Answer 3 out of the following 4 questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 2 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 2 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.
1. Consider the set of 2-dimensional planes in $\mathbb{R}^3$ that pass through the origin.

   (i) Show that this set is a real manifold by introducing coordinates and constructing an atlas.  
       [16 marks]

   (ii) Show that this manifold is diffeomorphic to $\mathbb{R}P^2$.  
       [4 marks]

       [Total 20 marks]
2. (i) Give the definition of an integral curve of a vector field $V$. What is a flow? Explain how a diffeomorphism $\sigma_V(\lambda)$ is generated by $V$. [5 marks]

(ii) Consider a chart where the coordinates of a point $p$ are $x^\mu(p)$ and we may write a vector field, $V$, in the coordinate basis as $V = V^\mu(x)\partial/\partial x^\mu$. Consider the integral curve of $V$ passing through $p$. Show that for small $\lambda$, we may write the coordinates of a point on this curve as,

$$x^\mu(p(\lambda)) = x^\mu(p) + \lambda V^\mu(p) + \frac{1}{2}\lambda^2 V^\nu(p) \frac{\partial V^\mu(x)}{\partial x^\nu} \bigg|_p + O(\lambda^3)$$

where $p(\lambda) = \sigma_V(\lambda) \cdot p$. [5 marks]

(iii) Consider a function $f$. By considering the integral curve equation, prove that the following differential equation holds,

$$\frac{d}{d\lambda} f(p(\lambda)) = V[f]_{p(\lambda)} \cdot$$

A solution to this differential equation can be written in terms of an exponential of the vector field as,

$$f|_{p(\lambda)} = e^{\lambda V} f|_p = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left( V^\mu \frac{\partial}{\partial x^\mu} \right)^n f(x) \bigg|_{x = x(p)}.$$

Show that this does indeed solve the above differential equation. [5 marks]

(iv) Consider two vector fields $U$ and $W$, and denote $f_{(1)} = f(\sigma_U(\lambda) \cdot \sigma_W(\lambda) \cdot p)$ and $f_{(2)} = f(\sigma_W(\lambda) \cdot \sigma_U(\lambda) \cdot p)$. Using the result that $f|_{\sigma_V(\lambda) \cdot p} = e^{\lambda V} f|_p$ from part iii), take $\lambda$ to be small and compute the difference $f_{(2)} - f_{(1)}$ up to and including quadratic dependence on $\lambda$. Suppose now that $\lambda$ is not small. Prove that if the Lie bracket of $U$ and $W$ vanishes everywhere then $f_{(2)} = f_{(1)}$. [5 marks]

[Total 20 marks]
3. Consider a non-coordinate basis $e_a(p)$ of vector fields for points $p$ in a chart $U$ of a manifold.

(i) Show that the Lie derivative of one basis vector with respect to another is of the form

$$\mathcal{L}_{e_a} e_b = C_{ab}^c e_c$$

(3.1)

for some functions $C_{ab}^c(p)$ and that these are antisymmetric

$$C_{ab}^c = -C_{ba}^c$$

(3.2)

[3 marks]

(ii) Define the action $e_a[f]$ of a basis vector field on a function $f$. Show that the functions $C_{ab}^c(p)$ satisfy

$$e_c[C_{ab}^d] - C_{[ab}^c C_{c]}^d = 0$$

(3.3)

where $[abc]$ denotes total antisymmetrization in the indices $a, b, c$, i.e. show they satisfy

$$e_c[C_{ab}^d] - C_{ab}^c C_{ce}^d + (cyclic \ permutations \ in \ (abc)) = 0$$

(3.4)

Deduce that if the $C_{ab}^c$ are constant in $U$, then they are the structure constants of a Lie algebra.

[6 marks]

(iii) Consider a metric $g$ with components

$$g(e_a, e_b) = g_{ab}$$

(3.5)

for some functions $g_{ab}(p)$. Calculate the Lie derivative of $g$ with respect to $e_a$ for general functions $C_{ab}^c(p)$. In the case in which the $C_{ab}^c$ are constant, find the condition for the Lie derivative to vanish,

$$\mathcal{L}_{e_a} g = 0$$

(3.6)

[3 marks]

(iv) Suppose the $C_{ab}^c$ are constant, and introduce a connection $\nabla$ with components

$$\nabla_{e_a} e_b = \lambda C_{ab}^c e_c$$

(3.7)

for some constant $\lambda$. Calculate the curvature and torsion for this connection. What happens to the curvature and torsion when $\lambda = -1$? What is the holonomy group for $\nabla$ for closed curves in $U$ when $\lambda = -1$?

[8 marks]

[Total 20 marks]
4. (i) Write down the equations of electromagnetism in terms of the field strength 2-form \( F \) and the current 1-form \( j \) in the language of differential forms. What condition must \( j \) satisfy? \([3\text{ marks}]\)

(ii) State Poincaré’s Lemma. Using this, show that on the manifold \( \mathbb{R}^4 \) the field strength \( F \) may be written in terms of a 1-form \( A \) as \( F = dA \). \([4\text{ marks}]\)

(iii) Take the Minkowski metric on \( \mathbb{R}^4 \) in spherical polar coordinates, so,

\[
g = -dt \otimes dt + dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2 \theta d\phi \otimes d\phi.
\]

Keeping the metric unchanged, let us now remove the line \( r = 0 \) from \( \mathbb{R}^4 \) to form a new manifold which we denote \( \mathcal{M} \).

Consider the field strength \( F = q \sin \theta d\theta \wedge d\phi \) for some constant \( q \) defined on \( \mathcal{M} \). Confirm that \( F \) satisfies the vacuum electromagnetic equations (i.e. the equations with \( j = 0 \)) on \( \mathcal{M} \). (This is the solution for a static magnetic monopole with world-line \( r = 0 \).) \([4\text{ marks}]\)

(iv) Compute the integral of \( F \) over the 2-sphere \( t = 0, \ r = r_0 \) for some constant \( r_0 \). Why does this confirm that \( F \) is not exact? \([4\text{ marks}]\)

(v) Consider a field strength \( F' \) solving the vacuum electromagnetic equations now on \( \mathbb{R}^4 \) rather than \( \mathcal{M} \).

Using your answers to part ii) and iv) show that one cannot find a smooth 2-form field strength tensor field \( F' \) on \( \mathbb{R}^4 \) that agrees with the solution \( F \) on \( \mathcal{M} \) for large enough radius \( r \), i.e. that there is no solution \( F' \) to the vacuum electromagnetic equations on \( \mathbb{R}^4 \) such that \( F' = q \sin \theta d\theta \wedge d\phi \) for \( r > r_1 \) for some constant \( r_1 \). \([2\text{ marks}]\)

(vi) What does this tell us about the cohomology of \( \mathbb{R}^4 \) and \( \mathcal{M} \)? \([3\text{ marks}]\)

[Total 20 marks]