Recall \( u = t - r^2 \) \( \quad dU = dt - 2r \, dr \)
\( v = t + r^2 \) \( \quad dV = dt + 2r \, dr \)

\[
dU \, dV = dt^2 - (2r)^2 \]
\[
= dt^2 - (1 - \frac{2m}{r})^2 \, dr^2
\]

\[
\Rightarrow ds^2 = -(1 - \frac{2m}{r}) \, dt^2 + (1 - \frac{2m}{r})^{-1} \, dr^2 + r^2 \, d\Omega
\]

\[
\frac{ds^2}{(1 - \frac{2m}{r})} = du \, dv + r^2 \, d\Omega
\]

(Note as an aside: \( q_{uv} = \frac{1}{2} \left( \frac{-2m}{r} \right) \)
\( \uparrow \) note the \( \frac{1}{2} \)!

\[
U = -e^{-r/2m} \Rightarrow dU = e^{-r/2m} \left( \frac{4m}{4m} \right) = \frac{1}{4m} \, UdU
\]
\[
V = e^{r/2m} \Rightarrow dV = \frac{1}{4m} \, VdV
\]

\[
\Rightarrow ds^2 = -(1 - \frac{2m}{r}) \left( \frac{dU}{4m} \frac{dV}{4m} \right) \frac{4m}{V} dU \, dV + r^2 \, d\Omega
\]
\[
= -16m^2 \left( 1 - \frac{2m}{r} \right) \frac{1}{V \, V} \, du \, dv + r^2 \, d\Omega
\]

Also have \( UV = -e^{r/2m} = -e^{r/2m} \left( \frac{r}{2m} - 1 \right) \)

\[
\Rightarrow \frac{1}{2m} \, e^{r/2m} \left( 1 - \frac{2m}{r} \right)
\]

\[
\Rightarrow ds^2 = -32m^3 \frac{e^{-r/2m}}{r} dU \, dV + r^2 \, d\Omega
\]

With \( r = r(u, v) \).
In region coord transm. \((t,r,\theta,\phi) \rightarrow (\psi, \xi, \phi, \psi)\)

\[ k = \partial_t = \frac{\partial \psi}{\partial t} \partial \psi + \frac{\partial \xi}{\partial t} \partial \xi + \frac{\partial \phi}{\partial t} \partial \phi + \frac{\partial \psi}{\partial \phi} \partial \phi \]

to work out \(\partial \psi + \partial \xi\) note that we
are taking partial derivatives keeping \(r\) fixed
when \(\psi\) is fixed we have \(\psi = c\) for
constant \(c\).

We also have \(\frac{\partial \xi}{\partial \psi} = \frac{1}{\xi} \psi\)

\(\Rightarrow \frac{\partial \xi}{\partial \xi} = -c e^{t/2m}\)

\(\Rightarrow \frac{\partial \xi}{\partial \xi} = -c e^{-t/2m}\)

\(\Rightarrow 2 \xi \frac{\partial \xi}{\partial \xi} = -c e^{t/2m} \frac{1}{2m} = -\frac{\partial \xi}{\partial \xi}^{1/2m}\)

\(\Rightarrow \frac{\partial \xi}{\partial \psi} = \frac{\psi}{4m}\)

Also \(2 \psi \frac{\partial \xi}{\partial \psi} = +c e^{-t/2m} \left(\frac{1}{2m}\right) = -\frac{\partial \psi}{\partial \xi}^{1/2m}\)

\(\Rightarrow \frac{\partial \psi}{\partial \psi} = -\frac{\psi}{4m}\)

Hence

\[ k = \partial_t = \frac{1}{4m} \left( \psi \frac{\partial \psi}{\partial \psi} - \psi \frac{\partial \psi}{\partial \psi} \right) \]

In these coord \(k^\mu = \frac{1}{4m} (-\psi, \xi, 0, 0)\)

\(||k||^2 = k^\mu k^\nu \gamma_{\mu\nu} = 2 k^\psi k^\xi \gamma_{\psi\xi}\)

\[ = \frac{1}{(4m)^2} (-\psi)(\xi) \left( -\frac{16 M^2}{\xi^2} \right) e^{-\frac{t}{2m}} = -\left(1 - \frac{2M}{\xi} \right) \]
In Schwarzschild coordinates,

\[ k^m = (1, 0, 0, 0) \]

\[ \| k \| = g_{mn} k^m k^n = 9 \pi = \left( \frac{2M}{r} \right) \checkmark \]
2. 

![Diagram showing paths of photons and lines of constant u]

\[ \Delta u_e = \text{difference in the u-coordinate of 2 events where photons emitted} \]

\[ \Delta u_0 = \text{``...'' received} \]

But \( \Delta u_e = \Delta u_0 \) because photons travel along lines of constant \( u \).

\[ ds^2 = -(1 - \frac{2M}{r}) du^2 - 2 du dr + r^2 d\Omega \]

For observers at \( r = \infty \), have constant \( r, \theta, \phi \)

\[ ds^2 = -d\tau_0^2 = -du^2 \]

\[ \Rightarrow \Delta \tau_0 = \Delta u_0 \]

The frequency of the photons at the emitter is \( v_e \)

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Redshift \( \frac{\Delta \lambda}{\lambda_0} = (\frac{\Delta \tau}{\tau_0})^{-1} = \Delta \tau_0 = \Delta \tau_c = \frac{du}{dt} \bigg|_c \)

We want to calculate \( \frac{du}{dt} \) as spaceship approaches \( r = 2M \). We will write \( s = r - 2M \) and neglect \( o(s^2) \) terms. Note \( r = 2M \left( 1 + \frac{s}{2M} \right) \)

Now for spaceship \( -1 = -\left( \frac{d}{r} \right) \left( \frac{du}{dt} \right)^2 - \frac{2}{r} \frac{du}{dt} \frac{dr}{dt} \)

So we need \( \frac{dr}{dt} \)

Recall from lectures:

\[
\left( \frac{dr}{dt} \right)^2 = \left( 1 - \varepsilon^2 \right) \left( \frac{R_{\text{max}}}{r} - 1 \right)
\]

\[
= \left( 1 - \varepsilon^2 \right) \left( \frac{2M}{1 - \varepsilon^2} - (1 - \varepsilon) r \right)
\]

\[
= \varepsilon^2 \left( 1 - \frac{s}{2M} \right)
\]

\[
\Rightarrow \frac{dr}{dt} \approx \varepsilon \left( 1 - \frac{s}{4M\varepsilon^2} \right)
\]

Hence, from +

\[- \frac{s}{2M} \left( \frac{du}{dt} \right)^2 + 2 \varepsilon \left( 1 - \frac{s}{4M\varepsilon^2} \right) \frac{du}{dt} + 1 \approx 0 \]

\[
\Rightarrow \frac{du}{dt} = -2 \varepsilon \left( 1 - \frac{s}{4M\varepsilon^2} \right) \left[ 4 \varepsilon^2 \left( 1 - \frac{s}{4M\varepsilon^2} \right)^2 + 4 \frac{s}{2M} \right]^{1/2}
\]

\[-2 \left( \frac{s}{2M} \right) \]
expand square root and neglect $o(\delta^2)$:

\[
\frac{du}{dt} \approx -2\varepsilon \left(1 - \frac{\delta}{\delta/m^2}\right) \pm \left(4\varepsilon^2\right)^{1/2} \frac{-\delta}{\delta/m^2}
\]

\[
\frac{du}{dt} > 0 \text{ so take lower sign}
\]

\[
\Rightarrow \frac{du}{dt} \bigg|_e \approx \frac{4M\varepsilon}{\delta}
\]

\[
\text{The emitter crosses horizon } \ r = 2m \text{ at some value of } u, \ u_0 \text{ say.}
\]

At \( r \to 2m \)

\[
\Rightarrow u \to u_0
\]

\[
\Rightarrow t \to t_0 - \frac{u_0}{c}
\]

\[
\Rightarrow u \to u_0 - 2c
\]

\[
\Rightarrow \ u \to u_0 - 2 \left( -2M \ln \left( \frac{r_0 - 2M}{2M} \right) \right)
\]

and \( u \to +\infty \) as \( s \to 0 \) since \( \ln (r - 2m) = \ln s \)

\[
\Rightarrow U = u_0 - 2 \left( 2M - 4M \ln \left( \frac{r_0}{2M} \right) \right) = u_0 - 4M + 4M \ln eM - 4M \ln s
\]

\[
\Rightarrow U \approx \frac{\text{Constant}}{u_0} - 4M \ln s
\]

\[
\Rightarrow s^{-1} \approx e^{(u - u_0)/4M}
\]

Thus \[
\frac{du}{dt} \bigg|_e \approx 4M\varepsilon e^{(u - u_0)/4M} = \text{redshift}
\]

Recall that \( u = \) proper time at infinity. Hence, redshift increases exponentially in proper time at \( \infty \) and signals disappear in time of order \( 4M \).
3. \( x^0 = \cos \theta \)
\( x^\pm = \sin \theta \mu^\pm \)

\[ (x^0)^2 + (x^\pm)^2 = \cos^2 \theta + \sin^2 \theta = 1 \]

So parameters \( N \)

Choose \( 0 \leq \theta \leq \pi \) so that we cover \(-1 \leq x^0 \leq 1\)

\[ ds^2 = dx^0 dx^0 + dx^\pm dx^\pm \]

\[ dx^0 = -\sin \theta d\theta \]
\[ dx^\pm = \cos \theta d\mu^\pm + \sin \theta d\mu^\mp \]

\[ ds^2 = d\theta^2 \sin^2 \theta + \left[ \cos^2 \theta d\mu^\pm d\mu^\pm + 2 \cos \theta \sin \theta d\mu^\pm d\mu^\mp \right] \]

\[ = \sin^2 \theta d\theta^2 + \cos^2 \theta d\theta^2 + \sin^2 \theta (d\Sigma_{N-1}) \]

\[ + 2 \cos \theta \sin \theta d\theta d\mu^\pm d\mu^\mp \]

Note that \( \mu^\pm \mu^\mp = 1 \)
\[ \Rightarrow 2 \cos \theta \sin \theta d\mu^\pm d\mu^\mp = 0 \]

Hence

\[ ds^2 = d\theta^2 + \sin^2 \theta d\Sigma_{N-1} \]
4. As in lectures. Conformal compactification. \((\tilde{\mathcal{M}}, \tilde{\mathcal{g}})\) has

\[
\alpha \tilde{\mathcal{g}}^2 = -4 \, du \, d\bar{u} + \sin^2(\bar{\theta} - \bar{\alpha}) \, d\bar{u}
\]

\[-\pi/2 \leq \bar{u} \leq \bar{\alpha} \leq \pi/2\]

\[
\begin{align*}
\bar{u} &= \tan \bar{u} \\
\bar{\theta} &= \tan \bar{\theta}
\end{align*}
\]

\[
\text{eg \tfrac{r}{\text{fixed}} \& t \to \infty \implies u, \bar{u} \to \infty \implies \bar{u} = \pi / 2 \quad \bar{\theta} = \pi / 2
\]

\[
\text{Note that \bigg\| \text{the radius of the } S^1 \bigg\| = 0 \quad \text{Hence is a point}
\]

\[
\text{eg \tfrac{r}{\text{fixed}} \& t \to \infty \implies u, \bar{u} \to -\infty \implies \bar{u} = -\pi / 2 \quad \bar{\theta} = -\pi / 2
\]

\[
\sin (\bar{\theta} - \bar{u}) = 0 \quad \text{so \tfrac{r}{\text{hub}} is also a point}
\]

\[
\text{eg \tfrac{r}{t \text{ finite}} \& r \to \infty \implies u \to -\infty \implies \bar{u} = -\pi / 2 \quad \bar{\theta} = \pi / 2
\]

\[
\sin (\bar{\theta} - \bar{u}) = \sin (\pi) = 0 \quad \text{and scagum is a point.}
\]

\[
\text{eg \tfrac{u}{\text{finite}} \& u \to -\infty \implies \bar{u} = -\pi / 2 \quad \tfrac{r}{|\bar{\theta}| < \pi / 2}
\]

\[
\text{Notice \sin (\bar{\theta} - \bar{u}) \neq 0 \quad \text{Hence, } \mathcal{G}^+ \text{ is then the product of the open interval } |\bar{\theta}| < \pi / 2 \text{ \& } \mathbb{R} \times S^2
\]

\[
\text{\( \mathcal{G}^+ \text{ similar: } u \text{ finite } \& \bar{u} \to +\infty : |\bar{u}| < \pi / 2 \quad |\bar{\theta}| = \pi / 2
\]
Notice, e.g., for $\tilde{u} \to \frac{\pi}{2}$ that as $\tilde{u} \to -\frac{\pi}{2}$ the radius of the $S^2 \to 0$ and we reach the metric points $i^+$ and $i^-$, respectively.

5. The arrowed lines represent radial geodesics from the star to the observer on worldline $A$. Yes, they still see the star.

$\tilde{u} = \frac{\pi}{2}$

$\tilde{u} = -\frac{\pi}{2}$

$\tilde{u} = 0$

$\tilde{u} = \pi$

$\tilde{u} = 0$

$\tilde{u} = \pi$

The arrowed lines are radial null geodesics from feet to head. Yes, they see them.

$\tilde{u}$ represents worldline of head

$\tilde{u}$ represents worldline of feet.
It depends.

**Necessary Conditions:**

*No matter what $B$ does, they won't meet $A$.*

On the other hand, they can meet.

(Note: the object is accelerated.)

The worldlines of $A$ will maximize the possibility of meeting.
\[
\frac{1}{
\sqrt{g} \nabla_{\mu} \phi = \frac{1}{\sqrt{g}} \left( \frac{1}{\sqrt{-g}} \right) \nabla_{\mu} \phi \nabla_{\nu} \phi 
\]

\[
= \frac{1}{\sqrt{-g}} \left( \frac{1}{\sqrt{-g}} \right) \nabla_{\mu} \phi \nabla_{\nu} \phi 
\]

\[
= \frac{1}{2} \nabla_{\mu} \phi \nabla_{\nu} \phi 
\]

On the other hand, we have

\[
\Pi_{\mu \nu} = \frac{1}{2} g^{\rho \sigma} \left( \xi_{\rho \mu, \nu} + \xi_{\nu \mu, \rho} - \xi_{\rho \nu, \mu} \right) 
\]

antisymmetric on \( \rho \) and \( \sigma \)

\[
= \frac{1}{2} g^{\rho \sigma} \partial_{\mu} \xi_{\rho \sigma} 
\]

Hence

\[
\Pi_{\mu \nu} = \frac{1}{\sqrt{-g}} \partial_{\mu} \phi 
\]

i) \( D_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \xi_{\mu \sigma} V^{\nu} \)

\[
= \partial_{\mu} V^{\nu} + \left( \frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} \right) V^{\nu} 
\]

\[
= \frac{1}{\sqrt{-g}} \partial_{\mu} \left[ \sqrt{-g} V^{\nu} \right] 
\]

ii) \( D^{2} \phi = \xi_{\mu \nu} D_{\mu} D_{\nu} \phi \)

\[
= \partial_{\mu} \left( \xi_{\mu \nu} D_{\nu} \phi \right) 
\]

\[
= \frac{1}{\sqrt{-g}} \partial_{\mu} \left[ \sqrt{-g} \xi_{\mu \nu} D_{\nu} \phi \right] 
\]

\[
= \frac{1}{\sqrt{-g}} \partial_{\mu} \left[ \sqrt{-g} \xi_{\mu \nu} \xi_{\nu \rho} \phi \right] 
\]
\[ \partial_{\mu} F^{\mu \nu} = \partial_{\mu} F_{\nu} + \partial_{\nu} F_{\mu} + \Gamma^{\mu}_{\nu \rho} F^{\rho \mu} \]

\[ = \partial_{\mu} F_{\nu} + \partial_{\nu} F_{\mu} \]

\[ = \partial_{\mu} F_{\nu} + \frac{1}{\sqrt{g}} \partial_{\nu} \sqrt{g} F_{\mu} \]

\[ = \frac{1}{\sqrt{g}} \partial_{\nu} \left[ \sqrt{g} F_{\mu} \right] \]

Note it was crucial that \( F_{\mu} = -F_{\mu} \)
to get this cancellation, hence result.

iv) \( A^r = \phi / r \)  
\( \phi \)  
\( A^\phi = -P \cos \theta \)

\[ \Rightarrow F_{r\theta} = \partial_r A_{\theta} - \partial_{\theta} A_r = \frac{\phi}{r^2} \]

\[ F_{\theta\phi} = \partial_\theta A_\phi - \partial_\phi A_\theta = -P \sin \theta \]

\[ \text{Metric} \]

\[
\begin{aligned}
g_{rr} &= -f \\
g_{\theta\theta} &= f^{-1} \\
g_{\phi\phi} &= r^2 \\
g_{\phi\theta} &= r^2 \sin \theta \\
g_{\theta\phi} &= r^2 \sin \theta \\
\end{aligned}
\]

\[ F^\mu_{\nu} = g_{\mu\Sigma} g^{\nu\Sigma} F_{\Sigma} \]

\[ F_{r\theta} = g^{\theta\phi} g_{r\phi} F_{r\phi} = -F_{r\phi} = -\frac{\phi}{r^2} \]

\[ F_{\theta\phi} = g^{\theta\phi} g_{\phi\phi} F_{\phi\phi} = \frac{P \sin \theta}{r^4 \sin^2 \theta} = \frac{P}{r^4 \sin \theta} \]

\[ \sqrt{-g} F_{r\theta} = -r^2 \sin \theta \left( -\frac{\phi}{r^2} \right) = -\frac{\phi \sin \theta}{r^2} \]

\[ \sqrt{-g} F_{\theta\phi} = r^2 \sin \theta \frac{P}{r^4 \sin \theta} = \frac{P}{r^2} \]
Maxwell's eqns
\[ \nabla \cdot F = 0 \]
\[ \Rightarrow \ \partial_n \left( \frac{\mathbf{F} \cdot \mathbf{n}}{r} \right) = 0 \]
\[ \nu = t \quad \partial_n \left( \frac{\mathbf{F} \cdot \mathbf{t}}{r} \right) = \partial_r \left( \frac{\mathbf{F} \cdot \mathbf{t}}{r} \right) = 0 \]
\[ \nu = r \quad \partial_n \left( \frac{\mathbf{F} \cdot \mathbf{r}}{r} \right) = \partial_t \left( \frac{\mathbf{F} \cdot \mathbf{t}}{r} \right) = 0 \]
\[ \nu = \theta \quad \partial_n \left( \frac{\mathbf{F} \cdot \mathbf{\theta}}{r} \right) = \partial_t \left( \frac{\mathbf{F} \cdot \mathbf{\theta}}{r} \right) = 0 \]
\[ \nu = \phi \quad \partial_n \left( \frac{\mathbf{F} \cdot \mathbf{\phi}}{r} \right) = \partial_t \left( \frac{\mathbf{F} \cdot \mathbf{\phi}}{r} \right) = 0 \]
\[
\text{Schwarzschild: } \quad ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{2\mu}{r} dr^2 + r^2 d\Omega^2
\]

\[
\xi = v - r
\]

\[
ds^2 = d\xi^2 - dr^2 \quad \quad d\Omega = (df + dr)
\]

\[
ds^2 = -\left(1 - \frac{2M}{r}\right) \left( dt^2 + dr^2 + 2d\xi dr \right) + 2 dt dr + 2r^2 d\Omega
\]

\[
+ r^2 d\Omega
\]

\[
= -\left( dt^2 + dr^2 + 2d\xi dr \right) + 2 dt dr + 2r^2 d\Omega
\]

\[
+ \frac{2M}{r} \left( df + dr^2 + 2d\xi dr \right) + r^2 d\Omega
\]

\[
ds^2 = -dt^2 + dr^2 + r^2 d\Omega
\]

\[
+ \frac{2M}{r} \left( df + dr \right)^2
\]

We now introduce cartesian co-ords \( \{x + iy = r e^{i\phi} \sin \theta \} \)

\[
\begin{align*}
& x + iy = (dr + icd\phi) e^{i\phi} \\
& dx + idy = dr d\phi
\end{align*}
\]

\[
-\quad dx + idy = (d\cos \theta + r \cos \theta d\phi + r \sin \theta d\phi) e^{i\phi}
\]

\[
dt = dr \cos \theta - r \sin \theta d\phi
\]
\[ dx^2 + dy^2 = d\rho^2 + \rho^2 \sin^2 \phi \, d\phi^2 + \rho^2 \cos^2 \phi \, d\theta^2 - 2 \rho \sin \phi \cos \phi \, d\rho \, d\phi \]
\[ = dx^2 + dy^2 + \rho^2 \, d\phi^2 \]
\[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \frac{2m}{\rho} (d\phi + dr)^2 \]
\[ r = r(x,y,t) \quad \text{via} \quad x^2 + y^2 + z^2 \]
\[ \Rightarrow dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy + \frac{\partial r}{\partial t} dt \]

In these coord's
\[ \partial \mu = \partial \nu + f K_\mu K_\nu \]

where
\[ K_\mu = \left( 1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) \quad \text{and} \quad f = \frac{2m}{r} \]

First check conditions with respect to \( \partial \nu \): 1) Notice \( k^2 = -1 + \frac{x^2 + y^2 + z^2}{r^2} = 0 \quad \text{Null} \)

2) \[ k^\mu \partial_\phi k_\mu = k^\mu \partial_\phi k_\mu = \partial_\phi k_\mu + \frac{x}{r} \partial_x k_\mu + \frac{y}{r} \partial_y k_\mu + \frac{z}{r} \partial_z k_\mu \]

\[ = 0 \]
\[ \mu = 4: \quad k^4 \rho \cdot k = 0 \]

\[ k^4 = 1 \]

\[ \mu = \chi: \quad k \cdot k = \frac{x}{\rho} \]

\[ k^4 \rho \cdot k = \frac{x}{\rho} \cdot \left( \frac{1}{\rho} - \frac{x}{\rho^2} \right) + \frac{y}{\rho} \cdot \left( \frac{1}{\rho} - \frac{x}{\rho^2} \right) + \frac{z}{\rho} \cdot \left( \frac{1}{\rho} - \frac{x}{\rho^2} \right) \cdot \frac{1}{\rho} \cdot \frac{1}{\rho} \]

\[ = \frac{x}{\rho} - \frac{x^3}{\rho^4} - \frac{xy}{\rho^4} - \frac{yz}{\rho^4} \]

\[ = \frac{x}{\rho} - \frac{x \cdot \frac{1}{\rho^4}}{\frac{1}{\rho^4}} \]

\[ = \frac{x}{\rho} - x \cdot \frac{1}{\rho^4} \]

\[ = 0 \]

Similarly for \( \mu = y \) and \( \beta \) by symmetry.

Finally, check conditions with respect to \( g_{\mu \nu} \).

Firstly, \( g_{\mu \nu} = \eta_{\mu \nu} - f k^\mu k^\nu \)

where \( k^\mu = \eta_{\mu \nu} k^\nu \)

Check \( g_{\mu \nu} \partial \nu = (\eta_{\mu \nu} - f k^\mu k^\nu) (\partial \nu + f k^\nu k_\nu) \)

\[ = \delta^\nu \partial^\nu + f k^\mu k_\nu - f k^\nu k_\nu + 0 \]

\[ = \delta^\nu \partial^\nu \]

Thus, \( k^\mu = \eta_{\mu \nu} k^\nu = g_{\mu \nu} k^\nu \) also.

\[ k^2 = \eta_{\mu \nu} \eta_{\nu \sigma} k^\mu k_\sigma \]

\[ = (\eta_{\mu \nu} - f k^\mu k^\nu) (1 + f k^\nu k_\nu) \]

\[ = \eta_{\mu \nu} k^\mu k_\nu \]

\[ = 0 \]
\[ k^a \partial_a k^m = k^a (\partial_a k^m - \gamma_{a}^{\mu} k_\mu) \]

We have shown \( k^a \partial_a k^m = 0 \) already. So, just have to show

\[ k^a k_a \gamma_\mu = 0 \]

\[ \Rightarrow k^a k_a \frac{1}{2} g_{\mu\delta} (g_{\delta \mu} + \gamma_{\mu \delta} \gamma - \gamma_{\mu \nu} \gamma_{\nu \delta}) = 0 \]

\[ \Rightarrow k^a k_a \left( g_{\delta \mu} + \gamma_{\mu \delta} \gamma - g_{\mu \nu} \gamma_{\nu \delta} \right) = 0 \]

\[ \text{symmetric on } \gamma \text{ and antisymmetric on } \gamma \]

\[ \Rightarrow k^a k_a \frac{\partial}{\partial x^\mu} (\gamma_{\delta \mu}) = 0 \]

\[ \Rightarrow k^a k_a \frac{\partial}{\partial x^\mu} (\eta_{\delta \mu} + f_{k^a k^b}) = 0 \]

\[ \Rightarrow \frac{\partial}{\partial x^\mu} (f_{k^a k^b}) = 0 \]

and this is true since \( k^a = 0 \).
(1) \[ (A \wedge B)_{\nu_1, \ldots, \nu_p, \nu_r, \ldots, \nu_q} = \frac{(p+q)!}{p! q!} \left( A_{\nu_1, \ldots, \nu_p} B_{\nu_r, \ldots, \nu_q} \right) \]

\[= \frac{(p)! (q)!}{p! q!} \left( B_{\nu_r, \ldots, \nu_q} A_{\nu_1, \ldots, \nu_p} \right) \]

\[= \frac{(p+q)!}{p! q!} \left( B_{\nu_r, \ldots, \nu_q} A_{\nu_1, \ldots, \nu_p} \right) \]

\[= (B \wedge A)_{\nu_r, \ldots, \nu_q, \nu_1, \ldots, \nu_p} \]

\[= (B \wedge A)_{\nu_1, \ldots, \nu_p, \nu_r, \ldots, \nu_q} \quad \text{by } (1) \]

\[\Rightarrow A \wedge B = (1) \left. \rho \right|_{\nu_1, \ldots, \nu_p} B \wedge A \]

(ii) 
\[
d(A)_{\nu_1, \nu_r, \nu_{r+2}} = (p+2) \left. \rho \right|_{\nu_1, \nu_r} dA_{\nu_2, \ldots, \nu_{r+2}} \]

\[= (p+2) (p+1) \left. \rho \right|_{\nu_1, \nu_r} dA_{\nu_2, \ldots, \nu_{r+2}} \]

\[= 0 \quad \text{since } dA_{\nu_1, \nu_r} \text{ is symmetric on interchanging } \nu_1, \nu_2 \]

Note in the first step, we know \( dA \) is a \((p+1)\)-form and we take the \( d \) of that object.
\[(A \land B)_{\mu_1 \ldots \mu_p, v_1 \ldots v_q} = \frac{(p+q)!}{p!q!} A[\mu_1 \ldots \mu_p, v_1 \ldots v_q]\]

\[d(A \land B)_{\mu_1 \ldots \mu_p, v_1 \ldots v_q} = (p+q+1) \frac{\partial}{\partial \psi} (A \land B)_{\mu_1 \ldots \mu_p, v_1 \ldots v_q}\]

\[= (p+q+1) \frac{\partial}{\partial \psi} \left( \frac{(p+q)!}{p!q!} A[\mu_1 \ldots \mu_p, v_1 \ldots v_q] \right)\]

\[+ \frac{(p+q)!}{p!q!} \frac{\partial}{\partial \psi} \left( A[\mu_1 \ldots \mu_p, v_1 \ldots v_q] \right)\]

\[= \frac{(p+q+1)}{p!q!} \left[ \frac{(p+1)! q!}{(p+q+2)! (p+1)} (dA \land dB)_{\mu_1 \ldots \mu_p, v_1 \ldots v_q} \right.\]

\[\left. + \frac{p!(p+1)!}{(p+q+2)! (q+1)} (A \land dB)_{\mu_1 \ldots \mu_p, v_1 \ldots v_q} \right]\]

Where we call \[(dA)_{\mu_1 \ldots \mu_p} = (p+1) \frac{\partial}{\partial \psi} A[\mu_1 \ldots \mu_p]\]

\[(dB)_{v_1 \ldots v_q} = (q+1) \frac{\partial}{\partial \psi} B[v_1 \ldots v_q]\]

\[= (d(A \land B))_{\mu_1 \ldots \mu_p, v_1 \ldots v_q} + (A \land dB)_{\mu_1 \ldots \mu_p, v_1 \ldots v_q} \]

\[\text{(-1)}^p (A \land dB)_{\mu_1 \ldots \mu_p, v_1 \ldots v_q}\]

\[\Rightarrow A \land B = dA \land B + (-1)^p A \land dB\]
\[(\star \star A)_{\mu_1 \ldots \mu_p} = \frac{1}{(n-p)!} \sum_{\rho} \frac{\varepsilon_{\mu_1 \ldots \mu_p} \varepsilon_{\mu_{p+1} \ldots \mu_n}}{\rho_1!} \prod_{i=1}^p \varepsilon_{\rho_i \mu_{p+i} \ldots \mu_{n}} A_{\nu_1 \ldots \nu_p}^\rho \]

\[
= \frac{1}{(n-p)!} \frac{1}{\rho!} \sum_{\rho} \varepsilon_{\mu_1 \ldots \mu_p} \varepsilon_{\mu_{p+1} \ldots \mu_n} (-1)^{p \cdot (n-p)} \prod_{i=1}^p \varepsilon_{\rho_i \mu_{p+i} \ldots \mu_{n}} A_{\nu_1 \ldots \nu_p}^\rho
\]

\[
= \frac{1}{(n-p)!} \frac{1}{\rho!} \sum_{\rho} (-1)^{p \cdot (n-p)} \cdot \left( \pm \right) \rho \cdot (n-p)! \sum_{\rho} A_{\nu_1 \ldots \nu_p}^\rho
\]

\[
= \pm (-1)^{p \cdot (n-p)} A_{\mu_1 \ldots \mu_p}
\]

\[
\Rightarrow \star \star A = \pm (-1)^{p \cdot (n-p)} A
\]
Again, work "from the left" i.e., take $x$ of the $(n-p+1)$-form $(\wedge^A)^n x$.

$$\begin{align*}
(1) \quad (x \wedge A)_{\mu_1 \ldots \mu_{p-1}} = & \frac{1}{(n-p+1)!} \sum_{\nu_1 \ldots \nu_{n-p}} (x \wedge A)_{\nu_1 \ldots \nu_{n-p}} \\
= & \frac{1}{(n-p+1)!} \sum_{\nu_1 \ldots \nu_{n-p}} (x \wedge A)_{\nu_1 \ldots \nu_{n-p}} \\
= & \frac{1}{(n-p)!} \sum_{\nu_1 \ldots \nu_{n-p}} (x \wedge A)_{\nu_1 \ldots \nu_{n-p}} \\
= & \frac{1}{(n-p)!} (-1)^{(n-p)P} \sum_{\nu_1 \ldots \nu_{n-p}} (x \wedge A)_{\nu_1 \ldots \nu_{n-p}} \\
= & \pm (-1)^{(n-p)P} \sum_{\nu_1 \ldots \nu_{n-p}} (x \wedge A)_{\nu_1 \ldots \nu_{n-p}} \\
= & \pm (-1)^{(n-p)P} \sum_{\nu_1 \ldots \nu_{n-p}} (x \wedge A)_{\nu_1 \ldots \nu_{n-p}} \\
\end{align*}$$