1) a) Show that the region $U$ in $\mathbb{R}^{n+1}$ outside a sphere $S^n$ of radius $R$ given by $U = \{ x \in \mathbb{R}^{n+1} : |x| > R \}$ is an open subset of $\mathbb{R}^{n+1}$ with the usual (metric) topology.

b) Show that the sphere $S^n$ is a closed subset of $\mathbb{R}^{n+1}$ with the usual (metric) topology.

2) Show that an open ball in $\mathbb{R}^n$ is non-compact. Is a closed ball compact?

3) In this question, we will develop a treatment of $S^2$ that is similar to the discussion of $\mathbb{RP}^2$ given in the lectures.

a) Consider the set of directed lines through the origin in $\mathbb{R}^3$. These are lines with a direction. Show that each directed line is labelled by a point on the unit sphere $S^2$.

b) Take a point in $\mathbb{R}^3 (x_1, x_2, x_3)$ which labels a line provided $\neq (0, 0, 0)$, i.e. take a point in $\mathbb{R}^3 - \{0\}$. This labels the line from $(0, 0, 0)$ to $(x_1, x_2, x_3)$, with direction pointing towards $(x_1, x_2, x_3)$. The point $(-x_1, -x_2, -x_3)$ labels the line from $(0, 0, 0)$ to $(-x_1, -x_2, -x_3)$ which has the opposite direction and so is regarded as labelling a distinct directed line. However, $(x_1, x_2, x_3)$ labels the same line as $(\lambda x_1, \lambda x_2, \lambda x_3)$ for any $\lambda \in \mathbb{R}^+$ where $\mathbb{R}^+$ are the positive real numbers, so that $\lambda \in \mathbb{R}^+$ if $\lambda \in \mathbb{R}$ and $\lambda > 0$.

c) Consider charts:

$$U_{1+} : \text{ all } p \in S^2 \text{ s.t. } x_1 > 0$$

$$\psi_{1+} : S^2 \rightarrow \mathbb{R}^2$$

$$p = (x_1, x_2, x_3) \rightarrow \left( \frac{x_2}{x_1}, \frac{x_3}{x_1} \right) = (a_1, a_2)$$

and

$$U_{1-} : \text{ all } p \in S^2 \text{ s.t. } x_1 < 0$$

$$\psi_{1-} : S^2 \rightarrow \mathbb{R}^2$$
\[ p = (x_1, x_2, x_3) \rightarrow \left( \frac{x_2}{x_1}, \frac{x_3}{x_1} \right) = (a'_1, a'_2) \quad (6) \]

Construct similar charts \((U_{2+}, \psi_{2+}), (U_{2-}, \psi_{2-}), (U_{3+}, \psi_{3+}), (U_{3-}, \psi_{3-})\) where \(U_{i+}\) is the region \(x_i > 0\) and \(U_{i-}\) is the region \(x_i < 0\). Show that these 6 charts form an atlas for \(S^2\).

d) Check the transition functions are smooth.

e) Show that, on identifying the points \((x_1, x_2, x_3)\) and \((-x_1, -x_2, -x_3)\), the atlas constructed here becomes the atlas constructed in the lectures for \(\mathbb{RP}^2\).

e) What would the corresponding atlas be for \(S^n\)?

[Compare with the treatment of \(S^n\) in Nakahara. Nakahara uses the same open sets \(U_{i\pm}\) but different maps \(\psi_{i\pm}\).]

\[ a) \text{ The Lie derivative of a cotangent vector field } w \text{ with respect to a vector field } V \text{ can be defined by} \]

\[ \mathcal{L}_V w \big|_p \equiv \lim_{\epsilon \to 0} \left( \frac{\sigma_V(+\epsilon)^* w|_{p'} - w|_p}{\epsilon} \right) \quad (7) \]

Here the diffeomorphism \(\sigma_V(\epsilon)\) induces the pull-back map

\[ \sigma_V(\epsilon)^* : T_{p'}^* \mathcal{M} \rightarrow T_p^* \mathcal{M} \]

and \(p' = \sigma_V(\epsilon)p\). Using coordinates \(x^\mu\), show that the Lie derivative of the covector field \(w = w_\mu(x)dx^\mu\) is given explicitly in components by

\[ \mathcal{L}_V w = \left( V^\alpha \frac{\partial}{\partial x^\alpha} w_\mu + w_\alpha \frac{\partial}{\partial x^\mu} V^\alpha \right) dx^\mu \quad (8) \]

b) For vector fields \(V, Y\) and cotangent vector field \(w\), use the expressions for \(\mathcal{L}_V w\) and \(\mathcal{L}_V Y\) in a coordinate basis to show that

\[ \langle \mathcal{L}_V w, Y \rangle + \langle w, \mathcal{L}_V Y \rangle = V[\langle w, Y \rangle] \]
5) Consider a \((0,2)\) tensor field \(g \in \mathcal{X}_2^n\). The Lie derivative of \(g\) \(\mathcal{L}_V g\) with respect to a vector field \(V\) is defined as

\[
\mathcal{L}_V g|_p \equiv \lim_{\epsilon \to 0} \left( \frac{\sigma_{V}(+\epsilon)^* g|_{p'} - g|_p}{\epsilon} \right) 
\]

(9)

where \(p' = \sigma_V(\epsilon)p\). Using a coordinate chart, explicitly compute the components of the Lie derivative \(\mathcal{L}_V g\).

*An interesting aside: take \(g\) to be the metric - you will find that the components are actually the same as \((\mathcal{L}_V g)_{\mu\nu} = 2\nabla_{(\mu} V_{\nu)}\) where \(\nabla\) is the (Levi-Civita) covariant derivative you encounter early on in GR.*