1. In Minkowski spacetime let $\psi_p(t, x)$ and $\psi_p^*(t, x)$ be a basis of positive and negative frequency solutions, respectively, satisfying

\[
(\psi_p, \psi_q) = \delta^3(p - q), \quad (\psi_p, \psi_q^*) = 0, \quad (\psi_p^*, \psi_q^*) = -\delta^3(p - q),
\]

where $(\ , \ )$ is defined via

\[
(f, g) = i \int d^3x \ (f^*(t, x) \partial_t g(t, x) - g(t, x) \partial_t f^*(t, x))
\]

The scalar field operator $\phi(t, x)$ can be expanded

\[
\phi(t, x) = \int d^3p \ (a_p \psi_p(t, x) + a_p^\dagger \psi_p^*(t, x))
\]

Show that we can write $a_p = (\psi_p, \phi)$ and $a_p^\dagger = - (\psi_p^*, \phi)$. Using this and the equal time commutation relations for the scalar field $\phi$, show that

\[
[a_p, a_q^\dagger] = (\psi_p, \psi_q) = \delta^3(p - q), \quad [a_p, a_q] = (\psi_p, \psi_q^*) = 0
\]

2. Consider the transformation

\[
a'_i = \sum_j (A_{ij} a_j - B_{ij} a_j^\dagger)
\]

Show that if $a_i$, $a_i^\dagger$ satisfy the usual commutation rules then so do $a'_i$, $a'_i^\dagger$, provided that $AA^\dagger - BB^\dagger = 1$ and $AB^T - BA^T = 0$.

3. Show that a massless scalar field in a Schwarzschild black hole spacetime that satisfies $\nabla^2 \phi = 0$ can be expanded as

\[
\phi = \sum_{\omega,l,m} c_{\omega lm} \frac{1}{r} R_{\omega l}(r) e^{-i\omega t} Y_{lm}(\theta, \phi)
\]

where $Y_{lm}$ are spherical harmonics and

\[
\left(\frac{\partial^2}{\partial r^2} + \omega^2\right) R_{\omega l}(r) = \left(1 - \frac{2M}{r}\right) \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) R_{\omega l}(r)
\]

4. Let $k$ be an arbitrary Killing vector. If $\phi$ satisfies the Klein Gordon equation show that $L_k \phi$ does also (hint: use a convenient set of coordinates). Also show, just in flat spacetime, that if we take $k = \partial_t$ then $L_k$ is antihermitian with respect to the Klein-Gordon product i.e. $(L_k f, g) = -(f, L_k g)$ for any two solutions $f, g$ of the Klein-Gordon equation.
5. Consider the integral that we considered in lectures

\[ \tilde{g}_\omega^T(\omega') = \int_{-\infty}^{0} dv \exp(i\omega'v + \frac{i\omega}{\kappa} \log(-v)) \]

Prove that for \( \omega' > 0 \)

\[ \tilde{g}_\omega^T(-\omega') = -\exp(-\pi\omega/\kappa)\tilde{g}_\omega^T(\omega') \]

Hint: consider the integral as a contour integral in the complex \( v \) plane along the negative \( v \) axis, with a branch cut along the positive \( v \) axis because of the log. For \( \omega' > 0 \) argue that one can rotate the contour to the positive imaginary axis and then set \( v = ix \) to get an expression for the integral (don’t try to do it!). For \( \omega' < 0 \) argue that one can rotate onto the negative imaginary axis and then set \( v = -ix \). Compare the two integrals to get the result.

6. The Hawking temperature for a black hole is

\[ T = \frac{\kappa}{2\pi} \]

Add in fundamental constants \( G, \hbar, c, k_B \) to make this dimensionally correct. Also calculate the temperature of a Schwarzschild black hole in Kelvin expressed in terms of the mass of the black in units of solar mass.

The Bekenstein-Hawking entropy is given by

\[ S = \frac{1}{4} A \]

where \( A \) is the area of the event horizon. Add in fundamental constants to make this dimensionally correct. Calculate the entropy of a Schwarzschild black hole with solar mass, keeping an explicit factor of \( k_B \) in the expression.

7. Use the fact that a Schwarzschild black hole radiate at the Hawking temperature

\[ T_H = \frac{1}{8\pi M} \]

to show that the thermal equilibrium of a black hole with an infinite reservoir of radiation at temperature \( T_H \) is unstable.

A finite reservoir of radiation of volume \( V \) and temperature \( T \) has an energy \( E_{res} \) and entropy \( S_{res} \) given by

\[ E_{res} = \sigma V T^4, \quad S_{res} = \frac{4}{3} \sigma V T^3 \]

where \( \sigma \) is a constant. A Schwarzschild black hole is placed in the reservoir. Show that the total entropy \( S = S_{BH} + S_{res} \) is extremised for fixed total energy \( E = E_{res} + M \) when \( T = T_H \). Show that the extremum is a maximum only if \( V < V_c \) where the critical value of \( V \) is

\[ V_c = \frac{2^{20} \sigma^4 E^5}{5^5} \]

What happens as \( V \) passes from \( V < V_c \) to \( V > V_c \) or vice-versa?
8. The specific heat of a charged black hole of mass $M$ and fixed charge $Q$ is

$$C = T_H \frac{\partial S_{BH}}{\partial T_H} \bigg|_Q$$

With $S_{BH} = A/4$, show that for the electrically charged Reissner-Nordstrom black hole

$$C = \frac{2S_{BH} \sqrt{M^2 - Q^2}}{M - \sqrt{M^2 - Q^2}}$$

Hence show that $C^{-1}$ changes sign when $M$ passes through $2|Q|/\sqrt{3}$ and interpret the result.

9. Consider the Euclidean Schwarzschild black hole obtained by writing $t = i\tau$ to get

$$ds^2 = (1 - \frac{2M}{r})d\tau^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2d\Omega$$

Consider the metric close to $r = 2M$ by changing coordinates via $R^2 = r - 2M$ and approximating for small $R$. Show that at fixed coordinate values of $\theta, \phi$, the coordinate $\tau$ can be periodically identified with a certain period $\beta$, which should be determined.

Periodically identifying imaginary time is characteristic of a thermal state with temperature $\beta^{-1}$. What does this calculation suggest the temperature of the Schwarzschild black hole is? Compare this to the Hawking temperature.