Imperial College London

MSc TEST Jan 2016

PARTICLE SYMMETRIES AND UNIFICATION

For Students in Quantum Fields and Fundamental Forces

Thursday 14 Jan 2016: 10:00 to 12:00

Answer all questions in Part A and Part B
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 2 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 2 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

SECTION A: Particle Symmetries

1. Consider the first generation of quarks and leptons. Under the $SU(2) \times U(1)$ electroweak symmetry the left- and right-chirality states transform as

$$\begin{array}{rlllll} SU(2) \text{ doublets:} & \begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix} & Y = -1, & \begin{pmatrix} u_L \\ d_L \end{pmatrix} & Y = \frac{1}{3}, & \\ SU(2) \text{ singlets:} & e_R^- & Y = -2, & u_R & Y = \frac{4}{3}, & d_R \quad Y = -\frac{2}{3}, \end{array}$$

where Y denotes the $U(1)$ charge. Recall also that given a representation $\rho : G \to GL(n, \mathbb{C})$ one defines the conjugate representation ρ^* by

$$\rho^*(a) = [\rho(a)]^* \qquad \text{for all } a \in G,$$

where A^* is the complex conjugate of the matrix A.

(i) Show that a general element of $SU(2)$ can be parametrised as

$$a = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \in SU(2) \qquad \text{where } x, y \in \mathbb{C} \text{ and } |x|^2 + |y|^2 = 1.$$

The standard model doublets transform in the two-dimensional "defining" representation $\rho_{(2)}$. What is $\rho_{(2)}(a)$ for $a \in SU(2)$ in the form above? [6 marks]

(ii) Show that in general ρ^* defines a representation and that in particular $\rho^*_{(2)}$ is isomorphic to $\rho_{(2)}$. Hence derive how the conjugate quarks and leptons (e^+, $\bar{\nu}$, $\bar{u}$ and $\bar{d}$) transform under $SU(2) \times U(1)$. [10 marks]

(iii) Consider the Higgs scalar field $SU(2)$ doublet

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

with $Y = 1$, and assume it gets a vacuum expectation value (vev) with $\phi^+ = 0$ and $\phi^0 = V/\sqrt{2} \neq 0$.
Show that there is a $U(1)$ subgroup of $SU(2) \times U(1)$ that leaves the vev of Φ invariant. Find the charges of the quarks and leptons under this symmetry and comment on its physical meaning. [12 marks]

(iv) Show that one can unify the right-handed e_R^+, $\bar{\nu}_R$ and d_R states into a single module transforming as the defining representation of $SU(5)$. In particular, identify how the Standard Model $SU(3) \times SU(2) \times U(1)$ group embeds in $SU(5)$. [10 marks]

(v) Show that the remaining left-handed e_R^-, u_L, d_L and $\bar{u}_L$ states can be unified into a single $SU(5)$ module corresponding to the Young tableau $\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$.

[12 marks]

[Total 50 marks]

SECTION B: Unification and the Standard Model

2. The gamma matrices in $3+1$ spacetime dimensions satisfy the Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$, where the spacetime metric is $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

(i) Define the hermitian matrix γ^5 by $\gamma^5 = \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3$.

(a) Show that $e^{\mathrm{i}\theta\gamma^5} = \cos\theta\mathbf{1} + \mathrm{i}\sin\theta\gamma^5$ and hence that $e^{\mathrm{i}\theta\gamma^5}\gamma^\mu = \gamma^\mu e^{-\mathrm{i}\theta\gamma^5}$.

(b) Show that the axial transformation $\psi_\alpha \to \psi'_\alpha = \left[e^{\mathrm{i}\theta\gamma^5}\right]_{\alpha\beta}\psi_\beta$ is a rigid symmetry (constant θ) of the kinetic term $\mathcal{L}_{\mathrm{kin}} = -\mathrm{i}\,\bar\psi\partial_\mu\gamma^\mu\psi$ for a Dirac spinor field (where $\bar\psi = \psi^\dagger\gamma^0$), but not of the mass term $\mathcal{L}_{\mathrm{mass}} = -\mathrm{i}\,m\bar\psi\psi$.

[15 marks]

(ii) In 2-component notation, the basic $SL(2,\mathbb{C})$ transformations on the four types of spinor doublet representation (undotted and dotted, raised and lowered indices) are given by

$$\lambda_\alpha \to A_\alpha{}^\beta\lambda_\beta \qquad\qquad \bar\lambda_{\dot\alpha} \to \bar\lambda_{\dot\beta}A^{\dagger\dot\beta}{}_{\dot\alpha}$$
$$\psi^\alpha \to \psi^\beta A^{-1\alpha}_\beta \qquad\qquad \bar\psi^{\dot\alpha} \to (A^{-1})^{\dagger\dot\alpha}{}_{\dot\beta}\bar\psi^{\dot\beta}\ .$$

(a) Apply these rules to transform a bispinor $V_{\alpha\dot\beta}$ and use this to find the transformation behaviour of $V^\mu = -\frac{1}{2}\operatorname{Tr}(\tilde\sigma^\mu V)$. Show that this is the correct transformation behaviour of a contravariant Lorentz vector under a Lorentz transformation with $\Lambda^\mu{}_\nu = -\frac{1}{2}\operatorname{Tr}(\tilde\sigma^\mu A\sigma_\nu A^\dagger)$ where $(\sigma_\mu)_{\alpha\dot\beta} = (\mathbf{1}, \sigma^i)$, $(\tilde\sigma_\mu)^{\dot\alpha\beta} = (\mathbf{1}, -\sigma^i)$. You may use that $\sigma^\mu{}_{\alpha\dot\alpha}\tilde\sigma_\mu{}^{\beta\dot\beta} = -2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha}$.

(b) Show that $e^{\mathrm{i}\theta\sigma_2} = \cos\theta\mathbf{1} + \mathrm{i}\sin\theta\sigma_2$. Then show that the $\mathrm{S}L(2,\mathbb{C})$ transformation corresponding to a spatial rotation through an angle θ around the $y = x^2$ axis is

$$A = e^{\frac{\mathrm{i}}{2}\theta\sigma^2} = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}.$$

Thus, show that a spatial rotation by 2π around the y axis corresponds to the matrix $A = -\mathbf{1}$, and that one needs to make an $\mathrm{S}L(2,\mathbb{C})$ transformation by 4π radians in order to reobtain $A = \mathbf{1}$. What happens under a 2π transformation?

[15 marks]

[This question continues on the next page . . .]

(iii) Derive the Dirac equation of motion for a massless spinor field from the kinetic Lagrangian $\mathcal{L}_{\mathrm{kin}} = -\mathrm{i}\,\bar{\psi}\partial_\mu\gamma^\mu\psi$. You may use the fact that $(\gamma_\mu)^\dagger\gamma^0 = -\gamma^0\gamma_\mu$. Show that the $\delta\psi$ and $\delta\bar{\psi}$ variations just produce conjugated equations. Remember that spinor fields anticommute. Transform the Dirac equation into momentum space with coordinates p^μ.

(a) Show that solutions to the momentum-space Dirac equation satisfy

$$p^\mu p_\mu \psi(p) = 0 \quad \text{and} \quad \psi(p) = -\zeta\frac{p_i}{|\mathbf{p}|}\gamma^0\gamma^i\psi(p)$$

with $\zeta = 1$ if $p^0 > 0$ and $\zeta = -1$ if $p^0 < 0$, where the p_i are components of three-momentum, $i = 1, 2, 3$, and $|\mathbf{p}| = \sqrt{p^i p^i}$. The former solutions are called particle solutions; the latter are antiparticle solutions.

(b) The helicity Λ of a particle is the projection of its angular momentum 3-vector in the direction of the spatial momentum. Given that the Lorentz generators for spinors are $M_{\mu\nu} = -\frac{\mathrm{i}}{2}\gamma_{\mu\nu} = -\frac{\mathrm{i}}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ and the spatial angular-momentum 3-vector operator is $S_i = \epsilon_{ijk}M_{jk}$, show that the helicity operator Λ can be written as $\Lambda = -\frac{1}{2}\gamma^5\gamma^0\gamma^i\frac{p_i}{|p|}$. Hence, show that massless chiral fermions are eigenstates of helicity, and specifically that

$$\left.\begin{aligned} \Lambda\psi_L(p) &= \tfrac{1}{2}\zeta\psi_L(p) \\ \Lambda\psi_R(p) &= -\tfrac{1}{2}\zeta\psi_R(p) \end{aligned}\right\} \quad \zeta = \pm 1 \text{ for } p^0 \gtrless 0 \,,$$

where $\gamma^5\psi_{\substack{L\\R}} = \pm\psi_{\substack{L\\R}}$. [20 marks]

[Total 50 marks]

End of examination paper