1) Using
\[ \mathcal{L}_X f \equiv X[f] \] (1)
show that for any vector fields \( X, Y \) and function \( f \)
\[ \mathcal{L}_X (Y[f]) = (\mathcal{L}_X Y)[f] + Y[\mathcal{L}_X f] \] (2)

2) Recall that for \( H^r(\mathcal{M}) \), the vector structure is given by,
\[ \alpha[\omega_1] + \beta[\omega_2] = [\alpha \omega_1 + \beta \omega_2] \] (3)
where \( \alpha, \beta \in \mathbb{R} \) and \( \omega_{1,2} \in Z^r(\mathcal{M}) \). Show that this is independent of the choice of representatives in the equivalence classes above, ie. show that,
\[ \alpha[\omega'_1] + \beta[\omega'_2] = [\alpha \omega_1 + \beta \omega_2] \] (4)
where \( \omega'_{1,2} \in Z^r(\mathcal{M}) \) with \( \omega_1 \sim \omega'_1 \) and \( \omega_2 \sim \omega'_2 \).

3) Show that if one integrates an exact \( r \)-form \( \omega \) over an \( r \)-cycle \( c \) then \( \int_c \omega = 0 \). Recall an \( r \)-cycle is a closed \( r \)-dimensional submanifold with no boundary.

Consider a compact manifold with no boundary. A theorem states that given a closed \( r \)-form \( \omega \), then if \( \int_c \omega = 0 \) for all possible \( r \)-cycles then \( \omega \) is not just closed but is exact (this is implied by the relation of cohomology to topology which we did not discuss in lectures).

Use this to show that if for such a manifold all \( r \)-cycles are \( r \)-boundaries, then any closed \( r \)-form must be exact.

Hence determine the cohomology of \( S^m \) for any \( m \). Give representatives for a basis for the cohomology vector spaces \( H^r \).

4) For a 2-torus \( H^1(T^2) \) is of dimension two. Find two inequivalent closed 1-forms on \( T^2 \) which are representatives for a basis of \( H^1(T^2) \) and perform integrals over appropriate 1-cycles (ones that are not boundaries) to explicitly show they are not exact.
5) The Betti numbers of $T^3$ are $b_r(T^3) = \{1, 3, 3, 1\}$. Give representatives for a basis for the cohomology spaces $H^r(T^3)$. Again choose appropriate $r$-cycles to integrate these basis forms over to show they are closed but not exact.

6) Consider the manifold $\mathbb{R}^* = \mathbb{R}^3 - \{0\}$. Take the closed 2-form $F \in Z^2(\mathbb{R}^*)$;

$$F = \sin \theta \, d\theta \wedge d\phi$$

where $r, \theta, \phi$ are the usual spherical polar coordinates on $\mathbb{R}^3$ - with $\theta$ the ‘zenith’ and $\phi$ the ‘azimuth’ angles. Confirm that $F$ is indeed closed.

Consider the 2-cycle given by the unit sphere $r = 1$. Compute the period (ie. integrate!) $F$ over that 2-cycle, and hence show that $F$ is not exact, and that this 2-cycle is not a 2-boundary.

For the manifold $\mathbb{R}^3$ we know $H^2$ is trivial - there are no closed 2-forms which are not exact. How can this agree with what you have shown above for the 2-form $F$ on $\mathbb{R}^*$? [Hint: Does $F$ define a 2-form on $\mathbb{R}^3$?]

Extra question: This is a little harder. Please only attempt for interest.

7) Use Poincaré’s Lemma to determine that $H^1(S^2)$ is trivial.

[Hint: You may find it useful to take the Stereographic atlas on $S^2$. Consider the chart that covers the North pole and use the fact that a closed form must have smooth components there. Then consider a contractible set that covers all but the North pole.]