Answer three of the following four questions.

1.
(a) State the basic properties of the covariant derivative $D$ of a vector $V$ along a vector $U$. Find the transformation property of the connection components in a coordinate basis $\Gamma_{bc}^a$ under a coordinate change starting from the condition that $D_b V^b$ should transform as a tensor.

(b) Compute the commutator of two covariant derivatives acting on a vector $[D_a, D_b]V^c = R^c_{dab}V^d$ and hence find the expression for the curvature tensor in terms of $\Gamma_{bc}^a$.

(c) Consider a 1-form basis $e^i$ and generic $GL(n)$ connection 1-form $\omega^i_j$. Write down the expressions for the torsion 2-form and the curvature 2-form. Show that $D R^i_j = 0$.

(d) Assuming that the connection preserves the metric find the expression for the connection in coordinate basis in terms of the metric and torsion components.

2.
Consider a 2-dimensional space with metric $ds^2 = dx^2 + f^2(x)dy^2$.

(a) Choose a basis of 1-forms $e^i$ such that $ds^2 = e^i \otimes e^i$. Compute the components of the metric connection without torsion, and also the components of the curvature.

(b) Find such $f(x)$ for which the corresponding curvature scalar is constant $R = 2a = \text{const}$. Choose the integration constant so that the metric is flat $(dx^2 + x^2 dy^2)$ near $x = 0$. Show that the Euler characteristic $\left( \frac{1}{4 \pi} \int d^2 x \sqrt{g} \ R \right)$ of a solution representing a compact space with $a > 0$ is equal to 2 (restrict the range of variation of $\sqrt{a}x$ and $y$ to $(0, \pi)$ and $(0, 2\pi)$ to get regular spherical geometry).

(c) Find the form of the 1-st order differential equations for the Killing vector components $V^a \ (L_V g_{ab} = 0)$ for this metric.

(d) Find the form of the geodesic equation for this metric by deriving it from the variational principle $\int ds[\dot{x}^2 + f^2(x)\dot{y}^2]$. Show that these equations can be solved explicitly in the case of the constant curvature metric in (b).
3.
(a) Define a symplectic manifold, skew gradient, and Poisson bracket; show that the 
set of smooth functions on a symplectic manifold forms a Lie algebra with the Poisson 
bracket as a Lie product.

(b) show that a Hamiltonian vector field $U$ on a symplectic manifold ($\mathcal{L}_U \omega = 0$) 
satisfies $\omega(U) = dH$ (you may use the expression for the Lie derivative of a 2-form in 
terms of exterior differentials).

(c) Find equations for integral curves of a Hamiltonian vector field. Which is the 
Lagrangian they follow from?

(d) Show that the Hamiltonian vector fields form a Lie algebra which is a subalgebra 
of Lie algebra of vector fields on a manifold (use that $[\mathcal{L}_U, \mathcal{L}_V] = \mathcal{L}_{[U,V]}$).

4.
(a) Define the notion of Lie algebra, its basis, structure constants and write down the 
component form of the Jacobi identity.

(b) Starting with the Lie-algebra valued 1-form basis $e = g^{-1} dg \equiv e^i t_i$ where $g$ is a 
group element and $t_i$ are generators of its algebra, compute the connection and curvature 
of the group manifold in terms of the structure constants $C^i_{kj}$.

(c) Consider the Euclidean group $ISO(n, R)$ in n dimensions as acting on functions on 
$R^n$, i.e. $f'(x) = f(Kx + a)$, where $K$ is a matrix of $SO(n)$ and $a$ is a constant n-vector. 
Obtain the commutation relations of the corresponding Lie algebra by showing that the 
basis of generators of the algebra can be chosen as $P_a = \partial_a$, $M_{ab} = x_a \partial_b - x_b \partial_a$.

(d) Consider a 3-form $A$ in 11 dimensions with $F = dA$. Find the equations of motion 
for $A$ by starting with the action $S = \int \left( \frac{1}{4} F \wedge F^* + \frac{1}{3} A \wedge F \wedge F \right)$.