Differential Geometry: Example sheet 2 solutions

1) a) Show that the region $U$ in $\mathbb{R}^{n+1}$ outside a sphere $S^n$ of radius $R$ given by $U = \{x \in \mathbb{R}^{n+1} : |x| > R\}$ is an open subset of $\mathbb{R}^{n+1}$ with the usual (metric) topology.

b) Show that the sphere $S^n$ is a closed subset of $\mathbb{R}^{n+1}$ with the usual (metric) topology.

Solution

a) Consider a point in $x \in U$. Let $|x| = r$; then $r > R$ for $x \in U$. The open ball $B(x)$ with radius $(r - R)/2$ and centre $x$ is an open set containing $x$ that is in $U$. For every $x \in U$ there is an open ball $B(x)$ with radius $(|x| - R)/2$ in $U$ containing $x$ and $U$ is the union of all such open balls, $U = \cup_x B(x)$. Then $U$ is the union of open sets and so is open.

b) The complement of the sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = R\}$ is the union of two open sets, the open ball $B = \{x \in \mathbb{R}^{n+1} : |x| < R\}$ and the set $U = \{x \in \mathbb{R}^{n+1} : |x| > R\}$ which we have just seen is open. Thus the complement of $S^n$ is open, so $S^n$ is closed.

2) Show that an open ball in $\mathbb{R}^n$ is non-compact. Is a closed ball compact?

Solution

An open cover for the open ball $B = \{x \in \mathbb{R}^{n+1} : |x| < R\}$ is given by the open balls $B(x)$ with with centre $x$ and radius $(R - |x|)/2$. This has no finite sub-cover. A closed ball is closed and bounded and so by the Heine-Borel Theorem is compact.

4) a) The Lie derivative of a cotangent vector field $w$ with respect to a vector field $V$ can be defined by

$$\mathcal{L}_V w|_p \equiv \lim_{\epsilon \to 0} \left( \frac{\sigma_V(\epsilon)^* w|_{p'} - w|_p}{\epsilon} \right)$$

(1)

Here the diffeomorphism $\sigma_V(\epsilon)$ induces the pull-back map

$$\sigma_V(+\epsilon)^* : T^*_p \mathcal{M} \to T^*_p \mathcal{M}$$
and $p' = \sigma_V(\epsilon)p$. Using coordinates $x^\mu$, show that the Lie derivative of the covector field $w = w_\mu(x)dx^\mu$ is given explicitly in components by

$$
\mathcal{L}_V w = \left( V^\alpha \frac{\partial}{\partial x^\alpha} w_\mu + w_\alpha \frac{\partial}{\partial x^\mu} V^\alpha \right) dx^\mu
$$

(2)

Solution

The Lie derivative of a cotangent vector field $w$ with respect to a vector field $V$ is

$$
\mathcal{L}_V w \big|_p \equiv \lim_{\epsilon \to 0} \left( \frac{\sigma_V(\epsilon^\ast)w|_{p'}}{\epsilon} - w|_p \right)
$$

(3)

Let us use coordinates so $V = V^\mu(x) \frac{\partial}{\partial x^\mu}$ and $w = w_\mu(x)dx^\mu$ and

$$
x^\mu(p') = x^\mu(p) + \epsilon V^\mu(p) + O(\epsilon^2)
$$

(4)

Now we may evaluate the Lie derivative explicitly. Take the cotangent vector field $w$ evaluated at $p' = \sigma_V(\epsilon)p$, so

$$
w_{p'} = w_\mu(x(p')) \left( dx^\mu|_{p'} \right)
$$

(5)

and we can now pull back to obtain the cotangent vector we want at $p$,

$$
\sigma_V(\epsilon)^\ast w_{p'} = w_\mu(x(p')) \frac{\partial x^\mu(p')}{\partial x^\nu(p)} \left( dx^\nu|_p \right) \in T^*_p\mathcal{M}.
$$

(6)

Now using the expression for $x^\mu(p')$ above, $w(x(p'))$ can simply be expanded in the infinitesimal $\epsilon$ as

$$
w_\mu(x(p')) = w_\mu(x(p)) + \epsilon V^\alpha(x(p)) \left( \frac{\partial}{\partial x^\alpha} w_\mu(x) \bigg|_{x=x(p)} \right) + O(\epsilon^2)
$$

(7)

From

$$
x^\mu(p') = x^\mu(p) + \epsilon V^\mu(p) + O(\epsilon^2)
$$

(8)

we have

$$
\frac{\partial x^\nu(p')}{\partial x^\mu(p)} = \delta^\nu_\mu + \epsilon \left( \frac{\partial}{\partial x^\mu} V^\nu(x) \bigg|_{x=x(p)} \right) + O(\epsilon^2)
$$

(9)
Then we find
\[ \sigma_V(\epsilon)^* w_{p'} = w_{\mu}(x(p')) \frac{\partial x^\mu(p')}{\partial x^\nu(p)} \left( dx^\nu \right)_p \] (10)
\[ = \left( w_{\mu}(x(p)) + \epsilon V^\alpha(x(p)) \left( \frac{\partial}{\partial x^\alpha} w_{\mu}(x) \right)_{x=x(p)} \right) \frac{\partial x^\mu(p')}{\partial x^\nu(p)} \left( dx^\nu \right)_p \] (11)
\[ = \left( w_{\mu}(x(p)) + \epsilon V^\alpha(x(p)) \left( \frac{\partial}{\partial x^\alpha} w_{\mu}(x) \right)_{x=x(p)} \right) \left( dx^\nu \right)_p \] (12)
\[ \times \left( \delta^\nu_{\nu} + \epsilon \left( \frac{\partial}{\partial x^\nu} V^\mu(x) \right)_{x=x(p)} \right) + O(\epsilon^2) \left( dx^\nu \right)_p \] (13)
\[ = \left[ \left( w_{\mu}(x) \delta^\mu_{\nu} + \epsilon \left[ \delta^\mu_{\nu} V^\alpha(x) \frac{\partial}{\partial x^\alpha} w_{\mu}(x) + w_{\mu}(x) \frac{\partial}{\partial x^\nu} V^\mu(x) \right] + O(\epsilon^2) \right) dx^\nu \right]_{x=x(p)} \] (14)

We may now subtract \( w_p \) from this, divide by \( \epsilon \), and take the vanishing epsilon limit to give the Lie derivative in components at the point \( p \),
\[ \mathcal{L}_V w = \left( V^\alpha \frac{\partial}{\partial x^\alpha} w_{\mu} + w_{\alpha} \frac{\partial}{\partial x^\mu} V^\alpha \right) dx^\mu \] (15)

**4b)** For vector fields \( V, Y \) and cotangent vector field \( w \), use the expressions for \( \mathcal{L}_V w \) and \( \mathcal{L}_V Y \) in a coordinate basis to show that
\[ \langle \mathcal{L}_V w, Y \rangle + \langle w, \mathcal{L}_V Y \rangle = V[\langle w, Y \rangle] \]

**Solution**

We have
\[ \mathcal{L}_V w = \left( V^\alpha \frac{\partial}{\partial x^\alpha} w_{\mu} + w_{\alpha} \frac{\partial}{\partial x^\mu} V^\alpha \right) dx^\mu \] (16)
so that
\[ \langle \mathcal{L}_V w, Y \rangle = Y^\mu \left( V^\alpha \frac{\partial}{\partial x^\alpha} w_{\mu} + w_{\alpha} \frac{\partial}{\partial x^\mu} V^\alpha \right) \]

Also
\[ \mathcal{L}_V Y|_p = \left( V^\mu(x) \frac{\partial}{\partial x^\mu} Y^\nu(x) - Y^\mu(x) \frac{\partial}{\partial x^\mu} V^\nu(x) \right) \left. \frac{\partial}{\partial x^\nu} \right|_{x=x(p)} \] (17)
so that

$$\langle w, \mathcal{L}_V Y \rangle = \left( V^\mu(x) \frac{\partial}{\partial x^\mu} Y^\nu(x) - Y^\mu(x) \frac{\partial}{\partial x^\mu} V^\nu(x) \right) w_\nu$$

Then

$$\langle \mathcal{L}_V w, Y \rangle + \langle w, \mathcal{L}_V Y \rangle = Y^\mu \left( V^\alpha \frac{\partial}{\partial x^\alpha} w_\mu + w_\alpha \frac{\partial}{\partial x^\alpha} V^\mu \right)$$

$$= V^\alpha \left( Y^\mu \frac{\partial}{\partial x^\alpha} w_\mu + w_\mu \frac{\partial}{\partial x^\alpha} Y^\mu \right)$$

$$= V^\alpha \frac{\partial}{\partial x^\alpha} (w_\mu Y^\mu)$$

$$= V[(\langle w, Y \rangle)]$$

5) Consider a (0,2) tensor field $g \in \mathcal{J}_2^0$. The Lie derivative of $g \mathcal{L}_V g$ with respect to a vector field $V$ is defined as

$$\mathcal{L}_V g \vert_p \equiv \lim_{\epsilon \to 0} \left( \frac{\sigma_V(+\epsilon)^* g \vert_{p'} - g \vert_p}{\epsilon} \right)$$

where $p' = \sigma_V(\epsilon)p$. Using a coordinate chart, explicitly compute the components of the Lie derivative $\mathcal{L}_V g$.

**Solution**

Very similar to question 4a. We have

$$\mathcal{L}_V g \vert_p \equiv \lim_{\epsilon \to 0} \left( \frac{\sigma_V(+\epsilon)^* g \vert_{p'} - g \vert_p}{\epsilon} \right)$$

Let us use coordinates so $V = V^\mu(x) \frac{\partial}{\partial x^\mu}$ and $g = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu$ and

$$x^\mu(p') = x^\mu(p) + \epsilon V^\mu(p) + O(\epsilon^2)$$

Now we may evaluate the Lie derivative explicitly. Take the field $g$ evaluated at $p' = \sigma_V(\epsilon)p$, so

$$g_{\mu'} = g_{\mu}(x(p')) \left( dx^\mu \otimes dx^{\nu'} \vert_{p'} \right)$$
and we can now pull back to obtain \( g \) at \( p \),

\[
\sigma_V(\epsilon)^* g_{p'} = g_{\mu\nu}(x(p')) \frac{\partial x^\mu(p')}{\partial x^\rho(p)} \frac{\partial x^\nu(p')}{\partial x^\sigma(p)} \left( dx^\rho \otimes dx^\sigma \right)_{|p}
\]  

(22)

Now following similar steps to 4a) we find

\[
\mathcal{L}_V g = \left( V^\alpha \frac{\partial}{\partial x^\alpha} g_{\mu\nu} + g_{\alpha\nu} \frac{\partial}{\partial x^\mu} V^\alpha + g_{\mu\alpha} \frac{\partial}{\partial x^\nu} V^\alpha \right) dx^\mu \otimes dx^\nu
\]  

(23)