mechanics

\[ S = k_B \ln W \]

\# of microstates.

What are these?!

For this we need a theory of quantum gravity & is a major topic of research.

A state counting has been achieved for a special class of supersymmetric & near supersymmetric black holes in string theory. The key idea is the prediction that the black holes are "built" from configurations of intersecting branes & wrapped branes.
Black hole evaporation

Hawking's calculation neglected back reaction
but this should be ok for \( \Delta M \ll M \) but
will not be good in final stage when we need
a theory of Q gravity.

Can estimate time taken to evaporate.
using Stefan-Boltzmann law for rate of energy
loss for black body

\[
\frac{dE}{dt} = - \sigma A T^4
\]

\[
\sigma = \frac{\pi^2 \kappa \hbar^4}{30 c^3}
\]

\[
E = M c^2
\]

area of event horizon \( \approx \left( \frac{M \pi}{c^2} \right)^2 \)

\[
\approx \frac{1}{M} \sim \left( \frac{\hbar c^2}{G} \right) \frac{1}{M^2}
\]

\[
\approx T \sim \frac{c^2}{\hbar} \left( \frac{M}{M_0} \right)^3 \frac{1}{M_0^3}
\]

\[
\sim 10^{30} \left( \frac{M}{M_0} \right)^3 \text{s}
\]

\[
\sim 10^{-44} \left( \frac{M}{M_{\text{pl}}} \right) \text{s}
\neq 10^3 \text{ s}
\]