Imperial College London

MSc EXAMINATION April 2009

This paper is also taken for the relevant Examination for the Associateship

PARTICLES AND SYMMETRIES

For Students in Quantum Fields and Fundamental Forces

Wednesday, 29'th April 2009: 14:00 to 17:00

Answer 3 out of the following 6 questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) Define the following matrix Lie groups: (a) $SO(p,q)$, (b) $SL(n,\mathbb{C})$, (c) $SO(n,\mathbb{C})$, (d) $SU(n)$, (e) $Sp(n)$.

[5 marks]

(ii) Which of these groups are compact?

[3 marks]

(iii) Find the set of matrices that generate the matrix Lie algebras for (a) $SO(p,q)$, (b) $SL(n,\mathbb{C})$, (c) $SO(n,\mathbb{C})$, (d) $SU(n)$. Determine the dimensions of these groups.

[8 marks]

(iv) Choose a suitable basis for the matrix Lie algebra of $SO(p,q)$ and write the commutation relations in that basis.

[4 marks]

[Total 20 marks]

2. (i) Which Lie groups are symmetries of the standard model? Discuss briefly the role of each symmetry group.

[4 marks]

(ii) Which $SU(3)$ representation do the up, down and strange quarks fit into? What are the values of isospin and strangeness for each quark?

[4 marks]

(iii) Which $SU(3)$ representation do the pions fit into?

Explain how this representation can be understood as arising from the quark representations, given that the pions and other particles in this representation are made up of quarks. Show how this determines the values of isospin and strangeness for the pion multiplet

[9 marks]

(iv) Do all the particles in the pion multiplet have the same mass? What does this tell us about $SU(3)$ as a symmetry?

[3 marks]

[Total 20 marks]

3. (i) Show that for $SU(2)$ the 2×2 hermitian metric $\delta_{a\bar{b}}$ and the alternating tensor ϵ_{ab} are invariant tensors. (Here $a, b = 1, 2$.)

[3 marks]

(ii) Hence or otherwise show that $SU(2)$ has irreducible representations labelled by a single integer n. What is the physical significance of n? What is the dimension of the representation labelled by n?

[4 marks]

(iii) Show that for $SO(3)$ the 3×3 metric δ_{ij} and the alternating tensor ϵ_{ijk} are invariant tensors. (Here $i, j = 1, 2, 3$.)

[3 marks]

(iv) Hence or otherwise show that $SO(3)$ has irreducible representations labelled by a single integer m.

[4 marks]

(v) What is the relation between $SO(3)$ and $SU(2)$? Which representations of $SU(2)$ are also representations of $SO(3)$? Why is this, and what is the physical significance of this? Hence or otherwise show that the dimension of the representation labelled by the integer m is $2m + 1$. [6 marks]

[Total 20 marks]

4. (i) Show that the set of $2n \times 2n$ real matrices preserving an invertible antisymmetric matrix $\Omega_{ab} = -\Omega_{ba}$ form a group. (Here $a, b = 1, ..., 2n$.) Which group is this?

[5 marks]

(ii) Show that the group preserving Ω_{ab} is isomorphic to that preserving Ω'_{ab} for any two invertible antisymmetric matrices $\Omega_{ab}, \Omega'_{ab}$. You can assume there is an invertible matrix S such that $S\Omega S^t = \Omega'$ where S^t is the transpose of S.

[5 marks]

(iii) Show that the Lie algebra is generated by matrices $T^a{}_b$ such that $T_{ab} = \Omega_{ac}T^c{}_b$ is symmetric.

[5 marks]

(iv) Find the commutation relations for this Lie algebra.

[5 marks]

[Total 20 marks]

5. (i) What is a normal or invariant subgroup? Define the centre of a group. What is the centre of an abelian group? Show that the centre is always a normal subgroup.

[5 marks]

(ii) Show that the coset G/H is a group if H is a normal subgroup of G.

[6 marks]

(iii) Show that there is a homomorphism from G to G/H if H is a normal subgroup of G.

[5 marks]

(iv) What is the centre of $SL(2, \mathbb{C})$? What group is obtained by taking the quotient of $SL(2, \mathbb{C})$ by its centre?

[4 marks]

[Total 20 marks]

6. (i) How is the exponential of a matrix defined? Show that any matrix can be exponentiated to give a finite result. You can use the fact that any $n \times n$ matrix M can be written in the form

$$M = S(D + N)S^{-1}$$

where S is invertible, D is diagonal, N is nilpotent and $[D, N] = 0$.

[6 marks]

(ii) Show that the determinant of the exponential e^M of any matrix is

$$det(e^M) = e^{tr(M)}$$

where $tr(M)$ is the trace of M.

[6 marks]

(iii) Define a generator of a matrix Lie group. Show that if X is a generator of a matrix Lie group G, then MXM^{-1} is also a generator for any M in G. Show that the commutator of any two generators is a generator.

[8 marks]

[Total 20 marks]

 End of examination paper