Exam

M.Sc. in Quantum Fields and Fundamental Forces

Differential Geometry

2:00 – 5:00, Monday 28 April, 2014

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Answer THREE out of the four questions. Use a separate booklet for each question. Make sure that each booklet carries your name, the course title, and the number of the question attempted.

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Question (1)

(1.i). Give the definition of a real manifold and the definition of a complex manifold.

[4 marks]

(1.ii). The 2-sphere $S^2$ can be embedded in $\mathbb{R}^3$ as the surface

$$x^2 + y^2 + z^2 = 1$$

where $(x, y, z)$ are the usual coordinates for $\mathbb{R}^3$. Stereographic projection from the North pole $(0, 0, 1)$ maps a point $(x, y, z) \in S^2$ to a point $(X, Y) \in \mathbb{R}^2$ given by

$$X = \frac{x}{1 - z}, \quad Y = \frac{y}{1 - z}$$

Write down the corresponding formulae for the point $(X', Y') \in \mathbb{R}^2$ given by stereographic projection of $(x, y, z) \in S^2$ from the South pole $(0, 0, -1)$. Show that $S^2$ is a manifold by using the stereographic projection to explicitly construct a 2 chart atlas, and show the transition functions satisfy any requirements stated in part (1.i) above.

[4 marks]

(1.iii). Recall that the complex manifold $\mathbb{CP}^1$ can be thought of as the set of undirected complex lines in $\mathbb{C}^2$ that pass through the origin. Give a 2 chart atlas for $\mathbb{CP}^1$ and show the transition functions satisfy any requirements stated in part (1.i) above.

[4 marks]

(1.iv). Using your answers above and combining real coordinates of $S^2$ into a complex coordinate appropriately, show that $S^2$ is a complex manifold and that it can be thought of as $\mathbb{CP}^1$.

[4 marks]

(1.v). The stereographic projection construction gives a map from a point in $S^2$ with coordinates $(X, Y)$ to a point in $\mathbb{R}^3$ with coordinates $(x, y, z)$ given by

$$x = \frac{2X}{f}, \quad y = \frac{2Y}{f}, \quad z = 1 - \frac{2}{f}$$

where

$$f = 1 + X^2 + Y^2$$

This gives an inverse to the relations given in section (1.ii) above. Use this map to show that the pull-back of this map takes the flat metric

$$g = dx \otimes dx + dy \otimes dy + dz \otimes dz$$
on $\mathbb{R}^3$ to the following metric on $S^2$

$$g_{S^2} = \frac{4}{f^2} (dX \otimes dX + dY \otimes dY)$$

[4 marks]

[Total 20 marks]
Question (2)

(2.i). Give the definition of an integral curve of a vector field $V$. What is a flow? Explain how a diffeomorphism $\sigma_V(\lambda)$ is generated by $V$, where $\lambda$ is a parameter on the integral curve of $V$.

[4 marks]

(2.ii). Consider a chart where the coordinates of a point $p$ are $x^\mu(p)$ and where a vector field, $V$, is written in the coordinate basis as $V = V^\mu(x)\partial/\partial x^\mu$. Show that, for small $\lambda$, we may write the coordinates of a point $\sigma_V(\lambda) \cdot p$ as,

$$x^\mu(\sigma_V(\lambda) \cdot p) = x^\mu(p) + \lambda V^\mu(p) + O(\lambda^2).$$

[4 marks]

(2.iii). Given a tensor field $g$, we may define a new tensor field $g'$ as,

$$g'(p) \equiv (\sigma_V(\lambda))^* \cdot g(\sigma_V(\lambda) \cdot p).$$

Show that for small $\lambda$, $g'$ can be written in terms of $g$ using the Lie derivative, $\mathcal{L}$.

[4 marks]

(2.iv). Let $g$ be a $(0, 2)$ tensor field which is given explicitly as $g = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu$. Use the Leibniz rule for the Lie derivative and the fact that $\mathcal{L}_V(dx^\mu) = \partial V^\mu(x)/\partial x^\nu dx^\nu$ to determine the components of $\mathcal{L}_V g$ in the coordinate basis.

[4 marks]

(2.v). For $n$-dimensional space $\mathbb{R}^n$ with flat metric $g$ that has constant components $g_{\mu\nu}(x) = \delta_{\mu\nu}$, consider the vector field

$$V = \Lambda_{\mu\rho}g^{\mu\rho}x^\mu \frac{\partial}{\partial x^\rho}$$

where $\Lambda_{\mu\nu}$ is constant and anti-symmetric

$$\Lambda_{\mu\nu} = -\Lambda_{\nu\mu}.$$

Show that $\mathcal{L}_V g = 0$. What is the geometric significance of this result?

[4 marks]

[Total 20 marks]
Question (3)

(3.i). Define closed and exact differential forms and explain how these give rise to cohomology.  

(3.ii). For a compact $m$-dimensional Riemannian manifold, $\mathcal{M}$, give the definition of the Hodge star $\star$. Show that the inner product between 2 $r$-forms $\alpha, \beta$ that is given by 

$$(\alpha, \beta) \equiv \int \alpha \wedge \star \beta$$

is both symmetric and positive definite.  

(3.iii). Give the equations of electromagnetism in the language of differential forms, in terms of the forms $F$ and $j$, the field strength and current respectively. Show $d^\dagger j = 0$.  

(3.iv). Define co-closed, co-exact and harmonic forms. Show that on a compact Riemannian manifold a harmonic form is closed and co-closed.  

(3.v). State the Hodge decomposition theorem for a compact Riemannian manifold. Using this, show that there is a unique harmonic representative for any given cohomology class on a compact Riemannian manifold.  

(3.vi). Suppose we are given a closed 2-form $\omega$ and a current $j$ on $\mathcal{M}$ and wish to find a solution of the equations of electromagnetism such that $F$ is in the same cohomology class as $\omega$, $[F] = [\omega]$. Using the Lorentz gauge, show that we may reduce this problem to the problem of finding the solution of the Poisson equation $\triangle A = j$ for a 1-form $A$ where $\triangle$ is the Laplacian.  

[Total 20 marks]
Question (4)

(4.i). Consider a Riemannian manifold \( M \) with metric \( g \) and connection \( \nabla \). Give a definition of the torsion map and explain how it is related to the torsion tensor. What is meant by a metric connection? What is the Levi-Civita connection? [6 marks]

(4.ii). With the usual polar coordinates \( r, \theta \) in \( \mathbb{R}^2 \) \((0 \leq \theta < 2\pi, r \geq 0)\), the unit disk is the region \( r < 1 \). Consider the metric on the unit disk given by

\[
g = \frac{1}{(1-r^2)^2} \left( dr \otimes dr + r^2 \sin^2 \theta d\theta \otimes d\theta \right).
\]

Consider a radial line \( \theta = a \) for some constant \( a \). Show that, with respect to the metric \( g \), the length of this line from the origin \( r = 0 \) to the circle \( r = r_0 \) with \( r_0 < 1 \) is

\[
s_0 = \tanh^{-1} r_0
\]

and that the circumference of a circle \( r = r_0 \) is

\[
C_0 = \pi \sinh(2s_0)
\]

Show that for small \( s_0 \), this is approximately \( C_0 \approx 2\pi s_0 \). What happens to the length \( s_0 \) and the circumference \( C_0 \) in the limit \( r_0 \to 1 \)? [4 marks]

(4.iii). Given a non-coordinate basis \( \{e_a\} \) and its dual \( \{\theta^a\} \), define the connection 1-form, \( \omega^a_b \), and the torsion 2-form, \( T^a \) that obey the Cartan structure equation,

\[
T^a = d\theta^a + \omega^a_b \wedge \theta^b.
\]

[3 marks]

(4.iv). Show that for a metric connection this connection 1-form is antisymmetric, so that \( \omega_{ab} = -\omega_{ba} \). [3 marks]

(4.v). Consider the geometry from part (4.ii) above with Cartesian co-ordinates \( \{X,Y\} \) for \( \mathbb{R}^2 \) with the unit disk

\[
X^2 + Y^2 < 1
\]

so that the metric is

\[
g = \frac{1}{(1-X^2-Y^2)^2} \left( dX \otimes dX + dY \otimes dY \right).
\]
Write this metric in a non-coordinate dual basis \( \{ \theta^a \} \) as
\[
g = \delta_{ab} \theta^a \otimes \theta^b
\]

Consider the Levi-Civita connection of this metric and explicitly deduce its connection 1-form in the \( \{ X, Y \} \) coordinate basis using the antisymmetry property \( \omega_{ab} = -\omega_{ba} \) and the Cartan structure equation given above.

[4 marks]