First note, for $r > 2m$ can use $(r, \theta, \phi)$.

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + r^2 dr^2 \quad \text{(exercise)}$$

which is not good at $r = 2m$

Define Kruskal–Steklov's (KS) coordinates for $r > 2m$

$$U = -e^{-\frac{r}{4m}}, \quad V = e^{\frac{r}{4m}} \begin{cases} \begin{array}{ll} U < 0 & r > 2m \\ V > 0 & \end{array} \end{cases}$$

Note

$$UV = -e^{\frac{r}{2m}} = -e^{\frac{r}{2m}(-1)} \quad \bigstar$$

$$\frac{V}{U} = e^{\frac{r}{2m}}$$

Determine $t$ in terms of $U$ & $V$ ($r > 2m$)

$$ds^2 = -\frac{32m^3}{c^2} e^{-\frac{r}{2m}} dU dV + r^2 d\Omega$$

with $r = r(U, V)$

Now define $r(U, V)$ for $U > 0$ or $V < 0$ via $\bigstar$.

The metric can be analytically extended through $U > 0 \& V > 0$ to new regions.

Notice $\bigstar$ $r = 2m \implies U = 0$ or $V = 0$

$r = 0 \implies U = 0 \implies V = 1$

Hence null geodesics have $U$ or $V$ = constant.
Kruskal diagram

Regions I + II covered by ingoing E-T coads
I + III

Region IV new! Schwarzschild

The above is the maximally extended eternal black hole.

Regions III + IV not significant for gravitational physics

It can be obtained from region II by

\[ u \rightarrow -u \]

Notice that falling into the black hole, the singularity "happens."

(Extended Kruskal is important in other contexts)
Exercise: \( k = \frac{1}{4M} \left( \nabla^2 - \frac{\partial}{\partial u} \frac{\partial}{\partial u} \right) \) defined everywhere.

\( k^2 = - \left( \frac{1-2M}{r} \right)^2 = 0 \) null on \( u = 0 \) or \( v = 0 \)

Orbits of \( K \)

Also note \( k = 0 \) on "bifurcation 2-sphere" at \( u = v = 0 \)

The Einstein-Rosen bridge

Consider a slice of constant \( t \), corresponds to line through origin.

In region I we have the induced metric on this slice is

\[ ds^2 = \left( \frac{1}{r} \right)^2 dr^2 + r^2 d\Omega^2 \quad r > 2M \]

In region IV there is an identical slice

\( r = 2M \) minimal \( S^2 \)

\[ \text{at } u = v = 0 \]

They join precisely on \( S^2 \)
\[ dp = \left( \frac{1 - 2m}{r^2} \right)^{1/2} dr \]

\[ \Rightarrow p = \left( \frac{1 - 2m}{r^2} \right)^{1/2} r + m \log \left[ \frac{r - 1}{m} \left( \frac{1 - 2m}{r^2} \right)^{1/2} \right] \]

\[ p = 0 \text{ at } r = 2m \]

\[ ds^2 = dp^2 + r^2 d\Omega \]

\[ \text{at } r = 2m \text{ and } p = 0 \quad ds^2 \propto r^2 d\Omega = 4m^2 d\Omega \]
this is the E-R bridge.
An example of a wormhole
It is non-traversable!

Topology & $\mathbb{R} \times S^2$

Penrose Diagrams

A way to represent causal structure of spacetime

Define a conformal transformation is a map from a spacetime $(M, g)$ to $(M, \tilde{g})$ with

$$\tilde{g}_{\mu\nu}(x) = A^2 g_{\mu\nu}(x)$$

$A(x)$ smooth, $A(x) \neq 0$

Conformal transformations preserve causal structure, $V$ a vector then

$$g_{\mu\nu} V^\mu V^\nu \geq 0 \iff \tilde{g}_{\mu\nu} V^\mu V^\nu = 0 \text{ since } A^2 > 0.$$

Curvature not preserved. Explore on problem sheet.

Idea of Penrose diagram on $(M, g)$

a) Do coordinate trans'n so that "infinity" is brought to a finite coordinate location
b) Do conf trans'n $(M, g)$ so that $\tilde{g}$ is regular on the boundary.

Add points at $\infty$ of $M$ to form $(\tilde{M}, \tilde{g})$

- the conformal compactification of $(M, g)$
Ex. Minkowski space \( D = 1+1 \)

\[ ds^2 = -dt^2 + dx^2 \quad -\infty < t, x < \infty \]

a) \( u = t - x \)
\[ v = t + x \]
\[ ds^2 = -dv du \quad -\infty < u, v < \infty \]

b) \( u = \tan \tilde{u} \)
\[ v = \tan \tilde{v} \]
\[ -\pi/2 < \tilde{u}, \tilde{v} < \pi/2 \]

Range is open since \( u, v = \pm \infty \) are not points of \( M \).

\[ ds^2 = \frac{1}{(\cos \tilde{u} \cos \tilde{v})^2} \left[ -d\tilde{u} d\tilde{v} \right] \]

Conformal compactification \( (\hat{M}, \hat{g}) \) choose \( \lambda = \cos \tilde{u} \cos \tilde{v} \)

\[
\begin{cases}
  d\hat{s}^2 = (\cos \tilde{u} \cos \tilde{v}) ds^2 = -dv du \\
  \text{And} \quad -\pi/2 \leq \tilde{u}, \tilde{v} \leq \pi/2
\end{cases}
\]

Lines of constant \( t \)

Lines of constant \( \tau \)
\[(\tilde{\alpha}, \tilde{\beta}) = (\pi L, \pi L) \iff t \to \infty \quad \text{Future temporal} \quad \infty \]

\[(\tilde{\alpha}, \tilde{\beta}) = (\pi L, -\pi L) \iff t \to -\infty \quad \text{Past temporal} \quad \infty \]

\[(\tilde{\alpha}, \tilde{\beta}) = (-\pi L, \pi L) \iff t \text{ finite} \quad x \to \infty \quad \text{Space-like} \quad \infty \]

\[(\tilde{\alpha}, \tilde{\beta}) = (-\pi L, -\pi L) \quad \text{Future null} \quad \infty \]

\[\tilde{\alpha} = \pi L \iff u \to \infty \quad x \text{ finite}
\[|\tilde{\alpha}| < \pi L \quad u \text{ finite}
\]

\[\tilde{\alpha} = -\pi L \iff u \to -\infty \quad x \text{ finite}
\[|\tilde{\alpha}| < \pi L \quad u \text{ finite}
\]

\[\text{Null geodesics start at } g^- \text{ and end at } g^+ \]

\[\text{Timelike geodesics } \quad i^- \quad i^+ \]

For a particle to arrive at } \text{it should be}
continuously accelerated
Example

Minkowski spacetime in 1+3 dimensions
\[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \]
\[ ds^4 = -dt^2 + dr^2 + r^2 d\Omega \]

First
\[ u = t-r \quad -\infty < u < \infty \quad \text{since } r > 0 \]
\[ v = t+r \quad \text{range is open since } \quad u, v \neq \pm \infty \text{ are not points in } M \]
\[ ds^2 = -du dv + \frac{(u-v)^2}{4} d\Omega \]

Next
\[ u = \tan \tilde{u} \quad -\frac{\pi}{2} < \tilde{u}, \tilde{v} < \frac{\pi}{2} \]
\[ v = \tan \tilde{v} \]
\[ ds^2 = \frac{1}{(2 \cos \tilde{u} \cos \tilde{v})^2} \left[ -4 d\tilde{u} d\tilde{v} + \sin^2 (\tilde{v}-\tilde{u}) d\Omega \right] \]

(Alternatively, choose
\[ (\tilde{t}, \tilde{r}) \]

conformal compactification \[ \Lambda = 2 \cos \tilde{u} \cos \tilde{v} \]
\[ ds^2 = -4 d\tilde{u} d\tilde{v} + \sin^2 (\tilde{v}-\tilde{u}) d\Omega \]
\[ \text{and} \quad -\frac{\pi}{2} < \tilde{u}, \tilde{v} < \frac{\pi}{2} \]

Only difference for 1+1 case is that
\[ \tilde{r} \rightarrow \tilde{r} \rightarrow \tilde{r} \rightarrow \infty \]

Every point represents a point on
\[ S^2 \]
\[ \text{radial null line.} \]
Another picture. \[ T = \bar{w} + \bar{v} \]
\[ x = \bar{w} - \bar{v} \]
\[ \omega \Rightarrow \bar{w} \]
\[ \bar{u} \Rightarrow \bar{v} \]

\[ -\pi \leq t \leq \pi \]
\[ 0 \leq x \leq \pi \]

\[ ds^2 = -dt^2 + dx^2 + \sin^2 x \, d\Omega \]

Metric on round \( S^3 \)

"Einstein Static universe"

Radial null line ends up on antipode of \( S^2 \).
Kruskal

In region I

\[ ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 - \frac{1}{\left( 1 - \frac{2M}{r} \right)} dr^2 - r^2 d\Omega^2 \]

Let \( u = \tan \frac{\theta}{2} \)

\(-\pi/2 < \theta < \pi/2 \)

\( u = \tan \frac{\theta}{2} \)

\(-\pi/2 < \theta < \pi/2 \)

\[ ds^2 = (\cos u \cos \theta)^2 \left[ -4 \left( 1 - \frac{2M}{r} \right) du \, d\bar{u} + r^2 \cos^2 \theta \cos^2 \bar{\theta} \, d\Omega^2 \right] \]

\[ \Delta^2 = \frac{1}{4} (\bar{g} - g) = \frac{\sin (2\bar{u})}{\cos \theta \cos \bar{\theta}} \]

\[ ds^2 = (\cos u \cos \theta)^2 \, ds^2 \]

\[ = -4 \left( 1 - \frac{2M}{r} \right) du \, d\bar{u} + r^2 \sin^2 (\bar{\theta} - \theta) \, d\Omega^2 \]

An example of asymptotic flat -

As \( r \to \infty \) (for any \( \bar{u} \)) approaches \( \infty \). So we can add \( \bar{u} \bar{\theta} \) term as before.

Note:
Near \( r = 2M \) introduce \( k-s \) coords to separate

1) have adjusted \( \Delta^2 \) so that \( r = 0 \) is straight line

2) All \( r = \text{const} \) surfaces meet at \( i^+ \), including the singular \( r = 0 \) surface. \( i^+ \) is singular surface cannot be added. Same for \textit{i-}
Asymptotically flat

"looks like Minkowski: at \( \infty \)"

The Kruskal example with singular \( i^\pm \) means that we can only demand that a flat has same structure for null \( \infty \) \( i^\pm \) of space-like infinity.

\[ i^+ \]
\[ i^- \]

Caveat: Need to be careful about smoothness of \( i^0 \) - it won't be for Kruskal (not obvious).

See Wald for more details.

Comments: 1) \( i^0 \) asymptotically vectors \( \omega \) that approach Killing vectors near \( i^0 \).

2) Symmetries of \( g^{\pm} \) known as BMS group - topical!
Black holes & event horizons

- Chronological (future) of a point $x$ is the set of points which can be reached from $x$ by a directed timelike curve

- Causal (future) of a point $x$ is the set of points which can be reached from $x$ by a directed timelike or null curve

Clearly $I^+(x) \subseteq J^+(x)$

For set $U \in M$ define $I^+(U) = \bigcup I^+(x_i)$ $\forall x_i \in U$

For a given asymptotically flat region

- Define future event horizon $H^+$ to be boundary of $J^-(\mathcal{G}^+)$ and black hole region
- $M$ be region of $M$ with $\mathcal{G}^+$

Define past event horizon

$H^- = J^+(\mathcal{G}^-)$ & white hole to be region $M / J^+(\mathcal{G}^-)$
Singularity

Coordinate singularities - can be removed by coord transform
not physical! e.g. $r = 2m$ - Kruskal

Curvature singularities - some are physical contracted
from $R^4$ - Schwarzschild $\frac{\partial}{\partial r}$ singularity
from $R^4$ - Kruskal $\frac{\partial}{\partial r}$ singularity
possible also to have coordinate singularities in which
no scalar changes, yet $\frac{\partial}{\partial r}$ coords in which $R^n$ up
remains finite (infinite tidal force)

Other examples,

Conical singularity

$$dr^2 + \lambda r^2 d\phi^2 \left( \begin{array}{c} \lambda \leq 1 \text{ change coords to get } dx^2 + dy^2 \\ \lambda > 1 \text{: define } \Delta \phi = \frac{\phi}{\lambda} \text{ and } 2\lambda - 1 \\ dr^2 + r^2 d\phi^2 \end{array} \right)$$

Circle circumference = $\frac{2\pi \lambda \epsilon}{\pi} = 2\pi R$ \(\frac{\lambda - 1}{\lambda} \epsilon \rightarrow R^2 \rightarrow 0 \)

axis is singular as not locally $R^2$

To define in general: use the idea that a singularity is
captured by having geodesics that cannot be extended
to arbitrarily large affine parameter.
**Defn**: A geodesic is complete if an affine parameter exists extending to \( \pm \infty \).

A spacetime is geodesically complete if all maximally extended causal geodesics are complete.

- \( M \text{ - complete} \)
- \( M \text{-3d} \text{ - geodesically incomplete} \)
- Come incomplete

**Note**

Singularity is **clothed**

Singularity is **naked** - singularities are seen at \( g^+ \)

Negative mass Schwarzschild

\[
\frac{\text{d}r}{\text{d}t} = -(1 + 2M/r) \frac{\text{d}t}{\text{d}r} + (1 + 2M/r)^2 \text{d}\theta^2 + r^2 \text{d}\phi^2
\]

Can naked singularities form?

**eg**

If happened future would cease to be predictable beyond from data on \( \Sigma \).
Cosmic censorship - Penrose.

Naked singularities cannot form from generic gravitational collapse in asymptotically flat spacetime that is non-singular on some initial spacelike hypersurface ("Cauchy surface") defined (adv).

**Comments**

1) Need to exclude fine-tuned situations (e.g., scalar field + spherical symmetry).
2) Excludes cosmological singularities ("dominant energy").
3) Same assumptions on name of matter (needed).
4) Unproven. - Major goal in classical/quantum GR.

Singularity theorems

Penrose, Hawking.

1965

Show powerful results that black holes singularity will occur in certain situations, including gravitational collapse.

Together => Black holes provide beautiful consistent picture of final state of grav. collapse.

In particular, black hole GR predicts that black holes (and big bang) are places to look for physics => quantum gravity.
Reissner-Nordström Black Hole

Einstein-Maxwell theory

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - F_{\mu\nu}F^{\mu\nu} \right] \]  

(notice normalization)

\[ F_{\mu\nu} = 2\pi \Lambda_{\nu} - \partial_{\mu} \Lambda_{\nu} = \partial_{\mu} \Lambda \Lambda - \partial_{\mu} \Lambda \Lambda \]  

(check)

Eq's of motion

\[
\begin{align*}
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} - \frac{1}{4} \nabla_{\mu} F_{\nu\rho} F^{\rho\sigma} \\
\nabla_{\mu} F^{\mu\nu} &= 0
\end{align*}
\]

Generalisation of Reissner-Nordström:

Unique spherically symmetric solution of E-Maxwell
(with non-constant radius function) \( dS^2 \) the R-N solution

\[ dS^2 = \left(1 - \frac{2M + e^2}{r} \right) dt^2 + \left(1 - \frac{2M + e^2}{r} \right)^{-1} dr^2 + r^2 d\Omega_2^2 \]

\[ A_\tau = -\frac{Q}{r} \]

\[ A_\phi = -\frac{e}{r} \cos\theta \]

\[ F_{\phi\tau} = \frac{Q}{r^2} \]

\[ F_{\phi\phi} = P \sin\theta \]

\[ e = \sqrt{Q^2 + P^2} \]

\[ A_\phi = -\frac{e}{r} \cos\theta \]

\[ F_{\phi\phi} = P \sin\theta \]

\[ e = \sqrt{Q^2 + P^2} \]

Static: \( k = 2l \)

Asymptotically flat as \( r \to \infty \)

To analyse, define \( \Delta = r^2 - 2Mr + e^2 = (r - r_+)(r - r_-) \)

with \( r_\pm = M \pm \sqrt{M^2 - e^2} \)

What happens at \( r = r_\pm \)?

Singularity at \( r = 0 \), \( F_{\mu\nu} \to \infty \)

\( F_{\mu\nu} \to \infty \) easy calc!
3 cases

1) "super extreme" \( e > M \) no coordinate singularities

2) subextremal \( e < M \) \( r_+ \neq r_- \) distinct

3) extremal \( e = M \) \( r_+ = r_- \)

1) \( e > M \)

\[
t, r, \theta, \phi \quad \text{for all } r > 0 \\
\Rightarrow r = 0 \text{ a time-like singularity for the } \pm \text{ mass Schwarzschild.}
\]

Non-physical

Cannot appear as end pt of collapse (cosmic censorship)

Note: electron has \( e \neq M \) but it is quantum mechanical

2) \( e < M \)

\[ \Delta = 0 \text{ at } r = r_+ , r_- \]

Proceed as we did for Schwarzschild

Start with \( r > r_+ \) and define

\[
dr_+ = \frac{\Delta}{\Delta} dr
\]

\[ \Rightarrow \delta_+ = r + \frac{1}{2k_+} \log |r - c| + \frac{1}{2k_+} \log |r - c| + \text{const.} \]

where \( k_+ = \frac{r_+ - r_-}{2r_+} \)

Define \( u, v \)

\[ u = t - r_+ , \quad v = t + r_+ \]

Ingoing E\(-\)F \((u, r, \theta, \phi)\)

\[
ds^2 = -\frac{\Delta}{\delta_+} dv^2 + 2dvdr + r^2 d\Omega
\]
smooth $V > 0$. Hence can extend to $\phi < 0$.

Can also define outgoing $E = F$ $(u, r, \theta, \phi)$

$$ds^2 = -\frac{\Delta}{\rho^2} du^2 - 2dudr + r^2 d\Omega^2$$

$\phi$ can extend to a different region $0 < r < r_f$.

Kruskal Coords

$$u^\pm = -e^{-kr} u \quad \tau^\pm = \tau \pm e^{kr}$$

In region $r > r_f$, use $(u^+, v^+, \theta, \phi)$

$$ds^2 = -\frac{r_c r}{k^2} e^{-2kr} \left( \frac{r-\frac{r_f}{2}}{r+\frac{r_f}{2}} \right)^{1+\frac{k_f}{1+k_f}} \, du^+ dv^+ + r^2 d\Omega^2$$

$r(u^+, v^+)$ defined via

$$-u^+ v^+ = e^{2kr} \left( \frac{r-\frac{r_f}{2}}{r+\frac{r_f}{2}} \right)^{1+\frac{k_f}{1+k_f}}$$

RHS monotonically increasing for $r > r_f$.

Initially:

$$\begin{align*}
U^+ &< 0 \quad \text{(which} \Rightarrow r > r_f) \\
v^+ &> 0
\end{align*}$$

Now can hence $U^+ > 0 \quad \text{or} \quad V^- \leq 0$

$$\Rightarrow \quad \begin{array}{c}
\text{II} \\
\text{IV} \\
\text{III} \\
\text{I}
\end{array}$$

$U^+$ maximal $V^+$ trans.$

Again I treason bridge for $t = \text{const.}$.

Also, again the null surfaces generated by $\nabla V$ intersect on the $r = r_f$.

Also, again the null surfaces generated by $\nabla V$ intersect on the $r = r_f$.
Major difference here is no singularities in $\Pi + \Pi$

Since $r(U^i, V^i) > r_-$

Let $r(U^i, V^i) = r - \infty$ when $UV = +\infty$

Now we know from EF coods that we can extend to $r < r_-$. Hence, must be able to extend.

Said differently, radial null geodesics reach $r = r_-$ in finite affine parameter $u$ hence thus will reach.

$U^iV^i = +\infty$ in finite affine parameter.

So need new coords.

Start in region $\Pi$ and use $(v, r, \theta, \phi)$.

Now define $u$ in region $\Pi$ as follows.

First define $t = u - r_+$ in region $\Pi$.

Then in region $\Pi$ use $(t, r, \theta, \phi)$ for metric as known with $\delta \leq r < r_+$.

Now define $u = t - r_+ = u^2 - 2r_+$.

Having got $u, v$ in region $\Pi$ we use

$U^{-} < 0 \quad \text{and} \quad V^{-} < 0 \quad \text{in} \quad \Pi$.

$$
\Gamma = \frac{\Gamma + r_+ \kappa k}{k^2 r^2} e^{2(k-1)} \left( \frac{\Gamma + r}{r} \right)^{k+1} d\Omega d\Omega + r^2 d\Omega
$$

$R(U, V) < r_+$ given by

$$
U^{-} V^{-} = e^{-2(k-1)r} \left( \frac{r-r_+}{r+r} \right) \left( \frac{r_+}{r} \right)^{k+1}
$$

This can be continued to $U^{-} > 0$ or $V^{-} > 0$. 

$$
\begin{array}{c}
\text{III'} \\
\text{II} \\
U^{-} V^{-} = -1
\end{array}
$$
Notice $r = 0 \Rightarrow V^- V^- = -1$ which is timelike.

Time reversal on $\mathcal{I}' \rightarrow \mathcal{I}$ which is connected to $I, \mathcal{I}, \mathcal{I}'$.

\[ \Rightarrow \]

Hence get 3 new regions.

Keep going!

New asymptotic region $\mathcal{I}'$.

Consider ingoing radial geodesic.

Think of as a collapsing shell.

Naively, hits $r = 0$ in region $\mathcal{I}$.

But suppose accelerate to go to region $\mathcal{I}'$.

Must cross a Cauchy horizon.
Define: Partial Cauchy surface \( \Sigma \), for spacetime \((M, g)\) is a hypersurface which no causal curve intersects more than once.

Define: Causal curve is past inextendable if it has no past end point in \( M \).

Define: Future domain of dependence \( D^+(\Sigma) \), of \( P \in M \) for which every past inextendible curve through \( P \) intersects \( \Sigma \).

Past domain of dependence \( D^-(\Sigma) \) similarly defined.

Domain of dependence \( D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma) \).

\( \Sigma \) is a region of \( M \) where we can determine what happens by data on \( \Sigma \).

Solutions of hyperbolic p.d.e.'s are uniquely determined by initial data on \( \Sigma \).

Define: A spacetime is globally hyperbolic if it has a Cauchy surface, i.e., a partial Cauchy surface so that \( D(\Sigma) = M \).

If globally hyperbolic, can predict what happens everywhere from data on \( \Sigma \).
Examples

Minkowski:

If \(\mathcal{M}\) is not globally hyperbolic, then \(\Sigma\) will have a boundary called the past Cauchy horizon for \(\Sigma\) at \(\partial^+ \Sigma\).

Existence of Cauchy horizons makes most of the physical questions we assumed in the symmetry of analyticity. If we relax these conditions, events could extend beyond Cauchy horizon in as many ways as the given data on \(\Sigma\).

Note also that \(B\) sends light signals to \(A\) at a finite proper time \(\tau\) signals from \(B\) undergo a redshift. Hence small perturb in region will have enormous energy at Cauchy horizon as measured by \(A\).
One possibility is that for realistic set ups, the r = 0 Cauchy Horizon is replaced by a singularity or so, and the resulting spacetime would be globally hyperbolic.

Collapsing charged matter

3) Extreme RN e = M, \( r_+ = M \)

\[ ds^2 = -\left(\frac{M}{r}\right)^2 dt^2 + \left(\frac{M}{r}\right)^2 dr^2 + r^2 d\Omega \]

Start with \( r > M \)

\[ dr^+ = \frac{dr}{(1-M/r)^2} \]

\[ r_* = r + 2M \log \frac{r-M}{M} - \frac{M^2}{r-M} \]

\[ u = t - r_* \]

\[ ds^2 = -\left(\frac{M}{r}\right)^2 du^2 + 2du dr + r^2 d\Omega \]

and can extend to \( 0 < r < M \) \rightarrow blackhole

Inside region

\[ u = t - r_* \]

can extend to get white hole

I & II

isometric

\( \phi \) keeps A.F.

region to future of \( t = 0 \).

I & III

\( \phi \) keeps A.F.

\( t^\prime \) future event horizon

\( t^\prime \) Cauchy horizon

\( t^- \) past event horizon

\( t^- \) Cauchy horizon
$k = \frac{\partial}{\partial \epsilon} = \frac{\partial}{\partial u}$ outgoing $E-F$

or $\frac{\partial}{\partial u}$ ingoing $E-F$

$\|\phi\|^2 < 0 \quad r \neq m \quad \text{time-like everywhere}$

$= 0 \quad r = m$
Novel feature:

\[ t = \text{constant} \quad \text{surface do not a } E\text{-Rosen bridge} \]
\[ \text{connecting two flat regions but on an infinite throat} \]
\[ t = \text{const}, (\Theta, \Phi) = \text{const} \quad \text{proper length } \int_{r_0}^{r} \frac{dr}{\sqrt{1 - m/r}} = \infty \]

Extremal Multi-Black Hole Solutions

Majumder - Luna Petrov (1943)

Set \( p = 0 \) so \( e = \Phi \)

\[ f = r - m \]

\[ ds^2 = -H^{-2} dt^2 + H^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]
\[ A = -H^{-1} dt + \frac{dt}{f} \]

\( H = 1 + \frac{M}{r} \)

\( f \) is a gauge transform

\( H \) is the first solution to the Einstein-Maxwell equations

Special case of

\[ \left\{ \begin{array}{l}
  ds^2 = -H^{-2} dt^2 + H^2 (dx^1)^2 \\
  A = -H^{-1} dt \\
\end{array} \right. \]
with \( d = 2 \); \( H(x) = 0 \). This solves \( \text{Maxwell} \) eq's.

Choose \( H = 1 + \sum_{i=1}^{N} \frac{M_i}{|x-x_i|} \)

1. static solution
   - \( N \) extremal K-N black holes with Mass = \( M_i \)
     (charge)
   at "positions" \( x_i \). In fact, each of these is an S^2
     not a point in spacetime

2. Cancellation of gravitational attraction
   and electromagnetic repulsion

3. Can embed this in \( D = 4 \) supergravity theory
   the solutions are supersymmetric (S)
   there exists "Killing spinors" \( \xi \) satisfying
   \[
   (\Sigma \gamma_\mu + F_{\mu\nu} \gamma^\nu) \gamma_5 \xi = 0
   \]