and:

singlet η' 957 MeV.

Note:

- π-mesons: very different mass
- poor symmetry (u, d, s)

There is a similar octet and singlet with ~~spin~~ spin 1 (vector mesons).

3. LIE ALGEBRAS AND SEMI-SIMPLE LIE GROUPS

3.1 The exponential map and Lie algebras

The ~~basic idea~~ key (and remarkable) point to understanding Lie groups is that actually one only needs to understand a related object known as a Lie algebra.

The Lie algebra captures the structure of the Lie group infinitesimally close to the identity element. However this structure is (essentially) enough to determine the whole Lie group!

To define the Lie algebra we start with the notion of the exponentiation of a matrix. Let X be an $n \times n$ complex or real matrix, then we define

$$e^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \qquad X^0 = \mathbb{1}_{n\times n}$$

One finds

- series converges for any X
- defines a continuous function.

It has the following properties:

i) $e^0 = \mathbb{1}_{n\times n}$

ii) e^X is invertible and $(e^X)^{-1} = e^{-X}$

iii) $(e^X)^* = e^{X^*}$, $(e^X)^T = e^{X^T}$

iv) $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta X}$ for $\alpha, \beta \in \mathbb{C}$

v) $e^{CXC^{-1}} = C e^X C^{-1}$

but

$$e^X e^Y \neq e^{X+Y}$$ unless X and Y commute.

Now we define the Lie algebra of a matrix group:

Def

Let G be a matrix group. The Lie algebra, $\mathfrak{g}$ of G is the set of all matrices X such that $e^{tX} \in G$ for all $t \in \mathbb{R}$.

Expanding for small t:

$$e^{tX} \simeq \mathbb{1} + tX + O(t^2)$$

so:

- Lie algebra corresponds to "infinitesimal transformations" near the identity.

We can calculate the ~~Lie algebra~~ form of the Lie algebra in some simple cases:

- $SO(n)$: $M^T M = \mathbb{1}$ or $M^{-1} = M^T$ so if $M = e^{tX}$

$$\Rightarrow e^{-tX} = e^{tX^T} \Leftrightarrow X^T = -X$$

so

$$so(n) = \{ X \in n \times n \text{ matrix} : X^T = -X \text{ (antisymmetric)} \}$$

- $U(n)$: $M^+ M = \mathbb{1}$ or $M^{-1} = M^+$ so if $M = e^{tX}$

$$\Rightarrow e^{-tX} = e^{tX^+} \Leftrightarrow X^+ = -X$$

so

$$u(n) = \{ X \in n \times n \text{ matrices} : X^+ = -X \text{ (anti-hermitian)} \}$$

(Note: for SU(n) (and some other groups) ~~physicists~~ physicists often define $M = e^{itX}$ so

$$su(n) = \{ X \in n\times n \text{ matrices} : X^{+} = X \}$$

Here we will always use the maths ~~no~~ definition.)

We can see the exponentiation in practice in a few simple cases:

<u>SO(2)</u>:

Lie algebra = antisymmetric 2×2 = $\left\{ \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \right\}$

Take $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in so(2)$ then $X^2 = -\mathbb{1}$ so

$$e^{tX} = \sum_{n=0}^{\infty} \frac{1}{n!} (tX)^n = \sum_{\text{even}} \frac{(-1)^{n/2}}{n!} t^n \cdot \mathbb{1} + \sum_{\text{odd}} \frac{(-1)^{(n-1)/2}}{n!} t^n \cdot X$$

$$= \cos t \cdot \mathbb{1} + \sin t \cdot X = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in SO(2)$$

<u>U(2)</u>:

Lie algebra = anti-Hermitian 2×2 = $\cancel{\left\{ \begin{pmatrix} ix_3 & x_1+ix_2 \\ -x_1+ix_2 & -ix_3 \end{pmatrix} \right\}}$

$$= \left\{ \begin{pmatrix} ix_3 & x_1+ix_2 \\ -x_1+ix_2 & ix_4 \end{pmatrix} \right\}$$

Take $X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ then:

$$e^{tX_3} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix}^n = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} (it)^n & 0 \\ 0 & (-it)^n \end{pmatrix}$$

$$= \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \in U(2).$$

Though it is harder to prove, we also have:

$$\det(e^X) = e^{\operatorname{tr} X}$$

trace $\sum_i X_{ii}$

(for diagonalizable matrices:

$$X = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix} \qquad e^X = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & e^{\lambda_2} & \\ 0 & & \ddots \ e^{\lambda_n} \end{pmatrix}$$

$$\det e^X = e^{(\lambda_1+\lambda_2+\dots+\lambda_n)} = e^{\operatorname{tr} X} \qquad)$$

so we also have

- $SU(n)$: $M^+M = \mathbb{1}$, $\det M = 1 \Rightarrow X^+ = -X$, $e^{\mathrm{tr}X} = 1 \Leftrightarrow \mathrm{tr}X = 0$

$$su(n) = \{X \in u(n) : \mathrm{tr}X = 0\}$$

Now we have the important property:

* Theorem

The Lie algebra g of a matrix ~~group G~~ Lie group G is a vector space ~~g~~ over $\mathbb{R}$ with a bracket $[\,,\,] : g \times g \to g$, that is

1) if $X, Y \in g$ then $aX + bY \in g$ $\quad \forall a, b \in \mathbb{R}$
2) if $X, Y \in g$ then $[X, Y] = XY - YX \in g$

The outline of the proof is as follows:

1) by definition it is clearly that if $X \in g$ then $aX \in g$. Next consider:

$$e^{tX/m}\, e^{tY/m} = \mathbb{1} + tX/m + tY/m + O(m^{-2})$$
$$= e^{(tX/m + tY/m)} + O(m^{-2})$$

so:

$$\left(e^{tX/m}\, e^{tY/m}\right)^m = e^{t(X+Y)} + O(m^{-1})$$

taking limits:

$$\lim_{m \to \infty} \left(e^{tX/m}\, e^{tY/m}\right)^m = e^{t(X+Y)} \in G$$

provided limits in G are in G if the matrix is invertible (ie uses matrix Lie group definition)

2) Since $Ce^{tY}C^{-1} = e^{tCYC^{-1}}$ taking $C = e^{tX} \in G$ we have

$$CYC^{-1} \in g$$

given g is a vector space (and we can take limits) then

$$\lim_{t \to 0} \frac{CYC^{-1} - Y}{t} = \lim_{t \to 0} \frac{(1+tX)Y(1-tX) - Y}{t}$$

$$= \lim_{t \to 0} \frac{t(XY - YX)}{t} = [X, Y] \in g.$$

Again it is useful to see this in an example:

$$su(2) = \left\{ \begin{pmatrix} ia_3 & a_1+ia_2 \\ -a_1+ia_2 & -ia_3 \end{pmatrix} = a_1\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + a_2\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + a_3\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\}$$

- manifestly a vector space.
- define: $X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ basis of $su(2)$

$$[X_1, X_2] = X_1X_2 - X_2X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & \phi \\ -1 & 0 \end{pmatrix}$$

$$= 2\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = 2X_3 \in su(2)$$

in general:

$$[X_i, X_j] = 2\varepsilon_{ijk} X_k$$

"structure constants"

Consider:

$$so(3) = \left\{ \begin{pmatrix} 0 & a_1 & -a_2 \\ -a_1 & 0 & a_3 \\ a_2 & -a_3 & 0 \end{pmatrix} = a_1\tilde{X}_1 + a_2\tilde{X}_2 + a_3\tilde{X}_3 \right\}$$

basis of $so(3)$

- now we have:

$$[\tilde{X}_1, \tilde{X}_2] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \tilde{X}_3$$

and in ~~de~~ general

$$[\tilde{X}_i, \tilde{X}_j] = \varepsilon_{ijk}\tilde{X}_k \quad \text{or} \quad [2\tilde{X}_i, 2\tilde{X}_j] = 2\varepsilon_{ijk}\, 2\tilde{X}_k$$

same Lie algebra ~~&~~ as $su(2)$! $2\tilde{X}_i \to X_i$

Actually we can abstract the notion of Lie algebra:

Definition

A finite-dimensional real or complex Lie algebra is a finite-dim. vector space g together with a map $[\cdot,\cdot]: g \times g \to g$ such that

1) $[\cdot,\cdot]$ is bilinear

ie $[aX+bY, Z] = a[X,Z] + b[Y,Z]$

$[X, aY+bZ] = a[X,Y] + b[X,Z]$

2) $[X,Y] = -[Y,X] \quad \forall X,Y \in g$

3) $[X[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad \forall X,Y,Z \in g$

Jacobi identity.

Note that the Jacobi identity holds automatically for $[X,Y] = XY - YX$ with matrices. Also ~~as~~ for Lie algebras from matrix groups we considered real vector spaces.

We can then define

<u>Definition</u>

A <u>Lie algebra homomorphism</u> between two Lie algebras g and h is a linear map: $\varphi : g \to h$ such that

$$\varphi([X,Y]) = [\varphi(X), \varphi(Y)] \quad \forall X,Y \in g$$

~~A~~ If φ is bijective then ~~it~~ is an <u>isomorphism</u> and we write $g \simeq h$.

From our examples above

- $su(2) \simeq so(3)$ $\quad \varphi : 2\tilde{X}_i \mapsto X_i$ $\quad$ isomorphic Lie algebras!
- $SU(2) \not\simeq SO(3)$ $\quad$ Lie groups are <u>not</u> isomorphic!

A important result (not proven here!) is that

<u>Theorem</u>

Every abstract ~~Lie algebra is~~ real Lie algebra is isomorphic to the Lie algebra of some Lie group.

We can also define sub-algebras:

<u>Definition</u>

A ~~sub-Lie~~ Lie sub-algebra is a subspace $h \subset g$ such that

$$[X,Y] \in h \quad \forall X,Y \in h$$

and the direct sum of Lie algebras:

We have

<u>Definition</u>

If g_1 and g_2 are Lie algebras, the direct sum $g_1 \oplus g_2$ has a Lie algebra structure with the bracket

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2])$$

$\forall X_1, Y_1 \in g_1, \; X_2, Y_2 \in g_2$

We also have representations of Lie algebras: we have:

$gl(n, \mathbb{C})$ = Lie algebra for $GL(n, \mathbb{C})$

= $n \times n$ complex matrices.

then

<u>Definition</u>

A (complex, finite dimensional) representation ρ of a Lie algebra g is a homomorphism $\rho : g \to gl(n, \mathbb{C})$

that is

$$\rho[X, Y] = [\rho(X), \rho(Y)] = \rho(X)\rho(Y) - \rho(Y)\rho(X) \qquad \forall X, Y \in g.$$

Just as before we have:

- left g-module is a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) with a product $g \times V \to V$ denote $av \in V$ such that
  1) $[a, b] \cdot v = a(bv) - b(av)$
  2) $a(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 av + \lambda_2 av$

  Choosing a basis for V fixes a representation.
- equivalent representations $\rho' = T^{-1} \rho T$, faithful, trivial.
- reducibility, decomposible
- ~~[illegible]~~ direct sum representations ~~and tensor products~~.

Note that:

* if $G = G_1 \times G_2$ then $g = g_1 \oplus g_2$

Suppose we have a <u>basis</u> T_i for $\mathfrak{g}$ so ~~and~~ any element $X \in \mathfrak{g}$ can be written

$$X = \sum_i a_i T_i$$

then

structure constants.
define Lie algebra.

$$[T_i, T_j] = c_{ijk} T_k$$

and

- $[X,Y] = -[Y,X] \iff c_{ijk} = -c_{jik}$
- Jacobi $\iff \sum_m c_{ijm} c_{mkl} + c_{jkm} c_{mil} + c_{kim} c_{mjl} = 0$

3.2 Baker-Campbell-Hausdorff formula and Lie theory

The central result of ~~Lie gp~~ the theory of Lie groups is the following theorem about homomorphisms.

<u>Theorem</u>

Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\varphi: \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is <u>simply connected</u>, then there exists a <u>unique</u> Lie group homomorphism $\Phi: G \to H$ such that

$$\Phi(e^X) = e^{\varphi(X)} \quad \text{for all } X \in \mathfrak{g}.$$

As a corollary:

<u>Corollary</u>

Let G and H be simply connected matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. If $\mathfrak{g}$ is isomorphic to $\mathfrak{h}$, then G is isomorphic to H.

Essentially the ~~"set" theory~~ "category" of simply-connected Lie groups (up to isomorphism) is the same as the "category" of Lie algebras — if we understand the Lie algebra, then we understand the Lie group.

To understand the theorem we need two elements:

We have:

1. Baker-Campbell-Hausdorff formula

   We have $e^X e^Y = e^Z$ where, provided X, Y "not too large"

   $$Z = X + Y + [X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \dots$$

   where ~~...~~ all terms can be written in terms of the Lie bracket.

2. topological notion of simple-connectedness

In essence:

① implies that (at least near the identity) the group product is encoded in the Lie algebra.

② gives the topological criterion for when this local structure completely determines the Lie group, without any "redundancy"

Note that we also have the converse of this theorem, which is easy to ~~be~~ prove.

Theorem

Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Suppose $\Phi : G \to H$ is a Lie group homomorphism. Then,

$$\varphi(X) = \frac{d}{dt}\Phi(e^{tX})\Big|_{t=0} \qquad \forall X \in \mathfrak{g}$$

defines a unique Lie algebra homomorphism

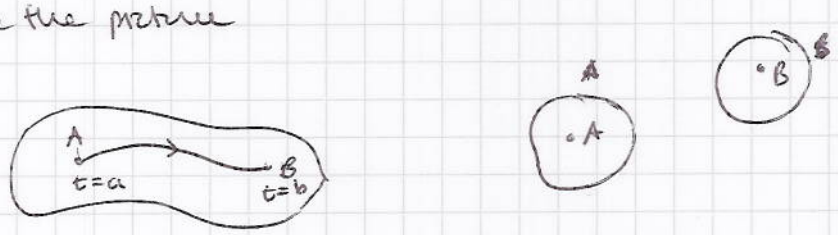
For the second component of the theorem we need a notion of connectedness and simple-connectedness. We start with the notion of connectedness:

Definition

A Lie group G, viewed as a manifold is connected if for any two points ~~in G~~ A and B in G there is a continuous path $A(t)$, $a \le t \le b$ in G with

$$A(a) = A \qquad A(b) = B$$

We have the picture



connected

not connected.

(Note this is what in topology is called "path connected". There is a weaker notion of "connected" but for manifolds to two notions are equivalent). For matrix groups:

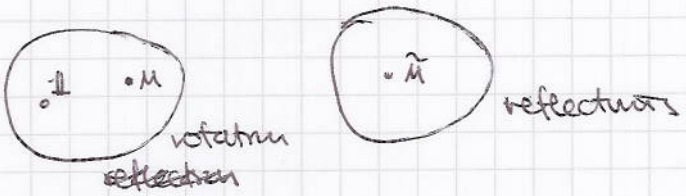
- $A(t)$ describes a continuous family of matrices.

Note:

$O(2)$: $M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ or $\tilde{M} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

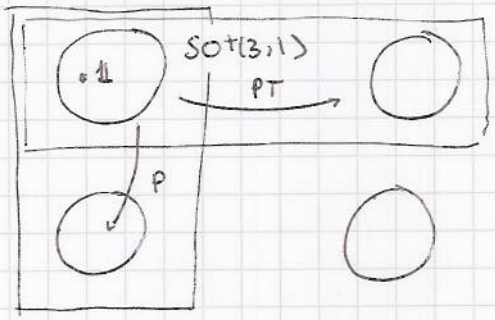
rotation reflection.

not connected.



same is true for $O(n)$.

$O(3,1)$: 4 different components:



$SO(3,1)$ proper

$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ time reversal

$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ parity.

$O^+(3,1)$

orthochronous.

in terms of product group:

$$O(2) \simeq SO(2) \times \mathbb{Z}_2$$

$$O(3,1) \simeq SO^+(3,1) \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$$

This is generally true of disconnected ~~group~~ Lie groups:

We have:

* $G_{con} \subset G$ the "<u>connected component</u> to the identity

  is a normal subgroup $G_{con} \triangleleft G$

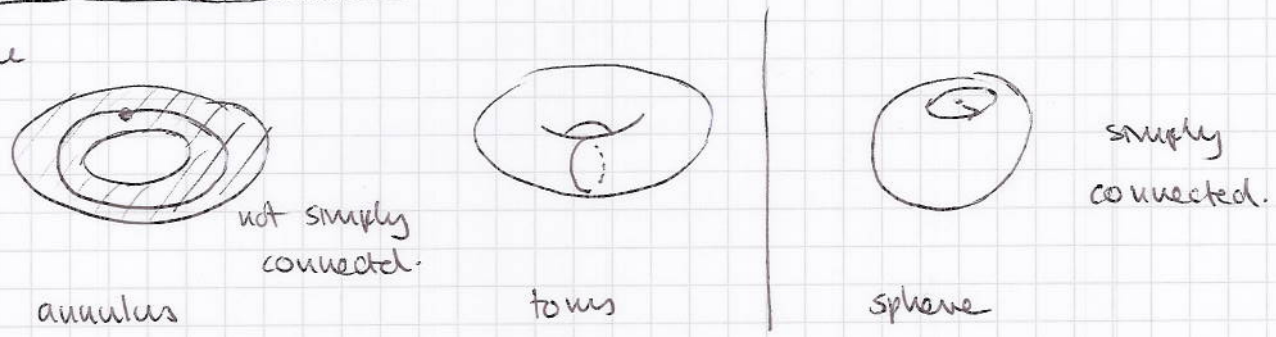
* $\Gamma = G / G_{con}$ is a <u>discrete group</u>

Next we have:

<u>Definition</u>

A Lie group G, viewed as a manifold, is <u>simply connected</u> if it is connected and <u>every loop</u> in G can be <u>shrunk continuously to a point</u>.

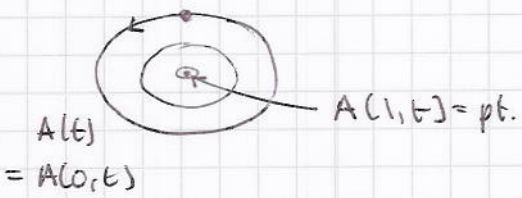
We have



More formally:

* loop: $A(t)$, $0 \le t \le 1$ lying in G with $A(0) = A(1)$

* contractible: continuous funct $A(s,t)$ $0 \le s,t \le 1$
  - $A(s,0) = A(s,1)$ for all s
  - $A(0,t) = A(t)$
  - $A(1,t) = A(1,0)$ for all t



We have a key example:

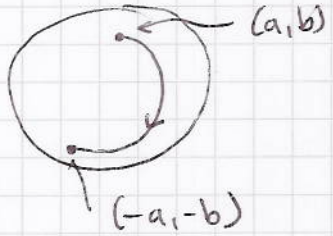
$$SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1 \right\} \simeq S^3$$

$SO(3) = S^3 / \mathbb{Z}_2$ we identify $(a,b) \leftrightarrow (-a,-b)$

then.

- S^3 : simply connected

- $S^3/\mathbb{Z}_2$: <u>not</u> simply connected:



- <u>closed</u> path on $S^3/\mathbb{Z}_2$
- cannot shrink to a point.

We see an example of what class of Lie groups we associate to a given Lie algebra:

- $su(2) \simeq so(3)$ but $SU(2) \neq SO(3)$
- we naturally associate the simply connected group $SU(2)$ to the Lie algebra, not $SO(3)$.

For another example:

- $u(1) = S^1$ <u>not</u> simply connected

loop ~~a~~ cannot ~~dec~~ shrink to

$u(1) = \{ \lambda \in \mathbb{C} \}$ $[\lambda, \lambda'] = 0$

The simply connected group is:

- $\mathbb{R}$ under addition: simply connected
- $S^1 = \mathbb{R}/\mathbb{Z}_n$ $(x + 2\pi n \sim x)$

In general one has:

- $SO(n)$: not simply connected
- $SU(n)$: simply connected
- $Sp(n)$: simply connected.

Leads to:

- $Spin(n)$ = simply connected group for $so(n)$ Lie algebra.

(~~SO(n)~~ $SO(n) = Spin(n)/\mathbb{Z}_2$, $Spin(3) \simeq SU(2)$)

In fact there are several accidental isomorphisms:

- $Spin(3) \simeq SU(2)$
- $Spin^+(3,1) \simeq SL(2,\mathbb{C})$
- $Spin(4) \simeq SU(2) \times SU(2)$
- $Spin(5) \simeq Sp(4)$
- $Spin(6) \simeq SU(4)$

Now we can define the universal cover:

<u>Definition</u>

Let G be a connected Lie group. The universal cover ~~of~~ of G is a simply connected Lie group $\tilde{G}$ ~~such that~~ with a Lie group homomorphism $\Phi: \tilde{G} \to G$ such that the associated Lie algebra homomorphism $\varphi: \tilde{\mathfrak{g}} \to \mathfrak{g}$ is an isomorphism.

~~Since~~ One finds:

<u>Theorem</u>

The universal cover always exists – though the universal cover of a matrix Lie group is not necessarily a matrix Lie group.

For our example: we have:

$$\widetilde{SO(3)} \simeq SU(2)$$

Note that ~~[illegible]~~ given $\Phi: \tilde{G} \to G$ one can define:

$$\ker \Phi = \{ a \in \tilde{G} : \Phi(a) = \text{identity} \}$$

by definition:

- $\ker \Phi$ is a normal, ~~Abelian~~, discrete subgroup of $\tilde{G}$ called the "fundamental group" ~~of~~ $\pi_1(G)$ of G.

(Note actually the fundamental group can be defined for any ~~top~~ path-connected topological space.) One have the quotient.

- $G = \tilde{G} / \ker \Phi$

In our example: $\Phi: SU(2) \to SO(3)$ since M and $-M$ are identified

$$\ker \Phi = \{ \mathbb{1}, -\mathbb{1} \} = \mathbb{Z}_2 \qquad \text{~~Gr~~} \quad SO(3) = SU(2)/\mathbb{Z}_2$$

3.3. Lie group and Lie algebra representations

Recall that

- Lie group: $\rho : G \to GL(n, \mathbb{C})$ homomorphism.
- Lie algebra: $\hat{\rho} : g \to gl(n, \mathbb{C})$ homomorphism.

By Lie theory we have

Theorem

For simply-connected Lie groups there is a one-to-one correspondence between representations of G and representations of g

However:

* for non-simply-connected Lie group G, we can have reps. of g with no corresponding rep. of G.

Essentially the correspondence is:

$$\rho(e^{tX}) = 1 + t\hat{\rho}(X) + \dots = e^{t\hat{\rho}(X)}$$

Note that for tensor products:

$$\rho_1(e^{tX}) \otimes \rho_2(e^{tX})$$
$$= (1 + t\hat{\rho}_1(X) + \dots) \otimes (1 + t\hat{\rho}_2(X) + \dots)$$
$$= 1 \otimes 1 + t(\hat{\rho}_1(X) \otimes 1 + 1 \otimes \hat{\rho}_2(X)) + \dots$$

hence we have:

Definition:

Given two g-modules V and W then the tensor product g-module $V \otimes W$ is given by

$$X(v \otimes w) = Xv \otimes w + v \otimes Xw \qquad \forall X \in g,\ v \in V,\ w \in W$$

or as given a hom:

$$(\rho_1 \otimes \rho_2)(X) = \rho_1(X) \otimes 1 + 1 \otimes \rho_2(X)$$

We can see all this for $SO(3)$ and $SU(2)$.

Recall:

$$SU(2) = \left\{ M = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1 \right\} \qquad SO(3) = SU(2) / \mathbb{Z}_2$$

$$M \leftrightarrow -M$$

$$so(3) \cong su(2) = \left\{ \begin{pmatrix} ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & -ix_3 \end{pmatrix} \right\}$$

Then we have:

- $\underline{2}$ rep:

$$\rho_2(M) = \begin{pmatrix} a & -\bar{b} \\ b & -\bar{a} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & -ix_3 \end{pmatrix} + \dots$$

$$= \mathbb{1} + \hat{\rho}_2(X) + \dots$$

but $\rho_2(-M) \neq \rho_2(M)$ so:

ρ_2 : rep of $SU(2)$ but <u>not</u> $SO(3)$

$\hat{\rho}_2$: rep of $su(2) \cong so(3)$

- $\underline{3}$ rep: $\omega^{ij} = \omega^{ji}$

$$\omega^{ij} \mapsto \cancel{\rho_2{}^i{}_k \rho_2{}^j{}_l \omega^{kl}} \; \rho_2(M)^i{}_k \, \rho_2(M)^j{}_l \, \omega^{kl}$$

$$= (\delta^i{}_k + \hat{\rho}(X)^i{}_k + \dots)(\delta^j{}_l + \hat{\rho}(X)^j{}_l + \dots)\, \omega^{kl}$$

$$= \omega^{ij} + (\hat{\rho}(X)^i{}_k \omega^{kj} + \hat{\rho}(X)^j{}_l \omega^{il}) + \dots$$

now:

$$\rho_2(-M)^i{}_k \, \rho_2(-M)^j{}_l \, \omega^{kl} = \rho_2(M)^i{}_k \, \rho_2(M)^j{}_l \, \omega^{kl}$$

so:

ρ_3 : rep of $SU(2)$ and $SO(3)$

$\hat{\rho}_3$: rep of $su(2) \cong so(3)$

We have.

$$\rho_n - \begin{cases} SU(2) \text{ and } SO(3) \text{ rep for } n \text{ odd} \\ SU(2) \text{ rep only for } n \text{ even} \end{cases}$$

Finally, it is useful to define the complexification of a Lie algebra. and similarly its "real forms".

We have:

Definition

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. We define its <u>complexification</u> $\mathfrak{g}_{\mathbb{C}}$ as the complex Lie algebra.

$$\mathfrak{g}_{\mathbb{C}} = \{ aX_1 + ibX_2 : X_1, X_2 \in \mathfrak{g} \}$$

such that

- $i(X_1 + iX_2) = -X_2 + iX_2$ (complex vector space)
- $[X_1 + iX_2, Y_1 + iY_2]$
$$= ([X_1, Y_1] - [X_2, Y_2]) + i([X_1, Y_2] + [X_2, Y_1])$$

We find: $(n = p+q)$

$$sl(n, \mathbb{R})_{\mathbb{C}} \simeq su(p,q)_{\mathbb{C}} \simeq su^*(n)_{\mathbb{C}} = sl(n, \mathbb{C})$$

$$so(p,q)_{\mathbb{C}} \simeq so^*(n)_{\mathbb{C}} \simeq so(n, \mathbb{C})$$

$$sp(p,q)_{\mathbb{C}} \simeq sp(2n, \mathbb{R})_{\mathbb{C}} \simeq sp(2n, \mathbb{C})$$

For example.

$$su(n) = \{ X \ n \times n \text{ complex matrix} : X^+ = -X, \ \operatorname{tr} X = 0 \}$$

$$sl(n, \mathbb{C}) = \{ Y \ n \times n \text{ complex matrix} : \operatorname{tr} Y = 0 \}$$

but given:

$$Y \in sl(n, \mathbb{C}) : Y = X + iX' \qquad X^+ = -X, \ X'^+ = -X^+$$

which is just $su(n)_{\mathbb{C}}$.

Now, naturally, we have:

Proposition.

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then every finite-dimensional complex representation $\hat{\rho}$ of $\mathfrak{g}$ has a unique extension to a complex-linear representation of $\mathfrak{g}_{\mathbb{C}}$ by

$$\hat{\rho}(X + iY) = \rho(X) + i\rho(Y)$$

and $\hat{\rho}$ is irreducible as a rep. of $\mathfrak{g}_{\mathbb{C}}$ if and only if it is an irrep of $\mathfrak{g}$.

For a simply-connected Lie group G we then have

$$* \quad \text{finite-dim. reps. of } G \overset{1:1}{\longleftrightarrow} \text{finite dim reps. of } \mathfrak{g}_{\mathbb{C}}$$

and if G is compact:

$$* \quad \text{finite dim unitary reps of } G \overset{1:1}{\longleftrightarrow} \text{finite dim. reps of } \mathfrak{g}_{\mathbb{C}}$$

Finally we need the precise definition of the adjoint rep: recall.

if $X \in \mathfrak{g}$ then $MXM^{-1} \in \mathfrak{g}$ for matrix Lie algebra.

$$e^{tMXM^{-1}} = Me^{tX}M^{-1} \in G \quad \text{if } M \in G$$

then:

Definition

If we view $\mathfrak{g}$ as a vector space V, it forms a G-module via the map: $G \times V \to V$

$$M \cdot X = MXM^{-1} \qquad \forall M \in G,\ X \in \mathfrak{g}$$

This is the "adjoint representation"

Sometimes the notation for the homomorphism is.

$$\mathrm{Ad}: G \to GL(\mathfrak{g}) \qquad \text{homomorphism}$$

$$M \longrightarrow \mathrm{Ad}_M \qquad \mathrm{Ad}_M \cdot X = MXM^{-1}$$

We also have the contragredient Lie algebra rep:

Definition

If we view $\mathfrak{g}$ as a vector space V, it forms a $\mathfrak{g}$-module via the map $\mathfrak{g} \times V \to V$

$$X \cdot \check{Y} = [X, \check{Y}] \qquad \forall X \in \mathfrak{g},\ \forall \check{Y} \in V = \mathfrak{g}$$

which is the Lie algebra adjoint rep.

We see that $\mathfrak{g}$ forms a $\mathfrak{g}$-module for itself!

To check we have a homomorphism:

$$M_1 \cdot (M_2 \cdot X) = M_1 (M_2 X M_2^{-1}) M_1^{-1} = (M_1 M_2) X (M_1 M_2)^{-1}$$

$$= (M_1 M_2) \cdot X$$