\( (\psi^1, \frac{d}{dt}) = z (\psi, 1) \)  
\[ d^3 x \left[ a_1 \psi + a_2 \frac{\partial \psi}{\partial x^1} \right] \]  
\[ = \frac{1}{s} d^3 x \sum a_2 \left( \frac{\psi, \frac{d}{dt}}{\frac{\partial \psi}{\partial x^1}} \right) \]  

since  
\( (\psi^1, \psi^1) = 0 \)  
\[ = a P \]  
\[ (\psi^1, \psi) = \frac{1}{s} d^3 x \sum a_2 \left( \frac{\psi, \frac{d}{dt}}{\frac{\partial \psi}{\partial x^1}} \right) \]  
\[ = -a P \]  
\[ \left[ a_1, a_2 \right] = -\left[ (\psi, \frac{d}{dt}), (\psi^1, \frac{d}{dt}) \right] \]  
\[ = - (i) \left[ \left( \psi, \frac{d}{dt} \right) - \left( \psi^1, \frac{d}{dt} \right) \right] \]  
\[ = \left[ \left( \psi, \frac{d}{dt} \right) - \left( \psi^1, \frac{d}{dt} \right) \right] \]  

only non-zero commutator:  
\[ \left[ \psi, \psi^1 \right] = i s^3 (\psi, \psi^1) \]  
\[ = \frac{1}{s} d^3 x \left[ \psi^1 \frac{d}{dt} \psi^1 - \psi \frac{d}{dt} \psi \right] \]  
\[ = \frac{1}{s} d^3 x \left[ \psi^1 \frac{d}{dt} \psi^1 - \psi \frac{d}{dt} \psi \right] \]  
\[ = \frac{1}{s} d^3 x \left[ \psi^1 \frac{d}{dt} \psi^1 - \psi \frac{d}{dt} \psi \right] \]  
\[ = \frac{1}{s} d^3 x \left[ \psi^1 \frac{d}{dt} \psi^1 - \psi \frac{d}{dt} \psi \right] \]  
\[ = \frac{1}{s} d^3 x \left[ \psi^1 \frac{d}{dt} \psi^1 - \psi \frac{d}{dt} \psi \right] \]  
\[ = \frac{1}{s} d^3 x \left[ \psi^1 \frac{d}{dt} \psi^1 - \psi \frac{d}{dt} \psi \right] \]
\[
\begin{align*}
\mathcal{L} &= \int d^3x \, d^3y \left\{ \rho^+(t, x) \, \dot{\psi}_-(t, x) + \rho^-(t, x) \, \dot{\psi}_+(t, x) + \delta^3(x - y) \left\{ \psi_+(t, x) \, \dot{\psi}_-(t, x) + \psi_-(t, x) \, \dot{\psi}_+(t, x) \right\} \right\} \\
&= i \int d^3x \, d^3y \left\{ \psi_+(t, x) \, \partial_t \psi_-(t, x) + \psi_-(t, x) \, \partial_t \psi_+(t, x) \right\} \\
&= (\psi_+, \psi_-) \\
&= \delta^3(x - y)
\end{align*}
\]

\[
\left[ a_\ell, a_{\ell'} \right] = \int d^3x \, d^3y \left[ \psi_+^\dagger(t, x) \, \dot{\phi}(t, x) - \phi(t, x) \, \dot{\psi}_+^\dagger(t, x) \right] \\
&= \int d^3x \, d^3y \left\{ \psi^\dagger_+(t, x) \, \dot{\phi}(t, x) - \phi(t, x) \, \dot{\psi}_+^\dagger(t, x) \right\} \\
&= \int d^3x \, d^3y \left\{ \psi_+^\dagger(t, x) \, \partial_t \psi_+(t, x) - \partial_t \psi_+^\dagger(t, x) \right\}
\]
\[ a_i = \sum_k (A_{ik} a_k - B_{ik} a_k^+) \]
\[ a_i^+ = \sum_k A_{ik} a_k^+ - B_{ik} a_k \]
\[ [a_i^+, a_j^+] = \sum_{k\ell} \left[ A_{ik} a_k - B_{ik} a_k^+ , A_{\ell j} a_{\ell} - B_{\ell j} a_{\ell}^+ \right] \]

only nonzero commutator is
\[ [a_k, a_k^+] = \delta_{k\ell} \]

\[ d \cdot [a_k^+, a_\ell] = -\delta_{k\ell} \]

\[ = \sum_{k\ell} \left( A_{ik} A_{\ell j} \delta_{k\ell} - B_{ik} B_{\ell j} \delta_{k\ell} \right) \]
\[ = \sum_k \left( \underbrace{A_{ik} A_{jk}}_{(AA^+)} - \overbrace{B_{ik} B_{jk}}^{(BB^+)} \right) \]
\[ = \left[ (AA^+) - (BB^+) \right]_{ij} \]
\[ = S_{ij} \quad \text{given condition} \]
\[ [a_i^+, a_j^+] = \sum_{k\ell} \left[ A_{ik} a_k - B_{ik} a_k^+ , A_{\ell j} a_{\ell} - B_{\ell j} a_{\ell}^+ \right] \]
\[ = \sum_{k\ell} \left( -A_{ik}^* B_{\ell j} + B_{ik}^* A_{\ell j}^* \right) \delta_{k\ell} \]
\[ = \sum_k \left( -A_{ik}^* B_{jk} + A_{jk}^* B_{ik}^* \right) \]
\[ = \left( -B^* A^+ + A^* B^+ \right)_{ji} \]
Hence

$[a'; q'] = 0 \Rightarrow -b^+A^+ + A^+B^+ = 0$

$\Leftrightarrow -AB^T + BAT = 0$

by taking the Hermitian conjugate.
\[
\begin{align*}
\text{ds}^2 &= \left(1 - \frac{2M}{r}\right) \left(dt^2 + dr^2 + r^2 d\Omega^2\right) \\
D^2 \Phi &= \frac{1}{\Theta} \frac{\partial}{\partial \nu} \left(G_{\nu \mu} \frac{\partial \Phi}{\partial \nu}\right) \\
\sqrt{-g} &= \left(1 - \frac{2M}{r}\right) r^2 \sin \Theta \quad \text{depends on r + 0} \\
\text{So } 0 &= D^2 \Phi = \frac{1}{\Theta} \frac{\partial}{\partial \nu} \left(G_{\nu \mu} \frac{\partial \Phi}{\partial \nu}\right) \\
\Rightarrow 0 &= g_{tt} \frac{\partial^2 \Phi}{\partial t^2} + g_{\phi \phi} \frac{\partial^2 \Phi}{\partial \phi^2} \\
&\quad + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 g_{\phi \phi} \frac{\partial \Phi}{\partial r}\right) + \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left(\sin \Theta g_{\phi \phi} \frac{\partial \Phi}{\partial \Theta}\right) \\
\Rightarrow 0 &= -\left(1 - \frac{2M}{r}\right) \frac{\partial^2 \Phi}{\partial t^2} + \frac{1}{r^2 \sin \Theta} \frac{\partial^2 \Phi}{\partial \phi^2} \\
&\quad + \frac{1}{r^2 \left(1 - \frac{2M}{r}\right)} \frac{\partial}{\partial r} \left(r^2 \left(1 - \frac{2M}{r}\right) \frac{\partial \Phi}{\partial r}\right) \\
&\quad + \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left(\sin \Theta r^2 \frac{\partial \Phi}{\partial \Theta}\right) \\
\Rightarrow \frac{\partial^2 \Phi}{\partial t^2} &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r}\right) \\
&\quad + \frac{1}{r^2} \left(1 - \frac{2M}{r}\right) \left[\frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left(\sin \Theta \frac{\partial \Phi}{\partial \Theta}\right) + \frac{1}{\sin^2 \Theta} \frac{\partial^2 \Phi}{\partial \phi^2}\right]
\end{align*}
\]

Solve by separation of variables.

E.g. \( \Phi = \chi(t) \psi(r, \Theta, \phi) \)

Then

\[
\frac{\partial^2 \chi}{\chi} = \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2} \left(1 - \frac{2M}{r}\right) \left[\frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left(\sin \Theta \frac{\partial \psi}{\partial \Theta}\right) + \frac{1}{\sin^2 \Theta} \frac{\partial^2 \psi}{\partial \phi^2}\right]\right\}
\]

\( \psi \)
EHS & RHS are both constants. \(-w^2\) \(sgn\)

Solve:

\[
\frac{\partial^2 X}{\partial t^2} = -w^2 \Rightarrow X = e^{-iwt}
\]

\(\Phi\)

Similarly, write

\[
\Phi = \frac{1}{r} \text{Re} w(r) Y_{\ell m}(\theta, \phi)
\]

where \(Y_{\ell m}\) satisfy

\[
\frac{1}{\sin \theta} \partial_\theta \left[ \sin \theta \partial_\theta Y_{\ell m} \right] + \frac{\ell^2}{\sin^2 \theta} Y_{\ell m} = -\ell(\ell + 1) Y_{\ell m}
\]

then we have

\[
\frac{1}{r^2} \partial^2 r \left[ \frac{1}{r^2} \partial r \left( \frac{1}{r} R \right) \right] + \frac{1}{r^2} \left( 1 - 2\ell \right) \frac{R}{r^2} \partial \Phi \left( e^{i\ell \Phi} \right) + \frac{w^2 R}{r^2} = 0
\]

\[
\frac{1}{r^2} \partial^2 r \left[ \frac{1}{r^2} \partial r \left( \frac{1}{r} R \right) \right] + \frac{1}{r^2} \left( 1 - 2\ell \right) \frac{R}{r^2} \partial \Phi \left( e^{i\ell \Phi} \right) + \frac{w^2 R}{r^2} = 0
\]

since \(\frac{dr}{dr} = 1 - 2\ell\)

Notice that \(R\) will only depend on \(w, \ell\) not \(m\) so have \(\text{Re} w\) in fact

\[
\Rightarrow \frac{1}{r^2} \partial^2 r \left[ \frac{1}{r^2} \partial r \left( \frac{1}{r} R \right) \right] + \frac{w^2 R}{r^2} = \frac{R}{r^3} \left( 1 - 2\ell \right) \frac{\ell(\ell + 1)}{r^2}
\]

\[
\frac{1}{r} (\partial^2 r + w^2 R) = \frac{R}{r^3} \left( 1 - 2\ell \right) \frac{\ell(\ell + 1)}{r^2} + \frac{2MR}{r^4} \left( 1 - 2\ell \right)
\]

\[
\Rightarrow \frac{1}{r} \left( \partial^2 r + w^2 R \right) = \frac{R}{r^3} \left( 1 - 2\ell \right) \frac{\ell(\ell + 1)}{r^2} + \frac{2MR}{r^4} \left( 1 - 2\ell \right)
\]

\[
\Rightarrow (\partial^2 r + w^2) \text{Re} w = \left( 1 - 2\ell \right) \left[ \frac{\ell(\ell + 1)}{r^2} + \frac{2M}{r^3} \right] \text{Re} w
\]

\[
\Phi = \sum_{\ell m} C_{\ell m} \frac{1}{r} \text{Re} w(r) Y_{\ell m}(\theta, \phi) e^{-i\ell \Phi}
\]

\(\Phi\) are constants.
Klein Gordon Eqn.

\[ \partial^2 \phi - m^2 \phi = 0 \]
\[ \Rightarrow \Box \phi = \partial^2 \phi - m^2 \phi = 0 \]

So we want to show that \( \Box \phi = \partial^2 \phi \) is true.

To show this first write \( \Box \phi = \frac{1}{\sqrt{g}} \partial_m (\sqrt{g} g^{mn} \partial_n \phi) \).

Next choose local coordinates \( x^a \) so that \( \Box \phi = \partial^2 \phi \) at one point of the manifold.

And we have \( \partial_a g^{mn} = 0 \) in these coordinates.

Thus \( \Box \phi = \partial^2 \phi \) in the special coordinates.

\[ \Box \phi = \partial_a \left[ \frac{1}{\sqrt{g}} \partial_m (\sqrt{g} g^{mn} \partial_n \phi) \right] \]
\[ = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{mn} \partial_n (\partial^a \phi)) \]
\[ = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{mn} \partial_n (\partial^2 \phi)) \]
\[ = \partial^2 (\partial^a \phi) \]

We have shown the result is true in these coordinates. But since the coordinates are independent of coordinates, it must be true for all coordinates.
\[(L_k f, g) = \int d^3x \left( L_k f^* \delta_t g \right) \]
\[= \int d^3x \left[ \Delta_k f^* \delta_t g - g \partial_t (L_k f^*) \right] \]
\[L_k f = \partial_t f \]
\[(L_k f, g) = i \int d^3x \left[ \partial_t f^* \delta_t g - \gamma \partial_t (\partial_t f^*) \right] \]
\[- \partial_t (f^* \delta_t g) \]
\[= -i \int d^3x \left( f^* \delta_t g + \delta_t g \left( \partial_t f^* \right) \right) \]
\[+ i \int d^3x \left( \partial_t (f^* \delta_t g) - \partial_t g \left( \partial_t f^* \right) \right) \]
\[= - (f, L_k g) \]
\[+ i \int d^3x \left( \partial_t (f^* \delta_t g) - \partial_t g \left( \partial_t f^* \right) \right) \]
\[+ f^* \partial_t \left( \delta_t g \right) - \delta_t g \left( \partial_t f^* \right) \]
\[= - (f, L_k g) \]
\[+ i \int d^3x \left( f^* \left( \partial_t \delta_t g \right) - \partial_t g \left( \partial_t f^* \right) \right) \]
\[i \int d^3x \left( f^* \delta_t g - \delta_t f^* \right) \]
\[i \int d^3x \left( 2f^* \delta_t g - \partial_t \right) \]
\[\Rightarrow (L_k f, g) = - (f, L_k g) \]

\[\text{Vanishes as total spatial derivative} \]

\[\Rightarrow (L_k f, g) = - (f, L_k g) \]
\[ \tilde{g}_{w}(w) = \int_{-\infty}^{\infty} \frac{1}{\lambda} e^{-\lambda x} (\frac{i}{w} - i \lambda \log(1 + i)) \]

For \( w > 0 \) then the integrand is exponentially suppressed in the upper half of the complex \( w \)-plane.

Then consider contour:

where we are closing the semi-circular arc on a limit at \( \infty \).

As integrand has no poles inside this contour, use Cauchy theorem to argue that \( \int_{c} = 0 \). In addition the integral \( \int_{c} \) vanishes because of exponential fall off of integrand.

Hence \( \int_{-} = -\int_{u} = \int_{c} \)

ie. We rotate contour clockwise.
So let's do this a with $v = i\omega$

\[ w > 0 \]

\[
\tilde{\delta}_{w}^T(\omega) = \int_{-\infty}^{\infty} \frac{dx}{x} \exp \left( i\omega \log(-ix) \right)
\]

\[ = -i \int_{0}^{\infty} dx \exp \left( -\omega x + i\omega \frac{\log x}{k} \right) \]

\[ = -i \exp \left( \frac{-\omega \pi}{2k} \right) \int_{0}^{\infty} dx \exp \left( -\omega x + i\omega \frac{\log x}{k} \right) \]

For $w < 0$

\[
\tilde{\delta}_{w}^T(\omega) = \int_{-\infty}^{\infty} \frac{dx}{x} \exp \left( i\omega \log(-ix) \right)
\]

\[ = -i \int_{0}^{\infty} dx \exp \left( \omega x + i\omega \frac{\log x}{k} \right), \text{ anticlockwise} \]

\[ = i \int_{0}^{\infty} dx \exp \left( \omega x + i\omega \frac{\log x + \pi i}{k} \right) \]

\[ = i \exp \left( -\frac{\omega \pi}{2k} \right) \int_{0}^{\infty} dx \exp \left( \omega x + i\omega \frac{\log x}{k} \right) \]

Thus with

\[ w' > 0, \quad \tilde{\delta}_{w'}^T(\omega') = i \exp \left( -\frac{\omega \pi}{2k} \right) \int_{0}^{\infty} dx \exp \left( -\omega x + i\omega \frac{\log x}{k} \right) \]

\[ = - \exp \left( -\frac{\omega \pi}{2k} \right) \tilde{\delta}_{w}^T(\omega') \]
\[ T = \frac{K}{2\pi} \quad \left[ K \right] = \frac{L}{T^2} \quad \text{acceleration} \]

Know \( \int k_B T \text{d}T = \text{Energy} = ML^2T^{-2} \)

Also \( \left[ h \right] = \text{Energy} \cdot T \Rightarrow \left[ \frac{hK}{c} \right] = \text{Energy} \)

Hence

\[
T = \frac{hK}{k_B c 2\pi}
\]

\((h\) appears as it involves quantum physics\)

For Schwarzschild

\[ K = \frac{1}{4M} \]

\[
\begin{align*}
\left[ c^2 \right] &= L^3M^{-1}T^{-2} \\
\left[ c \right] &= LT^{-1} \\
\left[ \frac{G}{c^4} \right] &= M^{-1}T^2L^{-1}
\end{align*}
\]

\[ \Rightarrow \left[ \frac{c^4}{GM} \right] = M^{-1}T^{-2}L^{-1} = \text{acceleration} \]

\[ \Rightarrow K = \frac{c^4}{4GM} \quad \text{(no } h\text{ appears here, just } G\text{)} \]

\[ \Rightarrow T_{\text{Sch}} = \frac{h}{2\pi c} \frac{c^4}{8\pi GM} = \frac{h c^3}{2\pi GM \Omega} \left( \frac{1}{M/M_0} \right) \]

\[ = 6 \times 10^{-8} K \left( \frac{1}{M/M_0} \right) \]
\[ S_{BH} = \frac{1}{4} A \]

Entropy \( \sim k_b \ln \frac{W}{\text{# of states}} \)

\[ \Rightarrow S_{BH} = k_b \frac{1}{4} \frac{A}{L^2} \quad L^2 = \frac{G\hbar}{c^3} \]

\[ = k_b \frac{1}{4} A \left( \frac{c^3}{G\hbar} \right) \]

For Schwarzschild, \( A = 16\pi M^2 \quad (\omega = h = G = c = 1) \)

\[ \Rightarrow A = k_B \frac{G^2 M^2}{c^4} \]

\[ \Rightarrow S_{BH} = k_B \frac{4\pi G^2 M^2}{c^4} \]

\[ = k_B \frac{4\pi G}{c^4} M_0^2 \left( \frac{M}{M_0} \right)^2 \]

\[ = k_B \left( \frac{10^7}{M_0} \right)^2 \]

\( \sim \log \text{ of number of states.} \)
outside reservoir is at fixed temperature $Th = \frac{1}{\delta t/m}$

the black hole a reservoir exchange energy, but the temperature of the reservoir stays fixed because it would take an infinite amount of energy to be emitted or absorbed by the black hole to change it.

If the black hole, by a small thermal fluctuation, radiates a small amount more than it absorbs, it will lose mass & its temperature will increase. Now it is hotter than the reservoir and more energy will flow from the black hole to the reservoir. The black hole loses more mass & gets hotter still & there is runaway evaporation.

Conversely, if the black hole absorbs a small amount of net energy in a thermal fluctuation, it will increase its mass & get cooler. This causes more energy to flow from the hotter reservoir to the black hole & it will get cooler still - with an infinite reservoir the black hole will grow without bound.
Now consider the black hole in a finite box of radiation

Entropy of system:

\[ S = S_{BH} + S_{res} \]

\[ = 4\pi M^2 + \frac{4\pi \sigma VT^3}{3} \]

Total energy is fixed: \( \text{d}E = 0 \)

\[ \Rightarrow \text{d}(M + E_{\text{res}}) = 0 \]

\[ \Rightarrow \text{d}M + \text{d}(\sigma VT^4) = 0 \]

\[ \Rightarrow \text{d}M + 4\sigma VT^2 \text{d}T = 0 \]

Now \( \text{d}s = \frac{8\pi M \text{d}m + 4\sigma VT^2 \text{d}T}{8\pi M (-4\sigma VT^3 \text{d}T)} \)

Hence \( \text{d}s = 0 \iff \frac{8\pi M T^3}{T} = T^2 \)

\[ \Rightarrow T = \frac{1}{8\pi M} = T_H. \]

To get the second derivative, consider derivatives with respect to a parameter \( \lambda \):

\[ \frac{\text{d}E}{\text{d}\lambda} = 0 = \frac{\text{d}M}{\text{d}\lambda} + 4\sigma VT^3 \frac{\text{d}T}{\text{d}\lambda} \]

\[ \frac{\text{d}s}{\text{d}\lambda} = 4\sigma VT^2 \left( 1 - 8\pi MT \right) \frac{\text{d}T}{\text{d}\lambda} \]

\[ \frac{\text{d}^2s}{\text{d}\lambda^2} = \frac{\text{d}}{\text{d}\lambda} \left[ 4\sigma VT^3 \left( 1 - 8\pi MT \right) \right] \frac{\text{d}T}{\text{d}\lambda} + \left[ 4\sigma VT^2 \frac{\text{d}T}{\text{d}\lambda} \right] \frac{d}{\text{d}\lambda} \left( 1 - 8\pi MT \right) \]

In equilibrium \( \frac{\text{d}s}{\text{d}\lambda} = 0 \iff T = \frac{1}{8\pi M} \)

\[ \Rightarrow \left. \frac{\text{d}^2s}{\text{d}\lambda^2} \right|_{T_H} = 32\pi \sigma \alpha V T^2 \frac{\text{d}T}{\text{d}\lambda} \left( \frac{M}{\text{d}\lambda} + \frac{T}{\text{d}\lambda} \frac{\text{d}M}{\text{d}\lambda} \right) \]

\[ \left. \frac{\text{d}^2s}{\text{d}\lambda^2} \right|_{T_H} = \frac{1}{8\pi M} \left( -4\sigma VT^3 \frac{\text{d}T}{\text{d}\lambda} \right) \]
\[ \frac{d^2 s}{dt^2} = -32\pi \delta V T^2 \left( \frac{dT}{dV} \right)^2 \left( \frac{V}{8\pi T} - 4\delta V T^4 \right) \]

\[ \Rightarrow V_c = \frac{1}{32\pi \delta T^5} \]

- minima

with \[ \frac{d^3 s}{dV^3} \bigg|_{TH} > 0 \] \(\Rightarrow V > V_c\) a hence unstable equilibrium,

\[ \frac{d^3 s}{dV^3} \bigg|_{TH} < 0 \] \(\Rightarrow V < V_c\) a hence stable equilibrium

(Note that from 1st part of question we expect that when \(V\) is very large a infinite reservoir + is unstable)

We can express \(V_c\) in terms of \(E\) too.

In \(E\) we have

\[ E = M + \delta V T^4 = \frac{1}{8\pi T} + \delta V T^4 \]

hence when \(V = V_c\)

\[ E = \frac{1}{8\pi T} (\delta + \frac{1}{2}) = \frac{5}{25\pi T} \quad \text{at } V = V_c \]

\[ \Rightarrow V_c = \frac{1}{25\pi T} \left( \frac{25\pi T}{5} \right)^5 = \frac{2^{20}\pi^4 E^5}{550} \]

When \(V\) of the black hole passes from \(V > V_c\) to \(V < V_c\), the thermal equilibrium becomes stable (at vice-versa)

Conclusion: the black hole can be in stable equilibrium with a small enough box so that if the black hole absorbs some energy and cools down, the thermal gas cools down to losing energy and its temperature drops more than the black hole.
\[ A = 4\pi \rho_+^2 \quad \rho_+ = M + \sqrt{M^2 - Q^2} \]

\[ \Rightarrow S = \frac{1}{4} A = \pi \left( M + \sqrt{M^2 - Q^2} \right)^2 \]

\[ T = \frac{K}{2\pi} = \frac{\rho_+ - \rho_-}{2\rho_+} = \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2} \]

\[ \frac{\partial S}{\partial T} \bigg|_Q = \frac{\partial S}{\partial M} \frac{\partial M}{\partial T} \bigg|_Q \]

\[ \frac{\partial S}{\partial M} \bigg|_Q = 2\pi \left( 1 + \frac{M}{\sqrt{M^2 - Q^2}} \right) \left( M + \sqrt{M^2 - Q^2} \right) \]

\[ \frac{\partial M}{\partial T} \bigg|_Q = \left( \frac{\partial T}{\partial M} \right)^{-1} \]

\[ \frac{\partial T}{\partial M} \bigg|_Q = \frac{M}{\sqrt{M^2 - Q^2}} \left( \frac{1}{(M + \sqrt{M^2 - Q^2})^2} - \frac{2\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^3} \right) \]

\[ = \frac{1}{\sqrt{M^2 - Q^2}} \left( \frac{1}{(M + \sqrt{M^2 - Q^2})^2} - \frac{2\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^3} \right) \]

\[ \Rightarrow \frac{\partial S}{\partial T} \bigg|_Q = 2\pi \frac{\sqrt{M^2 - Q^2 + M}}{\sqrt{M^2 - Q^2}} \frac{\sqrt{M^2 - Q^2} \left( M + \sqrt{M^2 - Q^2} \right)}{(M + \sqrt{M^2 - Q^2})^2} \]

\[ \Rightarrow \frac{T \partial S}{\partial T} \bigg|_Q = \sqrt{M^2 - Q^2} \left( \frac{2\pi \left( M + \sqrt{M^2 - Q^2} \right)}{(M + \sqrt{M^2 - Q^2})^2} - \frac{2\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^3} \right) \]

\[ \Rightarrow C = 2\pi \frac{\sqrt{M^2 - Q^2}}{(M - 2\sqrt{M^2 - Q^2})} \]
\[ C^{-1} = \left(2 \sqrt{M^2 - Q^2} \right)^{-1} (M - 2 \sqrt{M^2 - Q^2}) \]

When \( M = 2 |Q| / \sqrt{3} \) \( \Rightarrow M - 2 \sqrt{M^2 - Q^2} = 0 \)

Also, near \( M = 2 |Q| / \sqrt{3} \)
\[ 2 \sqrt{M^2 - Q^2} \] in +ve

Let \( f = M - 2 \sqrt{M^2 - Q^2} \)

\[ \frac{df}{dm} \bigg|_Q = 1 - \frac{2M}{\sqrt{M^2 - Q^2}} \]

\[ a \frac{df}{dm} \bigg|_Q \text{ at } M = 2 |Q| / \sqrt{3} \text{ is equal to } -3 < 0 \]

So \( f \) is monotonically decreasing at fixed \( Q \)

\[ C^{-1} > 0 \text{ for } M < 2 |Q| / \sqrt{3} \]
\[ C^{-1} < 0 \text{ for } M > 2 |Q| / \sqrt{3} \]

So, black hole has negative specific heat, at constant \( Q \), for \( M > 2 |Q| / \sqrt{3} \) which is similar to Schwarzschild (obtained as \( Q = 0 \)). As it radiates, it gets hotter.

If it doesn't lose any charge, its mass would decrease until \( M = 2 |Q| / \sqrt{3} \) and keep radiating to become a black hole with the specific heat with \( M < 2 |Q| / \sqrt{3} \)

If would then cool down as it loses energy and gradually approach an extremal black hole \( M \to |Q| \) (with \( T = 0 \)).
\[ ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \]

\[ R^2 = r - 2M \Rightarrow 2\pi dR = dr \]
\[ = 4\pi^2 dR^2 = d\pi^2 \]

Also \[ \frac{1 - \frac{2M}{r}}{r} = \frac{R^2}{R^2 + 2M} \approx \frac{R^2}{2M} \text{ as } r \rightarrow 2M \]

\[ \Rightarrow ds^2 \approx \frac{R^2}{2M} dt^2 + \frac{2M}{R^2} 4\pi^2 dR^2 + r^2 d\Omega^2 \]

as \[ R \rightarrow 0 \]

\[ = 8M \left[ dR^2 + \frac{R^2}{16M^2} dt^2 \right] + r^2 d\Omega^2 \]

\[ = 8M \left[ dR^2 + \frac{R^2}{(4M)^2} dt^2 \right] + r^2 d\Omega^2 \]

\[ \text{Theorectically}\]

As \[ R \rightarrow 0 \] this is like going to the origin in polar coordinates. It will be regular (no conical singularity) provided that \[ \frac{L}{4M} \]

is periodic with period \[ 2\pi \]

\[ \Rightarrow \Delta \Omega = 8\pi M \]

This establishes that metric is then regular at \[ r = 2M \]. Noting that as \[ r \rightarrow 0 \] the metric approaches

\[ ds^2 \approx dt^2 + dr^2 + r^2 d\Omega^2 \]

\[ \text{we can draw the following picture of the} \]
(r, t) part of the geometry is the "cigar"

\[ r \rightarrow \infty \]

\( r = M \) is a point

\( r = \text{constant} \) are circles

In this picture, the length of the \( r = \text{constant} \) circles is

\[ \int \left(1 - \frac{2M}{r}\right)^{1/2} \Delta t = \left(1 - \frac{2M}{r}\right)^{1/2} 8 \pi M \]

with period \( \beta \)

Note also that periodic Euclidean time is associated with temperature \( T = \beta^{-1} \).

In this case, it suggests

\[ T = \beta^{-1} = \frac{1}{8 \pi M} \]

which is precisely the Hawking temperature.

This "trick" also works for Kerr-Newmann and gives another important perspective on black hole thermodynamics.