Particles and symmetries: problem set 2

1. Recall that a (left) $G$-module, is a complex vector space $V$ with a linear map $G \times V \to V$ such that
   \[ a \cdot (b \cdot v) = (ab) \cdot v \]
   for all $a \in G$. Given a basis \{\(e_i\)\} so that $v \in V$ can be expanded as $v = v^i e_i$, define the corresponding representation $\rho : G \to GL(n, \mathbb{C})$ by $a \cdot e_i = \rho(a)^j e_j$ so that components transform as
   \[ v^i \mapsto \rho(a)^j v^j. \]
   (a) Show that the dual map $\bar{\rho} : G \to GL(n, \mathbb{C})$ defined by
      \[ \bar{\rho}(a) = \rho(a^{-1})^T \]
      forms a representation. Show that the conjugate map $\rho^* : G \to GL(n, \mathbb{C})$ defined by
      \[ \rho^*(a) = [\rho(a)]^* \]
      where $A^*$ is the complex conjugate of the matrix $A$, also forms a representation. Recall that if $G$ is a compact Lie group, then every complex representation is equivalent to a unitary representation. What does this imply about $\bar{\rho}$ and $\rho^*$?
   (b) The defining representation of $SU(2)$ is
      \[ \rho(2)(a) = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \quad \text{for} \quad a = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \in SU(2) \]
      where $xx^* + yy^* = 1$ be. Show that $\bar{\rho}(2) \sim \rho^*(2) \sim \rho(2)$.

2. (a) Recall that elements of the $(n + 1)$-dimensional irreducible $SU(2)$-module are symmetric tensors $w^{i_1 \cdots i_n}$ transforming as
    \[ w^{i_1 \cdots i_n} \mapsto \rho(2)^{i_1}_{j_1} \cdots \rho(2)^{i_n}_{j_n} w^{j_1 \cdots j_n}. \]
    Consider the $U(1)$ subgroup
    \[ U(1) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix} : aa^* = 1 \right\} \]
    Show that the module is decomposible as a $U(1)$-module and give its decomposition.
   (b) Let $\rho(3)$ be the defining representation of $SO(3)$, acting on the three-dimensional module $W$. Writing $w^i e_i \in W$ we have
    \[ w^i \mapsto \rho(3)^{i}_{j} w^{j}. \]
Consider $W \otimes W$ and show that it decomposes as

$$3 \otimes 3 = 1 \oplus 3' \oplus 5$$

where $n$ labels the $SO(3)$-module by its dimension.

(c) Show that the $3'$ module in the decomposition above is equivalent to the original $3$ module. [You can use the fact that, given an $d$-dimensional matrix $M$,

$$M^{i_1 j_1} \ldots M^{i_n j_n} \epsilon^{j_1 \ldots j_d} = (\det M) \epsilon^{i_1 \ldots i_d}$$

where $\epsilon^{i_1 \ldots i_d}$ is totally antisymmetric.]

3. Consider the proper orthochronous component $ISO^+(3,1)$ of the Poincaré group defined by the transformations

$$x^\mu \mapsto \Lambda^\mu_\nu x^\nu + a^\nu.$$  

with $\det \Lambda = 1$ and $\Lambda^0_0 > 0$. Let $S(\Lambda, a)$ be a unitary representation of $ISO^+(3,1)$.

(a) Let $|p^\mu\rangle$ transforms as an irreducible representation of the translation subgroup such that

$$S(1, a)|p^\mu\rangle = e^{-ip^\nu a^\nu}|p^\mu\rangle$$

and define $S(\Lambda, 0)|p^\mu\rangle = |\Lambda^\mu_\nu p^\nu\rangle$. Show that

$$S(\Lambda, a)|p^\mu\rangle = e^{-i(\Lambda p^\nu a^\nu)}|\Lambda^\mu_\nu p^\nu\rangle,$$

and hence that $S(\Lambda, a)$ forms a representation.

(b) Assuming that $p^2 = m^2$ and $p^0 > 0$, define a norm

$$\langle p^\mu | q^\nu \rangle = (2E_p)\delta^{(3)}(p - q)$$

where $E_p = \sqrt{p^2 + m^2}$. Show that $S(\Lambda, a)$ is unitary with respect to this norm.

(c) Consider a null momentum $p^\mu = (E, E, 0, 0)$. Show that Lorentz transformations of the form

$$\Lambda^\mu_\nu = \begin{pmatrix} 1 + r^2 & -r^2 & a & b \\ r^2 & 1 - r^2 & a & b \\ a' & -a' & \cos \theta & \sin \theta \\ b' & -b' & -\sin \theta & \cos \theta \end{pmatrix}$$

where $r^2 = \frac{1}{2}(a^2 + b^2)$, $a' = a \cos \theta + b \sin \theta$ and $b' = -a \sin \theta + b \cos \theta$ leave $p^\mu$ invariant. Defining

$$v^\mu = \begin{pmatrix} 1 + \frac{1}{2}(x^2 + y^2) \\ \frac{1}{2}(x^2 + y^2) \\ x \\ y \end{pmatrix}$$


show that acting with \( \Lambda \) transforms
\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
\mapsto
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
+ \begin{pmatrix}
a' \\
b'
\end{pmatrix}.
\]

Hence the little group of a null vector is isomorphic to the group of rotations and translations in the plane.

4. Consider the Georgi and Glashow Grand Unified model where \( SU(3) \times SU(2) \times U(1) \) is viewed as a subgroup of \( SU(5) \). Let \( V \) be the defining \( SU(5) \) module (the 5 irrep) so given \( v^i \in V \) we have
\[
v^i \mapsto \rho^i_j v^j,
\]
where \( \rho^i_\rho = 1 \) and \( \det \rho = 1 \). We also define the 10 module as an invariant subspace of \( V \otimes V \) formed by antisymmetric matrices \( u^{ij} = -u^{ji} \) transforming as
\[
u^{ij} \mapsto \rho^i_k \rho^j_l u^{kl}.
\]
Suppose the first generation of quarks and leptons embed in the 5 irrep as
\[
\begin{pmatrix}
e^+_R \\
\bar{\nu}_{e,R} \\
d_{R,1} \\
d_{R,2} \\
\bar{d}_{R,2}
\end{pmatrix}
\]
where the indices \( d_{R,i} \) label the three colours of quark.

(a) Find the subgroup \( SU(3) \times SU(2) \times U(1) \subset SU(5) \) such that the particles in the 5 transform appropriately.

(b) Show that decomposing the 10 module under this subgroup one finds the remaining first generation fields \( e^+_L, \bar{u}_L \) and \( (u_L, d_L) \).