Differential Geometry: Example sheet 4 solutions

3) Qu: Show that if one integrates an exact \( r \)-form \( \omega \) over an \( r \)-cycle \( c \) then \( \int_c \omega = 0 \).

Since \( \omega \) is exact, we may write \( \omega = d\alpha \). Then by Stokes,

\[
\int_c \omega = \int_c d\alpha = \int_{\partial c} \alpha
\]

but since \( c \) is a cycle it has no boundary so \( \partial c = 0 \), and hence,

\[
\int_c \omega = 0
\]

Qu: Consider a compact manifold with no boundary. A theorem states that given a closed \( r \)-form \( \omega \), then if \( \int_c \omega = 0 \) for all possible \( r \)-cycles then \( \omega \) is not just closed but is exact. Use this to show that if for such a manifold all \( r \)-cycles are \( r \)-boundaries, then any closed \( r \)-form must be exact.

If any \( r \)-cycle is an \( r \)-boundary, so that for any cycle \( c \) we may write \( c = \partial v \) for some \( v \), then given a closed \( r \)-form;

\[
\int_c \omega = \int_{\partial v} \omega = \int_v d\omega = 0
\]

where we use Stokes, and then the fact that since \( \omega \) is closed, then \( d\omega = 0 \). Hence we see for any \( r \)-cycle on such a manifold, then a closed form \( \omega \) obeys \( \int_c \omega = 0 \). Using the theorem quoted in the questions, this implies that the closed form \( \omega \) must be exact.

Note that if any closed \( r \)-form is exact then the \( H^r \) is trivial.

Qu: Hence determine the cohomology of \( S^m \) for any \( m \). Give representatives for a basis for the cohomology vector spaces \( H^r \).

Since \( S^m \) is connected we have \( H^0(S^m) = \mathbb{R} \). A representative for a basis for \( H^0 \) is any constant function.
Now, Poincare duality tells us that $H^0 = H^m = \mathbb{R}$. A representative for the non-trivial class of $H^m$ is any $m$-form $\omega$ such that $\int_{S^m} \omega \neq 0$. For example, one can take any volume form.

For a sphere any $r$-cycle for $r > 0$ and $r < m$ is a $r$-boundary; any $r$-cycle is topologically an $r$-sphere, and is the boundary of an $(r + 1)$-ball. Hence from the discussion above we have $H^r = 0$ are trivial for $0 < r < m$. 

2
4) Qu: For a 2-torus $H^1(T^2)$ is of dimension two. Find two inequivalent closed 1-forms on $T^2$ which are representatives for a basis of $H^1(T^2)$ and perform integrals over appropriate 1-cycles (ones that are not boundaries) to explicitly show they are not exact.

Take our usual circle Altas, and construct $T^2$ as the product $S^1 \times S^2$. Then on $T^2$ our product manifold Altas has two angle coordinates $(\theta_1, \theta_2)$.

Two inequivalent closed 1-forms are $\omega_1 = d\theta_1$ and $\omega_2 = d\theta_2$. (More than one patch is needed for each circle, with the volume 1-forms on each given by $d\theta_i$ locally in terms of local coordinates $\theta_i$.) Note that $\omega_{1,2}$ are volume forms on the two $S^1$ factors of $T^2$. They are clearly closed as $d^2 = 0$. Note they are clearly inequivalent; there is no choice of $\alpha$ such that $\omega_1 - \omega_2 = d\alpha$.

To show $\omega_1$ is closed but not exact we integrate over the cycle $c_1 = \{\theta_1 = [0, 2\pi), \theta_2 = \text{const}\}$. Then,

$$\int_{c_1} \omega_1 = \int_0^{2\pi} d\theta_1 = 2\pi \neq 0$$

(4)

and since this does not vanish, then $\omega_1$ cannot be exact. Similarly for $\omega_2$ integrate over $c_2 = \{\theta_2 = [0, 2\pi), \theta_1 = \text{const}\}$.

Thus $d\theta_1$ and $d\theta_2$ are representatives for a basis for $H^1(T^2)$. 

3
5) Qu: The Betti numbers of $T^3$ are $b_r(T^3) = \{1, 3, 3, 1\}$. Give representatives for a basis for the cohomology spaces $H^r(T^3)$. Again choose appropriate $r$-cycles to integrate these basis forms over to show they are closed but not exact.

Again we think of $T^3 = S^1 \times S^1 \times S^1$ and take a product Atlas with coordinates $(\theta_1, \theta_2, \theta_3)$.

Poincare’s theorem tells us that $H^0(T^3) = \mathbb{R}^{b_0} = \mathbb{R}$. As it is connected, any non-zero constant function is a representative for a basis. Integrating over a 0-cycle which is just a point simply returns the value of the constant function.

We have $H^1(T^3) = \mathbb{R}^{b_1} = \mathbb{R}^3$. Extending the answer to the previous question representatives for a basis are $\omega_{1,2,3} = d\theta_{1,2,3}$. These are closed as $d^2 = 0$.

Taking a cycle $c_1 = \{\theta_1 = [0, 2\pi), \theta_2 = \text{const}, \theta_3 = \text{const}\}$, then,

$$\int_{c_1} \omega_1 = \int_0^{2\pi} d\theta_1 = 2\pi \neq 0$$

so $\omega_1$ is not exact. Take a similar construction for cycles for $\omega_{2,3}$.

We have $H^2(T^3) = \mathbb{R}^{b_2} = \mathbb{R}^3$. Now, representatives for a basis are;

$$\alpha_1 = d\theta_2 \wedge d\theta_3, \alpha_2 = d\theta_3 \wedge d\theta_1, \alpha_3 = d\theta_1 \wedge d\theta_2,$$

(6)

Clearly these are closed since $d^2 = 0$.

Taking a cycle $a_1 = \{\theta_2 = [0, 2\pi), \theta_3 = [0, 2\pi), \theta_1 = \text{const}\}$, then,

$$\int_{a_1} \alpha_1 = \int d\theta_2 \wedge d\theta_3 = \int_0^{2\pi} d\theta_2 \int_0^{2\pi} d\theta_3 = 4\pi^2 \neq 0$$

(7)

so $\alpha_1$ is not exact. Take a similar construction for cycles for $\alpha_{2,3}$.

Finally $H^3(T^3) = \mathbb{R}^{b_3} = \mathbb{R}$. A representative is the volume form $\omega = d\theta_1 \wedge d\theta_2 \wedge d\theta_3$. Note that $d\omega = 0$ since any top form is closed. Integrating over the 3-cycle which is the entire $T^3$ gives $(2\pi)^3$ indicating the form is not exact.
6) **Question**: Consider the manifold \( \mathbb{R}^* \times \mathbb{R}^3 = \mathbb{R}^3 - \{0\} \). Take the closed 2-form \( F \in Z^2(\mathbb{R}^3) \);

\[
F = \sin \theta \, d\theta \wedge d\phi
\]  
(8)

where \( r, \theta, \phi \) are the usual spherical polar coordinates on \( \mathbb{R}^3 \) - with \( \theta \) the ‘zenith’ and \( \phi \) the ‘azimuth’ angles. Confirm that \( F \) is indeed closed.

Firstly one can compute,

\[
dF = d(\sin \theta \, d\theta \wedge d\phi) = \frac{\partial}{\partial x^\mu} \sin \theta \, dx^\mu \wedge d\theta \wedge d\phi = 0
\]  
(9)

since \( d\theta \wedge d\theta = 0 \).

**Question**: Consider the 2-cycle given by the unit sphere \( r = 1 \). Compute the period (ie. integrate!) \( F \) over that 2-cycle, and hence show that \( F \) is not exact, and that this 2-cycle is not a 2-boundary.

Integrate over the unit \( S^2 \) at origin;

\[
\int_{S^2} F = \int \sin \theta \, d\theta \wedge d\phi = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta = 2 \cdot (2\pi) = 4\pi \neq 0
\]  
(10)

Since the integral of \( F \) over a 2-cycle is non-zero, it cannot be exact.

**Question**: For the manifold \( \mathbb{R}^3 \) we know \( H^2 \) is trivial - there are no closed 2-forms which are not exact. How can this agree with what you have shown above for the 2-form \( F \) on \( \mathbb{R}^* \times \mathbb{R}^3 \)?

While \( F \) defines a smooth 2-form on \( \mathbb{R}^* \times \mathbb{R}^3 \), it does not on \( \mathbb{R}^3 \) since it is multivalued at the origin. Hence on \( \mathbb{R}^3 \) this \( F \) is not a smooth 2-form, and so doesn’t give an example of a closed but not exact 2-form.