Continuous symmetries
Constitutes the Poincaré invariance → rotations, boosts, translations

Scale invariance
Conformal invariance

Gauge symmetries, Standard Model: $SU(3) \times SU(2) \times U(1)$

Global symmetries, $SU(N_f)$, baryon symmetry, lepton symmetry...

Discrete Symmetries
$C, P, T$

Group Theory
- Lie group, continuous
- Non-abelian
- Lie algebras
- Representation theory

Group $G$ is a set of elements $g \in G$ with a binary composition law with 4 properties:

1. Closure \( g, g' \in G \Rightarrow g \cdot g' \in G \)
2. Associativity \( g, g', g'' \in G \)
   \( \text{then} \ (g \cdot g') \cdot g'' = g \cdot (g' \cdot g'') \)
3. Identity \( I \in G \) s.t. \( I \cdot g = g \cdot I = g \)
4. Inverse \( g \in G, \exists g^{-1}, \text{st.} \ g \cdot g^{-1} = g^{-1} \cdot g = I \)

Examples
- Rotations in various dimensions
- Integers with addition
- $GL(n, \mathbb{R})$ group of general linear $n \times n$ matrices with real entries
- $GL(n, \mathbb{C})$
- Boosts and rotations
- Translations
- Scale transformations...
Consider a vector space in \( \mathbb{R}^n \), \( V = (x_1, \ldots, x_n)^T \)

length \( \sum_{i=1}^{n} x_i^2 = \|x\|^2 \quad \|x\|^2 = V^T V \)

Consider the set of matrices \( M \) that preserve the length \[ V' = MV \quad \text{where} \quad M \in \text{GL}(n, \mathbb{R}) \]

\( (V')^T V' = V^T M^T M V = V^T V \quad \text{for any} \quad V \)

\( \Rightarrow M^T M = I_n \in \text{GL}(n, \mathbb{R}) \)

By definition, \( M \) is an orthogonal matrix \( M \in O(n) \)

\( n=2 \quad V = (x, y)^T \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \)

\( M^T M = I \quad \det M^T M = 1 = \det M^T \det M = (\det M)^2 \quad \det M = \pm 1 \)

divide into two cases, one for \( n \) odd

\( n \) even

\( n=2 \)

\( M = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad \Rightarrow \det M = 1 \)

Another discrete parameter \( M = JM \)

define \( J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

*definition the set of all matrices \( M \in O(n) \) s.t. \( \det M = 1 \) is called \( SO(n) \)

\( \theta \in [0, 2\pi] \), the set of all possible elements of \( SO(2) \) is parameterized by a circle

Group manifold: the set of all values that the parameters of the group can admit

group manifold of \( SO(2) \) is \( S^1 \) the circle
Aside: more generally $S^n$ is a rotation for the $n$-dimensional sphere, the set of all \( x_i, \quad i = 1, \ldots, n+1 \) s.t.
\[
\sum_{i=1}^{n+1} x_i^2 = r^2
\]
for \( r \) the radius of the $n$-sphere

\[ n=3, \quad O(3) - \text{the set of 3x3 orthogonal matrices} \]
\[ M \in O(3) \quad M^T M = I_3 \]
compute the number of continuous parameters
\( M \) has $n^2$ real entries and is subject to $\frac{n(n+1)}{2}$ equations

\[ \text{# off diagonal entries is } \binom{n}{2} \]
add $n$ diagonal entries
For \( n=3 \) we have 3 parameters, specialising to $SO(3)$

Example:
Euler angles
\[
\begin{pmatrix}
\cos \theta_1 & -\sin \theta_1 & 0 \\
\sin \theta_1 & \cos \theta_1 & 0 \\
0 & 0 & 1
\end{pmatrix} \rightarrow [\alpha]
\]

Problem: Not unique
\[ \rightarrow [\alpha] \]
\[
\begin{pmatrix}
\cos \theta_2 & 0 & \sin \theta_2 \\
0 & 1 & 0 \\
-\sin \theta_2 & 0 & \cos \theta_2
\end{pmatrix} \rightarrow [\beta]
\]
\[ \rightarrow [\beta] \]
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_3 & -\sin \theta_3 \\
0 & \sin \theta_3 & \cos \theta_3
\end{pmatrix} = M
\]
5. Observables \( A = A' \)
\( \hat{\alpha}, \hat{\rho} \) etc.
\[ [\hat{\alpha}, \hat{\rho}] = i\hbar \]

(quantization, \( \{ \alpha, p \} = 1 \rightarrow [\hat{\alpha}, \hat{p}] = i\hbar \))

6. Measurement
\( \psi \xrightarrow{\text{measurement}} \) eigenstate of \( A \)

"collapse of the wavefunction"

7. Composite systems
\( \psi(\mathbf{r}_1, \mathbf{r}_2) = \psi_A(\mathbf{r}_1) \psi_B(\mathbf{r}_2) \) if uncorrelated
Lecture 2

$SO(n)$, $O(n)$

$\{ \det \neq 1 \}$

special

for $\det = 1$

# of parameters in $SO(n)$

\[
\binom{n}{2} = \frac{n(n-1)}{2}
\]

called the dimension of $SO(n)$

Look at infinitesimal transformations

\[
\Psi(X), \quad \Psi(X + a) = \Psi(X) + a \frac{\partial \Psi}{\partial X} + \frac{a^2}{2} \frac{\partial^2 \Psi}{\partial X^2} + \ldots + \frac{a^n}{n!} \frac{\partial^n \Psi}{\partial X^n} + \ldots
\]

\[
a \frac{\partial}{\partial X} = \left( 1 + a \frac{\partial}{\partial X} + \frac{a^2}{2} \frac{\partial^2}{\partial X^2} + \ldots \right) \Psi
\]

\[
e^a \frac{\partial}{\partial X} \Psi
\]

\[
e^{ia \frac{\partial}{\partial X}} \Psi(X)
\]

$p = -i \hbar \frac{\partial}{\partial X}$, $p$ is the generator of space translations

Conservation law: translation invariance $\rightarrow$ conserved current, conserved charge, $p$

This conserved charge generates symmetry we started with

time translations, $H$,

$e^{i \frac{\hbar}{\kappa} H t} \Psi(X, t) = \Psi(X, t + \Delta t)$

[3 vectors! rotations, generators of rotations: angular momenta, $L$]

$SO(2)$,

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} = M \in SO(2)
\]

$M = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} + \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \theta + O(\theta^2)
$

$X$ is the generator of infinitesimal rotations in $SO(2)$

\[
e^{\theta X} = \sum_{n=0}^{\infty} \frac{(\theta X)^n}{n!} = \sum_{n=0}^{\infty} \frac{(\theta X)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\theta X)^{2n+1}}{(2n+1)!} = [\alpha]
\]
\[
X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
[\alpha] = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + X \sum \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}
\]

\[
I \cos \theta + X \sin \theta
\]

Look at \(SO(3)\),

\[
L_x = Y \frac{\partial}{\partial z} - Z \frac{\partial}{\partial y}
\]

\[
L_i = \varepsilon_{ijk} X_j \frac{\partial}{\partial x_k}
\]

\[
[L_x, L_y] = L_z
\]

\[
\begin{pmatrix}
\cos \theta_3 & -\sin \theta_3 & 0 \\
\sin \theta_3 & \cos \theta_3 & 0 \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} + \theta_3
\begin{pmatrix} 0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + O(\theta_3^2)
\]

\[
X_1 = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\]

Def. a \(n \times n\) matrix that represents a group element is called an \(n\) dimensional representation of the group.

\[
e^{X_1}X + \theta_2 X_2 + \theta_3 X_3
\]

\[
e^{A+B} = e^A e^B + \frac{1}{2} [A,B] + \frac{1}{12} [A,[A,B]] + \frac{1}{12} [[A,B],B] + \ldots
\]

Baker-Hausdorff Formula
\([X_1, X_2] = X_3\)
\([X_2, X_3] = X_4\)
\([X_3, X_4] = X_1\)

\([X_i, X_j] = \varepsilon_{ijk} X_k\)

3 \(X_i\)'s that form an algebra, Lie algebra
3 dimensional representation of this algebra

**Lie Algebras?** A Lie Algebra is a set of elements \(\mathfrak{g}\) which form a vector space, with a binary operation \([\cdot, \cdot] \in \mathfrak{g}\)

(i) **Closure:** \(A, B \in \mathfrak{g}, [A, B] \in \mathfrak{g}\)

(ii) **Bilinearity:**
\[
\]
\[
[C, \alpha A + \beta B] = \alpha [C, A] + \beta [C, B]
\]
\[
\forall \alpha, \beta \in \mathbb{C}
\]

(iii) **Alternativity:**
\[
[A, B] = -[B, A]
\]
\(\forall A, B \in \mathfrak{g}\)

(iv) **Jacobi identity:**
\[
[[A, B], C] + [[B, C], A] + [[C, A], B] = 0
\]
\(\forall A, B, C \in \mathfrak{g}\)

---

**Set of all Lie Algebras?**

How many finite dimensional Lie Algebras are there?

**reps**

[Ex: \(2 \times 2\) matrices to satisfy Lie Algebra - can they be found? \(4 \times 4\)?]
Recap

\[ \text{SO}(n), \text{MM}^T = I_n, \quad \det M = 1 \]

\[ M \in \text{SO}(n) \] preserves the length of an \( n \)-dimensional vector

\[ \mathbb{R} \ni x_i \quad i = 1, \ldots, n \quad \sum_i x_i^2 = \sum_i M_{ij} x_j \quad \sum_i x_i^2 = \sum_i x_i^2 \]

Algebra of \( \text{SO}(3) \) \[ [x_i, x_j] = \varepsilon_{ijk} x_k \quad x_i \text{ are } 3 \times 3 \text{ matrices} \]

For unitary groups, take a complex \( n \)-dimensional vector

\[ V = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \]

\[ v_i \in \mathbb{C}, \text{ a length } V^\dagger V, \quad V = UV \quad U \in \text{GL}(n, \mathbb{C}) \]

\[ (V')^\dagger (V') = V^\dagger U^\dagger UV = V^\dagger V, \quad U^\dagger U = I_n \]

\( U(\mathbb{n}) \) is the set of all matrices in \( \text{GL}(n, \mathbb{C}) \) s.t. \( UU^\dagger = I_n \)

\[ \det (UU^\dagger) = 1 \quad \det(UU^\dagger) = \det(U) \det(U^\dagger) = |\det U|^2 \]

\[ (\det U)^* \]

So \( \det U = e^{i \theta} \) for some \( \theta \), a phase

For \( U(1) \), the matrix is \( |X| \) can be written as \( e^{i \theta}, \theta \in [0, 2\pi] \)

group manifold is a circle \( S^1 \)

\( U(1) \cong \text{SO}(2) \)

[\( \text{SU}(n) \) is the set of unitary matrices s.t. \( \det U = 1 \)]

One can construct for any matrix in \( U(n) \) a unique decomposition

\[ U = e^{i \theta} \tilde{U} \quad U \in U(n), \quad \tilde{U} \in \text{SU}(n) \]

\[ \rightarrow U(n) = U(1) \times \text{SU}(n) \]

\( \in [\text{SU}(2) \ : \ c \in \text{SU}(1) \text{ with } n = 2] \)

Take \( n = 2 \) \( 2 \times 2 \) matrices \[ U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C} \]

[What conditions must this matrix satisfy to belong to \( \text{SU}(2) \)?]

\[ U^\dagger \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = U^\dagger \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \Rightarrow c = -b^*, \quad d = a^* \]

\[ \Rightarrow U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \]
\[UU^t = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \quad |a|^2 + |b|^2 = 1\]

\[a = Y_0 + iY_3 \quad \Rightarrow \quad (Y_0)^2 + (Y_3)^2 + (Y_2)^2 + (Y_3)^2 = 1\]

Group manifold of SU(2) is \(S^3\)

\[U = \begin{pmatrix} Y_0 + iY_3 \\ -iY_2 + Y_3 \end{pmatrix} = Y_0 \mathbb{1}_2 + iY_i \mathbf{s}_i\]

\[\mathbf{s}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{s}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \mathbf{s}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\]

\[\mathbf{s}_i \mathbf{s}_j = \delta_{ij} \mathbb{1}_2 + i\epsilon_{ijk} \mathbf{s}_k\]

\[T_i = -\frac{i}{2} \mathbf{s}_i \quad [T_i, T_j] = \epsilon_{ijk} T_k\]

2 representations of SU(2) 2 dimensional 3

\[\rightarrow\] Lie algebra of SU(2) is the same as the Lie algebra of SO(3)

For any matrix \(U \in SU(2)\)

Consider the 3x3 matrix, \(O_{ij} = \frac{1}{3} \text{Tr}(\mathbf{s}_i U \mathbf{s}_j U^{-1})\)

- Tr of a generator in the algebra is 0

*Exercise: show that \(O_{ij} \in SO(3)\)

Note that \(U\) and \(-U\) give the same element in SO(3)
therefore we have a 2 to 1 mapping from SU(2) to SO(3)

Recall the group manifold of SU(2) is \(S^3\)

SO(3) \(\sim S^3/\mathbb{Z}_2 = \mathbb{RP}^3\)

If \(Y_0, Y_1, Y_2, Y_3\) s.t. \(\sum_{i=0}^3 Y_i^2 = 1\), is a point in \(S^3\) or represents an element in SU(2), then \(Y_i - Y_j\) represent a single point in SO(3)

What is the set of all possible representations of SU(2)?
For an integer \( n = 2j \), \( n = 0, 1, 2, \ldots \), \( n \) is non-negative, there exists a representation of dimension \( n+1 = 2j+1 \), \( j \) is the spin.

A matrix of dimension \( (n+1) \times (n+1) \)

each such representation has a weight \( m \) \(-j \leq m \leq j\)
Representation Theory

SU(2) \[ [L_i, L_j] = \varepsilon_{ijk} L_k \]

\[ L_i = -\frac{i}{2} \sigma_i \]

2d representation of this algebra

\[ J_3 = i L_3 \quad J^\pm = \frac{i}{\sqrt{2}} (L_1 \pm i L_2) \]

\[ [J_3, J^\pm] = \pm J^\pm \]

\[ [J_+, J_-] = J_3 \]

J's act as a finite dimensional vector space

\[ J_3 \phi = m \phi \]

\[ J_3 J^\pm |\phi\rangle = (\pm J^\pm + m J_3) |\phi\rangle = (m \pm 1) J^\pm |\phi\rangle \]

*Highest weight j s.t. For m = j, \( J^+ |m = j\rangle = 0 \) requirement for finite dimensional

*There is a lowest weight state \( j' \) s.t. For \( J^- |m = j'\rangle = 0 \)

The number of steps from the lowest weight \( j' \) to the highest weight \( j \) is a non-negative integer, call it \( n \), \( j - j' = n \)

There is an operator which commutes with all 3 operators:

\[ J^2 = \frac{1}{2} (J_+ J_- + J_- J_+) + J^2_3 \]

\[ [J^2, J^\pm] = 0 \]

\[ [J^2, J_3] = 0 \]

\[ J^2 J^+ |m\rangle = J^+ J^2 |m\rangle = \alpha J^+ |m\rangle \]

\[ J^2 |m\rangle = \alpha |m\rangle \quad \alpha \text{ is the eigenvalue of } J^2 \text{ for all m} \]

\[ J^2 (J^+ |m\rangle) = \langle J^+ |m\rangle \]
For: \[ J^2 |m = j\rangle = (J_+ J_- + J_- J_+ + J^2_3) |m = j\rangle \]

\[ \text{but } J_+ |m = j\rangle = 0 \]
\[ \Rightarrow J^2 |m = j\rangle = ([J_+, J_-] + J_- J_+) |m = j\rangle + J^2 |m = j\rangle \]
\[ \Rightarrow J^2 |m = j\rangle = (j^2 + j) |m = j\rangle \]
\[ \Rightarrow \lambda = j^2 + j \]

Second Casimir operator of \( SU(2) \)

Similarly:
\[ J^2 |m = j\rangle = (J_+ J_- + J_- J_+ + J^2_3) |m = j\rangle = (j^2 - j^2) |m = j\rangle \]
\[ \text{but } J_+ J_- |m = j\rangle = 0 \]
\[ \Rightarrow \lambda = j^2 + j = j^2 - j^2 \]

Solutions: \( j' = -j \) \( \text{(i)} \)

2) \( \text{Put } j' = j - n \)
\[ \Rightarrow j^2 + j = (j - n)^2 - (j - n) \]
\[ = j^2 - 2j n + n^2 - j + n \]
\[ \Rightarrow 2j = n - 2j n + n^2 \]
\[ \Rightarrow 2j(n + 1) = n(n + 1) \]
\[ \Rightarrow j = \frac{n}{2} \quad n > 0 \]

Any representation of \( SU(2) \) is characterized by 2 quantum numbers

1) Spin \( j \) (can replace by highest weight, an invariant of this representation, constant over all states)

2) \( m \), eigenvalue of \( J_3 \), \( j \leq m \leq j \)

There are infinitely many representations for \( SU(2) \) characterized by highest weight \( n \), \( n = 0, 1, 2 \)
the dimension of each representation is \( n + 1 = 2j + 1 \)
There are \( n + 1 \) weights which are characterized by \( m \)
where \( -j \leq m \leq j \)
Recall SO(3) has the same Lie algebra
All reps of SO(3) are in this set but a rotation operator
around the \( L_3 \) looks like
\[
e^{iJ_3 \theta} | m \rangle = e^{i m \theta} | m \rangle
\]
\( \theta \) is an angle taking values \([0, 2\pi]\)
\( \Rightarrow e^{i m 2\pi} = 1 \) \( \therefore m \) is an integer
\( \therefore j \) is an integer
\( \Rightarrow n \) must be even

\( \Rightarrow \) For SO(3) \( n \) is even

- SU(2): the highest weight \( n \) is any non-negative integer
- SO(3): \( n \) is an even integer

Normalisations of States

- the SU(2) lattice, \( n = 0 \), \( j = 0 \), \( m = 0 \)
  \( n = 1 \), \( j = \frac{1}{2} \), \( m = \pm \frac{1}{2} \)
  \( n = 2 \), \( j = 1 \), \( m = -1, 0, 1 \)
  \( n = 3 \), \( j = \frac{3}{2} \), \( m = -\frac{3}{2} \ldots \frac{3}{2} \)

\( \Rightarrow \) mapping of 2d lattice to all weights of SU(2)

(All points in the positive quadrant)
Lecture 4

Irreducible representations of \( SU(2) \)

Highest weight \( n-2j \) \( n=0,1,2,3, \ldots \)
of dimension \( n+1 \)

\( (n+1) \) dimensional vector space \( \langle j, \nu \rangle \)

\( j \leq \nu \leq j \)

\( J_3, J_\pm \)

Consider the matrix \( M_{\nu, \mu} = \langle j, \mu | J_3 | j, \nu \rangle = \text{diag}(j, j-1, j-2, \ldots, -j) \)

\( \text{Tr}(M) \)

\( M \)

\( J_\pm \)

\[ \text{Tr} \ e^{i J_3 \theta} = e^{ij \theta} + e^{i(j-1) \theta} + e^{i(j-2) \theta} + \ldots + e^{i(-j) \theta} \]

Def: If \( M \) is a \( d \) dimensional representation of a group element then \( \text{Tr}(M) \) is called \( \chi \) character(s).

Set \( X = e^{i \theta} \)

\[ \chi_n(X) = \text{Tr} \ e^{i J_3 \theta} = \sum_{n=-\frac{n}{2}}^{\frac{n}{2}} e^{i n \theta} = \sum_{n=-\frac{n}{2}}^{\frac{n}{2}} X^{2n} \]

\( \chi_n(X) \) (the character of the \( SU(2) \) rep of highest weight \( j \))

Ex: show that the character is an invariant of the representation

Can replace \( J_3 \) by \( J_\pm \) and still get the same character

Examples: For \( n=0 \) \( \chi_0(X) = 1 \), \( \chi_1(X) = X + \frac{1}{X} \), \( \chi_2(X) = X^2 + 1 + \frac{4}{X^2} \)

\[ \chi_3(X) = X^2 + X + \frac{4}{X} + \frac{4}{X^2} \]

[is realized by]

Tensor product decomposition, multiplication of characters

\[ \chi_1(X) \chi_1(X) = \left( X + \frac{1}{X} \right)^2 = X^2 + 2 + \frac{4}{X^2} = \chi_2(X) + \chi_0(X) \]

\[ \frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0 \rightarrow \text{[working with characters of the group to obtain the same result]} \]

\[ \chi_1(X) \chi_2(X) = \left( X + \frac{1}{X} \right) \left( X^2 + 1 + \frac{4}{X^2} \right) = X^3 + X + \frac{4}{X} + X + \frac{4}{X} + \frac{1}{X^2} \]

\[ \rightarrow \chi_3(X) + \chi_4(X) \]

[6 weights here and here]

\[ \frac{1}{2} \otimes 1 = \frac{3}{2} \oplus \frac{1}{2} \]

\[ 1 \otimes 1 = 2 \oplus 1 \oplus 0 \]

[In general]: \( j \otimes j' = \oplus \)

\( j^+, j^- \)

\( j'' = |j' - j| \)

[\( \ominus = \) direct sum]
\[ |\Psi_n \rangle = (J^-)^m |\Psi \rangle \quad \text{[assuming]} \quad \langle \Psi | \Psi \rangle = 1 \]

\[ \langle \Psi_n | \Psi_n \rangle = \frac{(2j)!}{2^{2n}(2j-2n)!} N_n \quad \text{[normalization factor]} \quad J_3^+ = J_3, J_\pm = J_\mp \]

\[ \langle \Psi_n | \Psi_n \rangle \cdot \langle \Psi | (J^+)^n (J^-)^n | \Psi \rangle \]

\[ |j, m \rangle = \frac{1}{N_{j-m}} |\Psi_{j-m} \rangle \quad \text{Clebsch-Gordan coefficients} \]

\[ J_- |j, m \rangle = \frac{1}{\sqrt{2}} \sqrt{(j+m)(j-m+1)} |j, m-1 \rangle \quad \frac{1}{\sqrt{2}} \sqrt{(j+m)(j-m+1)} |j, m \rangle \quad \text{[or]} \]

\[ J_+ |j, m \rangle = \frac{1}{\sqrt{2}} \sqrt{(j+m+1)(j-m)} |j, m+1 \rangle \quad |j, m \rangle \quad \text{[easily valid]} \]

\[ J_- |j, m \rangle = \frac{1}{\sqrt{2}} \sqrt{(j+m+1)(j-m)} |j, m+1 \rangle \]

\[ \text{[Can use this calculation to evaluate the tensor products]} \quad |\frac{1}{2}, \frac{1}{2} \rangle \otimes |\frac{1}{2}, \frac{1}{2} \rangle \]

\[ |1, 1 \rangle = |\frac{1}{2}, \frac{1}{2} \rangle \otimes |\frac{1}{2}, \frac{1}{2} \rangle \]

\[ J_- |1, 1 \rangle = (J_- |\frac{1}{2}, \frac{1}{2} \rangle \otimes |\frac{1}{2}, \frac{1}{2} \rangle + |\frac{1}{2}, -\frac{1}{2} \rangle \otimes (J_- |\frac{1}{2}, \frac{1}{2} \rangle) \]

\[ |1, 0 \rangle = \frac{1}{\sqrt{2}} |\frac{1}{2}, -\frac{1}{2} \rangle \otimes |\frac{1}{2}, \frac{1}{2} \rangle + \frac{1}{\sqrt{2}} |\frac{1}{2}, \frac{1}{2} \rangle \otimes |\frac{1}{2}, -\frac{1}{2} \rangle \]

\[ \text{[RHS]} \quad R_{d_1} \otimes R_{d_2} \]

\[ \text{[Can act with J_- on both sides]} \quad J_- \text{acts on all states} \]

\[ |1, 0 \rangle = |\frac{1}{2}, \frac{1}{2} \rangle \otimes |\frac{1}{2}, -\frac{1}{2} \rangle \]

\[ J_- |1, 0 \rangle = |1, -1 \rangle \]

\[ |1, -1 \rangle = |\frac{1}{2}, -\frac{1}{2} \rangle \otimes |\frac{1}{2}, -\frac{1}{2} \rangle \]