Lecture 26

Particle Symmetries

Roots → positive roots, simple roots \( \alpha_i \). Fundamental weights

\[ A_{ij} = 2 \langle \alpha_i, \alpha_j \rangle, \quad 2 \langle \lambda, \alpha_j \rangle = \delta_{ij} \quad \lambda = (\lambda^{\alpha_i})_{ij} \alpha_i \] → give rise to basic rep of group [8]

The highest weight of irrep

\[ \lambda = \sum_{i=1}^{K} n_i \lambda_i \quad \text{if} \quad \lambda = \lambda_i \quad \text{each of its basic irrep} \]

[see later][8]

\[ \text{dim } \left[ 0, \ldots, 0, 1, 0, \ldots, 0 \right] = (n+1)^K \quad \text{K-th rank antisymmetric representation of } SU(n+1) \]

\[ \binom{n+1}{k} = \binom{n+1}{n+1-k} \]

Set \( N = n+1 \)

\[ \text{dim } \left[ v_1, \ldots, v_{N-1}, v_N \right] = \text{dim } \left[ v_{N-1}, \ldots, v_1 \right] \]

Example

\[ A_2 = D_3 = SU(4), SO(6) \]

\[ \text{dim } \left[ v_1, v_2, v_3 \right] = \text{dim } \left[ v_3, v_2, v_1 \right] \]

\[ \text{dim } \left[ 1, 0, 0 \right] = 4 = \text{dim } \left[ 0, 0, 1 \right] \quad \text{dim } \left[ 0, 1, 0 \right] = 6 \]

\[ [v_1, v_2, v_3] \text{ is in the tensor product of Sym}^{v_2} [1, 0, 0] \oplus \text{Sym}^{v_2} (\Lambda^2 [1, 0, 0]) \oplus \text{Sym}^{v_2} (\Lambda^3 [1, 0, 0]) \]

[when take K-th asym product move 1 along rows of \([0, 0, 0] \)]

For a representation of \( SU(N) \) of Dynkin labels \[ [v_1, \ldots, v_{N-1}, v_N] \]

the complex conj rep is \[ [v_{N-1}, \ldots, v_1] \]

\[ \left( \begin{array}{cccc}
    m_1 & m_2 & m_3 & m_4 \\
    m_5 & m_6 & m_7 & m_8 \\
    m_9 & m_{10} & m_{11} & m_{12}
\end{array} \right) \quad \text{complex conjugate} \]

For \( SU(4) \)

\[ \left( \begin{array}{c}
    \square \\
    \square \\
\end{array} \right)^c = \left( \begin{array}{c}
    \square \\
\end{array} \right) \]

\[ \left( \begin{array}{c}
    \square \\
\end{array} \right)^c = \left( \begin{array}{c}
    \square \\
\end{array} \right) \]

[Originally \( m_1, m_2, \ldots \) changed to \( m_3, m_4, \ldots \)]
All irreps \([0, n, 0]\) are real, symmetric traceless with rank tensors of \(SO(6)\).

\[
\begin{pmatrix}
 & \vdots \\
2,1 & \vdots \\
 & \vdots \\
4-1,4-2 & \vdots \\
(3,2) & \vdots \\
\end{pmatrix}
\]

i.e., \([1,1,0] \cong [0,1,1]\)

For \(SU(6)\), \([0,0,n,0,0]\) is real.

\([n_2, n_3, \ldots n_3, n_2, n_1]\) in general.

The \([\beta] \cong [\gamma]\) true for any group, now apply to \(B_n\).

\(B_n\) \([0, \ldots, 0, 1, 0, \ldots, 0]\)

\[\text{dim } [0, \ldots, 0, 1] = 2^n\] spinor of \(SO(2n+1)\)

Examples:

\(\text{B}_1 = \text{C}_1 = \text{A}_2 \cong SU(2)\) \(2^1 = 2\) fund. rep.

\[\det (A_{B_n}) = 2\] \(SO(3)\) \(Sp(1)\)

2 types of irreps \([n_2, \ldots, n_n]\): if \(n_n\) is even bosonic
odd fermionic

Tensor product of

bosonic \(\times\) bosonic gives bosonic
ferm \(\times\) ferm gives fermionic

\([\text{stay in bosonic lattice}]\)

\([\text{stay in fermionic lattice}]\)

\([\text{move from ferm to bos}]\)

\([\text{because of properties of determinant}]\)

\([\text{fundamental]}\)

\([\text{true root}]\)

\(A_1: \lambda = \frac{1}{2} \alpha\)

\(D_2 = A_1 \times A_1\)

\(\alpha \lambda\)

\(\lambda\)

\(\alpha\)

\(\lambda\)

[Area 4 times as large for \(D_2\) as area picked out by roots]
A representation that appears in physics

\[ [1, 0, \ldots, 0, 1] \]

\[ \text{vector} \sim \text{spinor} \]

\[ \begin{align*}
B_2 & : [0, 1] \quad 4, \text{pseudo-real} \\
B_3 & : [0, 0, 1] \quad 8, \text{real} \\
E_8 & : [0, 0, 0, 1] \quad 16, \text{real}
\end{align*} \]

Cf.

\[ A_{2n+1} \]

\[ A_5 \cong SU(6) \]

\[ \chi_{[1,0,0,0,0]} = \frac{Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + \frac{1}{Y_1}}{Y_1} \]

Set the fugacities of \( C_3 \) to be \( Z_i \)

\[ Z_1 = Y_1 = Y_5 \]
\[ Z_2 = Y_2 = Y_4 \]
\[ Z_3 = Y_3 \]

\[ \chi_{[1,0,0]}_{C_3} = \frac{Z_1 + Z_2 + Z_3 + \frac{1}{Z_2}}{Z_1} \quad \text{pseudo-real} \]

\[ [0, \ldots, 0, 1, 0, \ldots, 0] \quad \text{traceless K-th rank antisymmetric rep of Sp(n)} \]

[How know traceless?]

Recall for SO(n) 2 vectors \( V_i, W_i \quad i = 1, \ldots, n \)

\[ V_i W_j + V_j W_i \quad \text{second rank symmetric} \]

\[ V_i W_i \] is a singlet
\textbf{Note 1: Relation Between Determinant and Area (general formulation)}

In 2D, area of a parallelogram spanned by 
\[ \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} \]
\[ \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} \]
is given by
\[ \| \mathbf{a} \times \mathbf{b} \| = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \]

[see traceless by analogy]
Physics: $A_\mu$, $\mu = 0 \ldots 3$ so we are in $SO(4)$

e.g. $D_2 = A_1 \times A_1$

4-D rep: $[3,0]$, $[0,3]$ or $[1,1]$, or $[1,0] + [0,1]$

spin $\frac{3}{2}$ rep For the vector rep we want

one $SU(2)$

do not want this

2 spinors, which do not work properly as well

For the $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$\partial_\mu$ trans. in $[1,1]$ (i.e. as a vector $A_\mu$)

$F_{\mu \nu} \rightarrow -\Lambda^2 [1,1]$ anti-sym product of two vectors

$\dim \Lambda^2 [1,1] = 4 \choose 2 = 6$

Compute $\Lambda^2 [1,1]$ in irrep with $PE_F$

or $f(x_1, x_2) = (x_1 + x_1^{-1})(x_2 + x_2^{-1}) = \chi_{[1,1]}(x_1, x_2)$

then $\frac{1}{2} f^2(x_1, x_2) - \frac{1}{2} f(x_1^2, x_2^2) = (2,0)(0,2)(0,0)(0,0)(0,-2)(-2,0)$

from $(1,1)(1,-1)(-1,1)(-1,-1)$

Highest weight: $[2,0]$ so $\Lambda^2(1,1) = [2,0] + [0,2]$ (sym in the exchange of $[\cdots, \cdots]$)

So $F_{\mu \nu}$ is reducible

It's a reducible rep of $SO(4)$: the adjoint rep of $SO(4)$

Can then be written

$F_{\mu \nu} = F_{\mu \nu}^+ + F_{\mu \nu}^-$ which transforms without mixing

Using $\varepsilon_{\mu \nu \rho \sigma} F^\rho \sigma \equiv \tilde{F}_{\mu \nu}$, we can write $F_{\mu \nu}^{\pm} \equiv \frac{1}{2} (F_{\mu \nu} \pm \tilde{F}_{\mu \nu})$

and the $F_{\mu \nu} = F_{\mu \nu}^+ + F_{\mu \nu}^-$

Does this work?

$\varepsilon_{\mu \nu \rho \sigma} F_{\mu \nu}^+ = \varepsilon_{\mu \nu \rho \sigma} (\frac{1}{2} F_{\mu \nu} + \frac{1}{2} \varepsilon_{\mu \nu \alpha \beta} F^{\alpha \beta}) = \frac{1}{2} \tilde{F}_{\rho \sigma} + \frac{\sqrt{2}}{2} \varepsilon_{\rho \sigma} F^{\alpha \beta}$

$= F_{\rho \sigma}^+ \uparrow$ shows that
i.e. $F^\mu$ has an eigenvalue +1 under $E_{\mu}^\nu$ as well as $F_{\mu}^\nu$.

We can identify these self-dual and anti-self-dual tensors $F^\mu$ and $F_{\mu}^\nu$ as the elements transforming in $[2,0]$ and $[0,2]$.

The middle dimensional antisym tensor:

$$\frac{2n}{2} = n$$

SO($2n$): the $n$-th rank antisym rep of SO($2n$) is reducible into SD and ASD tensors.

Self dual $\Leftrightarrow F^+ = 0$, antiself dual $\Leftrightarrow F^- = 0$

$$D_\alpha: \quad \bigcirc\cdots\bigcirc$$

SO($2n$): basic rep are $[0, \ldots, 1, \ldots, 0]$ $

\uparrow_{kth}$

For $k=n-1$ or $n$: spinor rep of SO($2n$), dim $2^{n-1}$ distinguished by calling spinor and spinor (!)

Dirac spinor (on SO(4)) is reducible in $[1,0] + [0,1]$ (because massless part)

called two Weyl spinors, the L and R ones

(though Dirac spinor is L-R sym)

Let $A_{D_n} = 4$ so set the last two indices to be $n_{n-1}, n_n$ being even or odd (each)

giving $[n_1 \ldots n_{n-1}, n_n] \otimes [m_1 \ldots m_{n-1}, m_n] = [\ldots n_{n-1} + m_{n-1}, n_n + m_n]$

Selection rules:

$$[\ldots \text{odd}, \text{odd}] \otimes [\ldots \text{odd}, \text{odd}] = [\ldots \text{even}, \text{even}]$$

odd, even \quad odd, odd \quad even, odd

\vdots \quad \vdots
Decomposition of exceptional groups \( E_i \) \((i=6,7,8)\) 

\[ \text{dim } [1,0\ldots 0]_{E_6} = 27 = \text{dim } [0\ldots 0,1]_{E_6} \]

\[ \text{dim } \text{Adj } (E_6) = 78 \]

\[ \text{dim } [0,1,0\ldots 0] = 912 \]

\( E_7 \): \( \text{dim } \text{Fundamental } E_7 = 56 \) pseudoreal; \( \text{dim } \text{Adj } E_7 = 133 \)

\( E_8 \): \( \text{dim } \text{Adj } E_8 = 248 \)

Recall: every dot in Dynkin diagram is simple root and fundamental weight (one \( \alpha_i \) corresponds to one \( \Lambda_j \))

\[ \Lambda_i = (A^{-1})_{ij} \alpha_j \quad \text{with} \quad A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \]

[i.e. the roots allow us to form the Cartan matrix]

\[ 2(\Lambda_i, \alpha_j) = S_{ij} \]

\# every rep is represented by a highest weight \( \Lambda \) decomposition in \( \Lambda_i \):

\[ \Lambda = \sum_n n_i \Lambda_i \]

GUT:

\[ \text{SU}(5) \supset \text{SU}(3) \times \text{SU}(2) \times U(1) \]

\[ \text{SO}(10) \]

\[ \text{E}_6 \]

\( \text{How to couple a scalar field to a gauge field?} \quad \text{(Use covariant derivative)} \)

\[ D_{\mu} \phi = \partial_{\mu} \phi + i g A_{\mu} \phi \]

\( \phi \) is a vector space; \( A_{\mu} \) transforming it gives the interaction, i.e. the action of \( A_{\mu} \) on \( \phi \), and the mass of fields
Indeed, consider Dynkin diagram for $E_6$

```
 o---o---o---o
```

taking a subset

```
 o---o   o = D_5 = SO(10)
```

then

```
 o---o---o   o = A_4 = SU(5)
```

but need to understand how to extend the mechanisms: branching

**Branching Rules**

$SU(3) \supset SU(2) \times U(1)$

```
SU(3) \supset SU(2) \times U(1)
```

Suppose we have a group $SU(3)$ and we want a subgroup decomposition $SU(2) \times U(1)$

```
\begin{bmatrix}
2 \times 2 & 1 \times 2 \\
2 \times 1 & 1 \times 1
\end{bmatrix}
```

in $SU(3)$ rep

decomposed into $SU(2)$ different rep

or 3x3 matrix:

```
\begin{bmatrix}
2 \times 2 & 1 \times 2 \\
2 \times 1 & 1 \times 1
\end{bmatrix}
```

Hence $3 \rightarrow 2 + 1$

but charged under the other gp: $2 \uparrow 1 + 1 \downarrow$

2nd rank sym of $SU(3) = 6 \rightarrow ?$

Use the Fugacities:

$$\chi_{[1,0]}(Y_1, Y_2) = Y_1 + Y_2 + \frac{1}{Y_1} Y_2 = \chi_3$$

$$3 = \cdot \cdot \cdot = \text{doublet} \quad \text{with different charges} \quad \begin{cases}
\chi_3 = q^4 \chi_2 + q^{-2} \chi_1 \\
3 = 2 \uparrow 1 + 1 \downarrow
\end{cases}$$

Note: $SU(2)$ fugacity = $X$, $U(1)$ fugacity = $q$; \begin{align*}
3 = q^4 \chi_2 + q^{-2} \chi_1 \\
3 = q^{\frac{1}{2} \chi + X} + \frac{1}{q^2}
\end{align*}

set $SU(2)$ fugacity to $X$

* $U(1)$ --"-- \( = -q \)
Lecture 28

Particle Symmetries

\[
giving: \quad \frac{Y_1}{Y_2} = q_X, \quad \frac{Y_2}{X} = q, \quad 1 = \frac{1}{Y_2} \frac{q}{q_X} = (q_X \cdot q)^{-1} = (q^2)^{-1} \quad \text{ok}
\]

Fugacity map:

\[
Y_1 = q_X, \quad Y_2 = q^2
\]

For the branching rules

not choosing:

\[
X \quad \text{and} \quad \frac{1}{X} \quad \text{symmetry} \quad \text{for SU}(2) \quad \text{properties} \quad (q_X \Rightarrow q_X)
\]

product of terms with \(X\) or \(\frac{1}{X}\) = terms without it \(\Rightarrow\) \(\frac{1}{q_X^2}\)

Fugacity map not unique: 3 possible maps:

1. \(Y_1 = q_X, \quad Y_2 = q^2\)
2. \(Y_1 = q_X, \quad Y_2 = \frac{X}{q}\)
3. \(Y_1 = \frac{q}{X}, \quad Y_2 = q^2\)

For \(6\):

\[
\begin{array}{c}
\ldots \ldots \quad \text{triplet} \\
\ldots \quad \text{doublet} \\
\cdot \quad \text{singlet}
\end{array}
\]

\[
\chi_{[2,0]} = \frac{Y_1^2}{Y_2} + \frac{Y_2^2}{Y_1} + \frac{1}{Y_1} + \frac{Y_2}{Y_2} + \frac{1}{Y_2}
\]

Dynkin labels for \(E_6\)

with the first map: \((Y_1 = q_X, \quad Y_2 = q^2)\)

\[
\chi_{[2,0]} = q^2 X^2 + q^4 + \frac{1}{q^2 X^2} + \frac{q^2 + q^4}{q} + \frac{q^4 + 1}{q^2}
\]

\[
= q^2 (X^2 + 1 + \frac{1}{X^2}) + 1 \frac{X + 1}{q} + \frac{1}{q^4}
\]

i.e.

\[
\chi_{[2,0]} = q^2 \chi_{[2]} + q^{-1} \chi_{[1]} + q^{-4} \chi_{[0]}
\]

\[
\Rightarrow \quad 6 = 3_2 + 2_{-1} + 1_{-4}
\]
Once you have chosen a map there are 2 ways of implementing this map (in this case).

For $X_n$ there are $n$ ways of choosing this map.
Lecture 29

Braching rules

\[ SU(3) = SU(2) \times U(1) \]

recap of lecture 28

Fugacity map

\[ \begin{align*}
3 & \rightarrow 2_1 + 1_{-2} \\
5 & \rightarrow (3,1) + (1,2) \\
& \text{triplet of} \\
& \text{SU(3)} = \\
& \text{singlet under} \\
& \text{SU(2)} \\
& \text{doublet under SU(2)} \\
& \text{singlet under SU(3)} \\
& \text{2-d} \\
& \text{3-d}
\end{align*} \]

[Used to be popular For SM represetations]

\[ \gamma \text{(singlet SU(3)}} \\
\text{doublet SU(2)} \]

[Cartan gens - trace zero]

[ singlet carries U(1) charges - arbitrary normalization - only care e- charge some integer # times some value ⇒ deal w lattice for non-Abelian groups, though this is Abelian]

Character \[ \chi_{(1,0,0,0)}^{SU(5)} = \chi_{(1,0,0,0)}^{SU(3)} \chi_{(0,0,0,0)}^{SU(2)} \chi_{(1,0,0,0)}^{SU(2)} \chi_{(0,0,0,0)}^{SU(2)} \chi_{(0,0,0,0)}^{SU(2)} \]

\[ [1,0,0,0] \]

\[ \frac{Y_1}{Y_2} + \frac{Z_1}{Z_2} + \frac{1}{Z_2} \]

\[ \begin{align*}
& \text{SU(3) fugacities } Z_1, Z_2 \\
& \text{SU(2)} \quad \bigtimes \quad \gamma
\end{align*} \]

Pick a fugacity rep:

\[ Y_4 = q^3, Y_3 = 0 \]

[Fundamental]

[Fundamental]

[Character]

Check:

\[ \begin{align*}
[Y_4 : SU(5) - \text{Fundamental}] \\
[Y_3 : SU(5) - \text{Fundamental}] \\
[Z_2 : SU(3) - \text{Fundamental}] \\
[3 - 3rd rank anti-sym rep SU(5)] \\
\Rightarrow 3 \times \text{charge} \quad U(1) \text{ charge]
\end{align*} \]
\[
5 \rightarrow (3,1)_2 + (1,2)_3 \quad \text{(because } \overline{1} = 2)\]

[cf \( Y_4 \)-possible to decompose to SU(2) rep, i.e. \( X_Y = Y_4 \) as above \( \Rightarrow \) coherent]

[can tell how SU(5) decompose to reps of SU(3)]

[4d lattice SU(5)-different ways to choose sublattices also choose fugacities + natural basis, \( Y_1 \) correspond to fund. weight etc. decomp doesn't depend on fugacity maps]

10 \( \rightarrow \) \( [0,1,0,0] \), \( SU(5) = (Y_2 + Y_3 Y_4 Y_5 + Y_4 Y_2 + Y_5 Y_2 Y_3 Y_4) \)

[2\( nd \) rank comm. rep \( SU(5) \)]

[\( 1 \quad \nabla \quad \nabla \) \& comm. simple -add squares on the sym]

[10 monomials]

\[\begin{align*}
\frac{Y_2}{Y_3} + \frac{Y_3 Y_4}{Y_3} + \frac{Y_4 Y_2 Y_3}{Y_3} + \frac{Y_5 Y_2 Y_3}{Y_3} \\
\sqrt{q} \frac{Z_2}{Z^2} + \sqrt{q} \frac{Z_3}{Z^2} (\text{by taking AS terms of product } \chi_{[1,0,0,0]} \chi_{[3,0,0,2]})
\end{align*}\]

\[\frac{Y_2}{Y_3} \quad \frac{Y_3 Y_4}{Y_3} \quad \frac{Y_4 Y_2 Y_3}{Y_3} \quad \frac{Y_5 Y_2 Y_3}{Y_3} \]

\[\text{dim} = 3 + (3 \times 2) + 1 = 10\]

10 \( \rightarrow \) \( (3,1)_2 + (3,2)_1 + (1,1)_6 \)

[just by writing down 1st term]

\[\left( \frac{1 \times Z_1}{q} \right) \text{ [this gives the } (3,2)_1 \text{ term]}\]

Interpretation:

\((3,2)_1 : \text{Fit naturally with LH quarks}\]

\((\overline{3},1)_4 : \quad \overline{\text{n---n---n---n---RH quarks}}\]

\((1,1)_6 : \text{take to predict massive particle not yet found} \quad \text{[or RH electron-singlet under both]}\]

[adjoint \( SU(5) \) contains adjoint \( SU(3) \) and \( SU(2) \)]

[leptons-singlet under \( SU(2) \) and doublet under \( SU(2) \)]

We have:

\[
\begin{align*}
U(N) & \supset U(n_2) \times U(n_2) \\
SO(N) & \supset SO(n_2) \times SO(n_2) \\
Sp(N) & \supset Sp(n_1) \times Sp(n_2)
\end{align*}
\]

[RANK: \( \frac{N-1}{2} \) \( n_2 = \frac{N-1}{2} \) \( 1 \)]

\[
SU(N) \supset S(U(n_2) \times U(n_2)) \supset SU(n_1) \times SU(n_2) \times U(1)
\]

[Removing Cartan element prop to id (non-zero trace) on both sides]

Note: \( \text{can go to larger groups: } SO(10) \supset SU(5) \) see eq (6) \( E_6 \supset SO(10) \) above

\[\text{traceless: } \begin{align*}
3 \text{ weights with } q^4 = 12 \\
3 \times 2 \quad -1 \quad -1 \quad -1 \quad -q^4 = -1 \Rightarrow -6 \\
1 \quad -1 \quad -1 \quad -q^4 = -6 \Rightarrow -6
\end{align*}\]

[\( n \)-dim real vector space]

\[\text{[natural embedding]} \quad U \text{-complex vector space} \]

\[\text{SO} \text{-real vector space} \]

\[\text{Sp} \text{-2n dim real vector space} \]

[removing traceless subgroup in Cartan algebra]
G ⊆ H

[How find such subgroup?]

\[
SO(8) \supset SO(4) \times SO(4) = SU(2)^4
\]

rank: 4

4 + 1

[rule: given G, what are all possible subgroups?

e.g. can I embed E_6 inside E_7?]

\[
E_7 \supset E_6 \times SU(2) \leftarrow [\text{correct?}] \ [\text{No}]
\]

\[
E_7 \supset E_6 \times U(1)
\]

\[
E_7 \supset SO(12) \times SU(2)
\]

Consider the Dynkin diagram:

\[
\begin{array}{c}
\text{E}_6: o \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv \equiv
\end{array}
\]

[rank]

[preserved in this decomposition]
Branching Rules:

\[ G \supset H \quad \text{and write} \quad R_G = \sum_i R_{H,i} \quad \text{with fugacity maps} \]

We have:

\[ U(N) \supset \prod_i U(n_i), \quad \text{with} \quad \sum_i n_i = N \]

\[ SU(N) \supset U(1)^{K-1} \prod_{i=1}^K SU(n_i), \quad \text{idem} \]

\[ SO(N) \supset \prod_i SO(n_i), \quad \text{idem} \]

\[ Sp(N) \supset \prod_i Sp(n_i), \quad \text{idem} \]

Dynkin method: to compute the decomposition (for classical and exceptional groups) use extended Dynkin diagram (for affine Lie algebra):

add one node that has a scalar product with the node of the adjoint rep

**Examples**

**B**\(_n\):

\[ \begin{array}{c}
\text{adj rep} \\
\text{(because dim adj = n+2)}
\end{array} \]

\[ \Rightarrow \quad \hat{B}_n: \]

extended diagram (has \(n+1\) nodes)

**D**\(_n\):

\[ \begin{array}{c}
\text{2nd rank AS} \\
\text{again is the adj}
\end{array} \]

\[ \Rightarrow \quad \hat{D}_n: \]

\[ \hat{A}_n: \]

\[ \Rightarrow \hat{C}_n: \]

\sim \text{adj} = [2, 0, \ldots, 0]

1.
because for $\text{SO}(n)$, $\text{adj} \equiv 2^{\text{nd}}$ rank AS

for $\text{Sp}(n)$, $\text{adj} \equiv 2^{\text{nd}}$ rank Sym

and because it is $[2, 0, \ldots, 0]$, we will put 2 lines from the node corresponding to $[1, 0, \ldots, 0]$

**Exceptionals**

$E_6$:

```
  adj
```

$E_7$:

```
  adj
```

$E_8$:

```
  adj
```

$G_2$:

```
  adj
```

$F_4$:

```
  adj
```

**Prescription:** remove a node from the extended Dynkin diag $\hat{G}$

the remaining diag is a subgp of the original gp $G$

**Example**

$\hat{D}_n$ gives by removing a node somewhere along the line

```
  o----o . o----o
  K nodes    n-K nodes
```

$\Rightarrow \text{D}_n \supset \text{D}_K \times \text{D}_{n-K}$

i.e. $\text{SO}(2n) \supset \text{SO}(2K) \times \text{SO}(2n-2K)$
Specific cases:

\[ D_n \supset \text{SO}(2n-4) \times \text{SU}(2) \times \text{SU}(2) \quad \text{SO}(4) \]

Note that this method is not complete and useful mainly for exceptional cases. 

- \( \hat{B}_n \) gives choice of \( \text{A}_n \) only \( \Rightarrow \) useless

- \( \hat{G}_2 \) gives \( \text{G}_2 \) the trivial answer

\[ \text{A}_1 \times \text{A}_1 = \text{SU}(2) \times \text{SU}(2) \quad \text{(via o-o)} \]

\[ \text{A}_2 = \text{SU}(3) \quad \text{(via o-o)} \]

So \( \text{SU}(2) \times \text{SU}(2) \subset \text{G}_2 \), \( \text{SU}(3) \subset \text{G}_2 \) (or \( \text{SO}(4) \))

- \( \hat{F}_4 \) gives \( \text{F}_4 \)

\[ \text{A}_1 \times C_3 \equiv \text{SU}(2) \times \text{Sp}(1) \]

\[ \text{A}_2 \times \text{A}_2 \equiv \text{SU}(3) \times \text{SU}(3) \]

\[ \text{A}_3 \times \text{A}_1 \equiv \text{SU}(4) \times \text{SU}(2) \]

\[ \text{B}_4 \equiv \text{SO}(9) \]

\( \hat{E}_8 \) (also called \( \text{E}_7 \)) gives \( \text{E}_8 \)

\[ \text{A}_1 \times \text{E}_7 \]

\[ \text{A}_2 \times \text{E}_6 \]

\[ \text{A}_3 \times \text{D}_5 \equiv \text{SU}(4) \times \text{SO}(10) \equiv \text{SO}(6) \times \text{SO}(10) \]

\[ \text{A}_4 \times \text{A}_4 = \text{SU}(5) \times \text{SU}(5) \quad \text{(because \( \text{A}_3 = \text{D}_5 \))} \]

\[ \text{A}_1 \times \text{A}_1 \times \text{A}_5 \]

\[ \text{A}_1 \times \text{A}_7 \]

\[ \text{D}_8 \equiv \text{SO}(16) \]

\[ \text{A}_8 \equiv \text{SU}(9) \]

So \( \text{A}_3 \times \text{D}_5 \) is not maximal (can embed it in another subgp)
\[ E_8 \supset SU(5) \times SU(5) \]

\[ 248 \rightarrow \underbrace{(24,1) + (1,24)} + (5,\bar{5}) + (\bar{5},5) \]

not enough

\[ \text{(fundamental + adj)} + (\quad) \]

still not enough

or \[ + (5,10) + (10,\bar{5}) \]

\[ + (5,10) + (\bar{5},10) \]

OK?

\[ E_6 \text{ gives } E_6 \]

\[ A_1 \times A_5 \quad \text{so } A_1 \times A_2 \times A_5 \text{ in } E_8 \]

\[ A_2 \times A_2 \times A_2 \quad \text{not maximal because} \quad (A_1 \times A_5) \times A_2 \subset A_2 \times E_6 \subset E_8 \]

(and again the same)

For real rep of \( G \) of dim \( n \), we can embed it in \( SO(n) \)

\[ G \supset SO(n) \]

\[ n \leftarrow n \]

Example

\[ SU(n) \subset SO(n^2 - 1) \quad \text{from the dim of the adj} \]

\[ \text{adj} \leftarrow (n^2 - 1) \text{rep} \quad \text{to the rep of } SO(\text{dim adj}) \]

[Notes copied from board p.6->end here]
\[ E_6 : \]
\[ E_7 : \]
\[ E_8 : \]
\[ F_4 : \]
\[ G_2 : \]
\[ A_n : \]
\[ B_n : \]

remove a node from the extended Dynkin diagram \( \hat{G} \), the remaining diagram is a subgroup of \( G \).

\[ D_n : \]
\[ D_n \supset D_k \times D_{n-k} \]
\[ \text{SO}(2n) \supset \text{SO}(2k) \times \text{SO}(2n-2k) \]

\[ B_n : \]
\[ \text{SO}(2n+1) \supset \text{SO}(2k) \times \text{SO}(2n-2k+1) \]

\[ G_2 \supset G_2 \]
\[ \text{SO}(2n+1) \supset \text{SO}(2n) \]
\[ \text{SU}(2) \times \text{SU}(2) \supset G_2 \]
\[ \text{SU}(3) \subset G_2 \]
\[ A_1 \times A_1 \]
\[ A_2 \]
Branching Rules

\[ F_4, A_2 \times C_3, A_2 \times A_2, A_3 \times A_2 \rightarrow B_4 \]

\[ E_6 \rightarrow E_7 \]

\[ \text{SO}(6) \rightarrow \text{SU}(4) \times \text{SO}(6) \]

\[ E_6, A_1 \times E_7, A_2 \times E_6, A_3 \times D_5, A_4 \times A_4 \]

\[ A_1 \times A_5, A_1 \times A_7, D_6, A_8 \rightarrow \text{not maximal} \]

\[ \text{SO}(16) \subset E_8 \]

\[ A_1 \times A_2 \times A_5 \subset A_2 \times E_6 \subset E_8 \]

**Dynkin**

\[ E_8 \Rightarrow \text{SU}(5) \times \text{SU}(5) \]

\[ 248 \rightarrow (24, 1) + (1, 24) + (5, 10) + (10, 5) + (5, 10) + (5, 10) \]

\[ E_6 \rightarrow \text{not maximal} \]

\[ \Lambda = \sum n_i \Lambda_i \]

\[ \{ n_i \} \]

\[ E_6, A_1 \times A_5, A_2 \times A_2 \times A_2 \]

\[ \text{SU}(3)^3 \subset E_6 \]

For a real representation of a group \( G \) of dimension \( n \):

\[ G \subset \text{SO}(n) \]

Example:

\[ \text{SU}(n) \subset \text{SO}(n^2 - 1) \]

\[ n \leftarrow n \]

\[ \text{adj} \leftarrow n^2 - 1 \]