1) As discussed in lectures, the metric allows us to define the maps:

\[ T_p M \rightarrow T^*_p M \]
\[ U \rightarrow \omega_U, \quad \text{where } \omega_U(X) = g(U, X), \forall X \in T_p M, \text{ and,} \]
\[ T^*_p M \rightarrow T_p M \]
\[ \omega \rightarrow U_\omega, \quad \text{where, } \omega(X) = g(U_\omega, X), \forall X \in T_p M \]

Confirm the components of \( \omega_U \) in a coordinate basis are given by the components of \( U \) with their index ‘lowered’ by the metric components. Likewise confirm the components of \( U_\omega \) are given by components of \( \omega \) with their index ‘raised’. We may then define a map,

\[ J^q_r(M) \rightarrow J^{q-1}_{r+1}(M) \]
\[ T \rightarrow T' \]

where \( T' \) is defined by,

\[ T'(\omega_1, \ldots, \omega_{q-1}, U_1, \ldots, U_{r+1}) = T(\omega_1, \ldots, \omega_{j-1}, \omega_{U_1}, \omega_{j+1}, \ldots, \omega_{q-1}, U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_{r+1}) \]

for all \( \omega_1, \ldots, \omega_{q-1} \in T^*_p M \) and for all \( U_1, \ldots, U_{r+1} \in T_p M \). Confirm the components of \( T' \) are given by the components of \( T \) with the \( j \)th upper index ‘lowers’ to the \( i \)th position. Find the analogous map for \( J^q_r(M) \rightarrow J^{q+1}_{r-1}(M) \) which ‘raises’ the \( i \)th lower index of the components of \( T \) to the \( j \)th position.

2) For \((M, g)\) the volume element is defined as \( \Omega_M \equiv \sqrt{|g|} \, dx^1 \wedge \ldots \wedge dx^m \) in some coordinate basis \( \{dx^\mu\} \), where \( g = \det g_{\mu\nu} \) with \( g_{\mu\nu} \) the metric components in these coordinates. Confirm that in a new coordinate basis \( \{dy^\mu\} \), the form of the volume element remains invariant, provided the coordinate transformation \( x \rightarrow y \) preserves orientation ie. \( \det \frac{\partial x^\mu}{\partial y^\nu} > 0 \). On an orientable manifold - ie. where we can pick coordinates covering the manifold so that \( \det \frac{\partial x^\mu}{\partial y^\nu} > 0 \) on all overlaps between different coordinates systems - the volume element defines a volume form and can be used to integrate functions. Show the components of the volume form are \( \frac{1}{m!} \sqrt{|g|} \epsilon_{\mu_1 \ldots \mu_m} \). These components are said to transform as a pseudo-tensor, since they transform as a
tensor provided the coordinate transform is oriented.

3) Show $** \omega = +(-1)^{r(m-r)} \omega$ for Riemannian signature $(0,m)$. Confirm that for the inner product $(\cdot, \cdot)$ on two $r$-forms, and $d^! \equiv (-1)^{mr+m+1} * d*$, that $(\beta, d\alpha) = (d^! \beta, \alpha)$ for $\alpha \in \Omega^{r-1}$ and $\beta \in \Omega^r$ on a compact manifold without boundary. Repeat the same for Lorentzian signature $(1,m-1)$ where you should find $** \omega = -(1)^{(r(m-r)} \omega$, and where now $d^! \equiv (-1)^{mr+m} * d*$.

4) A curve, $C$, can be parameterized in a chart as \{x^\mu(\lambda)\}. Show that a functional variation of the action,

$$I = \int d\lambda \left( g_{\mu\nu} \frac{dx^\mu(\lambda)}{d\lambda} \frac{dx^\nu(\lambda)}{d\lambda} \right),$$

with respect to the path $x^\mu(\lambda)$ gives the equation governing a geodesic curve for the Levi-Civita connection.

5) Taking coordinates \{x^\mu\}, explicitly compute the Laplacian acting on a function, $f$, to obtain,

$$\Delta f = -\frac{1}{\sqrt{|g|}} \partial_\mu \left( \sqrt{|g|} g^{\mu\nu} \partial_\nu f(x) \right).$$

You may find the following identity useful; on an $m$-dimensional manifold,

$$\epsilon^{\rho_\alpha_2...\alpha_m} \epsilon^{\sigma_\alpha_2...\alpha_m} = \frac{(m-1)!}{\det g_{\mu\nu}} \delta^\sigma_\rho.$$

6) Consider the Lorentzian EM equations, $dF = 0$ and $d^! F = j$. Show that $d^! j = 0$ and compute its components in a coordinate basis - you should find current conservation.

Consider a 4-submanifold $\mathbb{R} \times V$ of 4 dimensional spacetime, where $\mathbb{R}$ is time and $V$ is a 3-submanifold with 2-boundary $B$. Suppose no current flows into or out of the spatial volume $V$, so $j = 0$ on the boundary $B$ at all times. Use $d^! j = 0$ and Stokes theorem to show that the charge enclosed in the volume at a time $t$,

$$Q_t = \int_V *j|_t$$
is constant in time.

7) The (1,3) signature Schwarzschild metric is given by:

\[ g = -\left(1 - \frac{r_0}{r}\right)dt \otimes dt + \left(1 - \frac{r_0}{r}\right)^{-1}dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi) \]

Using the coordinate basis associated to the coordinates \(\{t, r, \theta, \phi\}\) compute the components of the torsion free metric connection (the Levi-Civita connection).

We define the Ricci \((0,2)\) tensor,

\[ Ric(X,Y) \equiv < dx^\mu , R (X, \frac{\partial}{\partial x^\mu} , Y) > , \forall X,Y \in T_p M. \tag{2} \]

Using your connection, compute the components of this in terms of components of the Riemann tensor in the coordinate basis. Show the Schwarzschild metric has vanishing Ricci tensor - ie. it solves the vacuum Einstein equations.

Schwarzschild wrote down his metric in 1915, very shortly after Einstein published his theory of General Relativity, and then died in 1916 while serving in the First World War.