Differential Geometry: Example sheet 3

1) Setup: Consider the flows, $\sigma_X(\lambda)$ and $\sigma_Y(\lambda)$, generated by two vector fields $X$ and $Y$. Consider starting at a point $p$, and flowing first by $\alpha$ along $X$, and then by $\beta$ along $Y$, arriving at $p_{XY} = \sigma_Y(\beta) \circ \sigma_X(\alpha) p$. Starting at $p$ and switching the order we arrive at $p_{YX} = \sigma_X(\alpha) \circ \sigma_Y(\beta) p$.

To do: Using a chart containing $p$, and with coordinates $\{x^\mu\}$, show for infinitesimal $\alpha, \beta$ that $x^\mu(p_{XY}) - x^\mu(p_{YX}) = \alpha \beta ([X,Y])^\mu$ to leading order in $\alpha, \beta$.

What have we learned? We say the flows generated by $X,Y$ ‘commute’ if $p_{XY} = p_{YX}$ for all starting $p$. The Lie derivative $\mathcal{L}_X Y = [X,Y]$ measures the ‘non-commutability’ of these two flows. We have shown that $[X,Y] = 0$ if and only if the flows commute.

2) Take 2 differential forms $\xi \in \Omega^a$, $\eta \in \Omega^b$. Confirm that
   i) $\xi \wedge \eta = (-1)^{ab} \eta \wedge \xi$,
   ii) $d(\xi \wedge \eta) = (d\xi) \wedge \eta + (-1)^a \xi \wedge (d\eta)$.

3) The exterior derivative elegantly generalizes the operations found in 3-d vector calculus; $\text{grad}(f) = \nabla f$, $\text{div}(\mathbf{v}) = \nabla \cdot \mathbf{v}$ and $\text{curl}(\mathbf{v}) = \nabla \times \mathbf{v}$, where $f$ is a function and $\mathbf{v}$ is a 3-vector. Consider the manifold $\mathcal{M} = \mathbb{R}^3$, and taking a coordinate basis, show:
   i) for $f \in \Omega^0(\mathcal{M})$, the components of $df \in \Omega^1(\mathcal{M})$ are the components of $\text{grad}(f)$.
   ii) for $w \in \Omega^1(\mathcal{M})$ with components given by those of a 3-vector in $\mathbb{R}^3$ the components of $dw \in \Omega^2(\mathcal{M})$ give the components of the $\text{curl}$ of that 3-vector.
   iii) for $w \in \Omega^2(\mathcal{M})$ with components given by those of a 3-vector, $dw \in \Omega^3(\mathcal{M})$ computes $\text{div}$ of that 3-vector.

4) For a vector field $V$ and tensor field $T$ of type $(q,r)$, show that for any constant $c$, the Lie derivative satisfies

$$\mathcal{L}_c V T = c \mathcal{L}_V T$$
5) The **pull-back of a function**. Suppose we have a map \( f : \mathcal{M} \to \mathcal{N} \) and a function on \( \mathcal{N}, \ g \in \mathcal{F}(\mathcal{N}) \),

\[
g : \quad \mathcal{N} \to \mathbb{R} \tag{1}
\]

Then we can **pull-back** the function \( g \) onto a function on \( \mathcal{M} \) by

\[
g \cdot f : \quad \mathcal{M} \to \mathbb{R} \tag{2}
\]

The pull-back of the function \( g \) by \( f \) is sometimes written as \( f^*(g) \), so \( f^*(g) \equiv g \cdot f \), and this is a function on \( \mathcal{M} \).

\( i \) Introduce local coordinates on \( \mathcal{M} \) and \( \mathcal{N} \) and find the form of \( f^*(g) \) as a function of the coordinates on \( \mathcal{M} \).

\( ii \) Suppose \( f : \mathcal{M} \to \mathcal{N} \) is a diffeomorphism. Let \( V \) be a vector field on \( \mathcal{N} \) and \( \omega \) be a cotangent vector field on \( \mathcal{N} \), so that \( \langle \omega, V \rangle \) is a function on \( \mathcal{N} \). Show that the pull-back of this function satisfies

\[
f^*(\langle \omega, V \rangle) = \langle f^*\omega, f^*V \rangle
\]