and:

\[ \text{singlet } y^{'} = 957 \text{ MeV.} \]

Note:

- \( \tau \)-mesons: very different mass
- poor symmetry (u, d, s)

There is a similar octet and singlet \( u \Delta = \bar{s} \Delta = \tau \Delta \) (vector mesons).

3. LIE ALGEBRAS AND SEMI-SIMPLE LIE GROUPS

3.1 The exponential map and Lie algebra

The key (and remarkable) point to understanding Lie groups is that actually one only needs to understand a related object known as a Lie algebra.

The Lie algebra captures the structure of the Lie group infinitesimally close to the identity element. However, this structure is (essentially) enough to determine the whole Lie group.

To define the Lie algebra, we start with the action of the exponential of a matrix. Let \( X \) be an \( n \times n \) complex or real matrix, then we define

\[ e^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \]

One finds

- series converges for any \( X \)
- defines a smooth function.

It has the following properties:

\[ i) \quad e^0 = I_{\text{real}} \]
\[ e^x \text{ is invertible and } (e^x)^{-1} = e^{-x} \]

\[ (e^x)^* = e^{x^*}, \quad (e^x)^T = e^{x^T} \]

\[ e^{(x+y)x} = e^{xy} e^{y^x} \quad \text{for } x, y \in \mathbb{C} \]

\[ e^{x C^{-1}} = C e^x C^{-1} \]

\[ e^x e^y \neq e^{x+y} \quad \text{unless } x \text{ and } y \text{ commute.} \]

Now we define the Lie algebra at a matrix group.

**Def**

Let \( G \) be a matrix group. The Lie algebra \( \mathfrak{g} \) of \( G \) is the set of all matrices \( X \) such that \( e^{tx} \in G \) for all \( t \in \mathbb{R} \).

Expanding for small \( t \):

\[ e^{tx} = 1 + tx + O(t^2) \]

so:

- The Lie algebra corresponds to "infinitesimal" transformations near the identity.

We can calculate the Lie algebra at \( G \) for some simple cases:

- **SO(n)**: \( M^TM = I \) or \( M^{-1} = M^T \) so if \( M = e^{tx} \)
  \[ e^{-tx} = e^{tx^T} \quad \Rightarrow \quad X^T = -X \]

  \[ \text{so} \]

  \[ \text{SO}(n) = \{ X \in \text{n x n matrices} : X^T = -X \} \quad (\text{anti-symmetric}) \]

- **SU(n)**: \( M^TM = I \) or \( M^{-1} = M^* \) so if \( M = e^{tx} \)
  \[ e^{-tx} = e^{tx^T} \quad \Rightarrow \quad X^T = -X \]

  \[ \text{so} \]

  \[ \text{SU}(n) = \{ X \in \text{n x n matrices} : X^T = -X \} \quad (\text{anti-Hermitian}) \]
(Note: for such (and some other guys) physics physicists often define $M = e^{tX}$ so

$su(3) = \{ X \in \text{non-zero \ matrix : } X^3 = 0 \}$

Here we will always use the matrix form definition.)

We can see the exponential map practice is a fast couple of cases:

$so(2)$:

Lie algebra = antisymmetric $2 \times 2 = \{ (0, 0) \}$

Take $X = (0, 1) \in so(2)$ then $X^2 = -1$ so

$e^{tX} = \sum_{n=0}^{\infty} \frac{1}{n!} (tX)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} t^{2n+1} \cdot X$

$= \cos t \cdot 1 + \sin t \cdot X = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in so(2)$

$u(2)$:

Lie algebra = anti-Hermitian $2 \times 2 = \{ (iX_3 / \lambda_1, iX_3 / \lambda_2) \}$

$= \{ \begin{pmatrix} \lambda_0 iX_3 & X_1 iX_2 \\ -X_1 iX_2 & \lambda_0 iX_3 \end{pmatrix} \}$

Take $X_3 = (0, 0)$ then:

$e^{tX_3} = \sum_{n=0}^{\infty} \frac{1}{n!} (0 - i)^n = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \in u(2)$.

Though it is harder to prove, we also have:

$\det(e^X) = e^{\text{trace } X}$

(for diagonalizable matrices):

$X = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$

$X = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix}$

$\det e^X = e^{(t+\lambda_2 t+\cdots+\lambda_n t)} = e^{tX}$
so we also have

\[ \text{SU}(n) : M^*M = 1, \ \text{det} M = 1 \Rightarrow X^* = -X, \ \text{e}^{\text{ix}} = 1 \Rightarrow \text{ix} = 0 \]

\[ \text{SU}(n) = \{ X \in \text{u}(n) : \text{ix} = 0 \} \]

Now we have the important property:

\* **Theorem**

The Lie algebra \( g \) of a matrix group \( G \) is a vector space \( k \) with a bracket \([L, J] = g \times g \rightarrow g\) that is

1) if \( X, Y \in g \) then \( aX + bY \in g \) for all \( a, b \in k \)

2) if \( X, Y \in g \) then \([X, Y] = XY - YX \in g \)

The outline of the proof is as follows:

1) by definition it is clearly that if \( X \in g \) then \( aX \in g \). Next consider:

\[
\left( e^{tX/m} e^{tY/m} \right)^m = \left( e^{tX/m} + e^{tY/m} + O(m^2) \right)^m
\]

\[
= e^{(tX/m + tY/m)} + O(m^2)
\]

so:

\[
\left( e^{tX/m} e^{tY/m} \right)^m = e^{t(x+y)} + O(m^{-1})
\]

Taking limit:

\[
\lim_{m \to \infty} \left( e^{tX/m} e^{tY/m} \right)^m = e^{t(x+y)} \in G
\]

provided limit \( m \in G \) are \( m \in G \) if the matrix is invertible (we use the inclusion Lie group definition)

2) Since \( Ce^{\text{sx} \cdot C^{-1}} = e^{\text{SCYC} \cdot C^{-1}} \) taking \( C = e^{tx} \in G \) we have:

\[ \text{CYC}^{-1} \in g \]

given \( g \) is a vector space (and we can take limit) then

\[
\lim_{t \to 0} \frac{\text{CYC}^{-1} - Y}{t} = \lim_{t \to 0} \frac{(1+tx)Y(1-tx) - Y}{t} = \lim_{t \to 0} \frac{t(\text{XY} - YX)}{t} = [X, Y] \in g
\]
Again it is useful to see this in an example:

\[ \mathfrak{su}(2) = \left\{ \begin{pmatrix} a_3 & a_1 + ia_2 \\ -a_1 - ia_2 & -a_3 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{C}, a_1^2 + a_2^2 = 1 \right\} \]

- manifesting a vector space.

- define: \( X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \) basis of \( \mathfrak{su}(2) \)

\[ [X_1, X_2] = X_1 X_2 - X_2 X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = 2X_3 \in \mathfrak{su}(2) \]

in general:

\[ [X_i, X_j] = 2 \varepsilon_{ijk} X_k \]

"structure constants."

Counter:

\( \mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & a_1 - ia_2 \\ a_1 + ia_2 & 0 \end{pmatrix} : a_1, a_2 \in \mathbb{R} \right\} \)

- now we have:

\[ [\tilde{X}_1, \tilde{X}_2] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = X_3 \]

and in general

\[ [\tilde{X}_i, \tilde{X}_j] = 2 \varepsilon_{ijk} \tilde{X}_k \] \( \tilde{X}_i \rightarrow X_i \)

same Lie algebra \( \mathfrak{so}(3) \) as \( \mathfrak{su}(2) \)!

Actually we can abstract the notion of Lie algebra.

**Definition**

A finite-dimensional real or complex Lie algebra \( \mathfrak{g} \) is a finite-dimensional vector space \( \mathfrak{g} \) together with a map \( [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \) such that

1) \([\cdot, \cdot]\) is bilinear

\[ c [aX+bY, Z] = a [cX, Z] + b [cY, Z] \]

\[ [X, aY+bZ] = a [X, Y] + b [X, Z] \]
2) \([X,Y] = -[Y,X] \quad \forall X, Y \in g\)

3) \([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \forall X, Y, Z \in g\)

Jacobi Identity.

Note that the Jacobi identity holds automatically for \((X,Y) = XY - YX\) with matrices. Also, for Lie algebras, such matrices generate us a countable real vector space.

We can then define:

**Definition**

A **Lie algebra homomorphism** between two Lie algebras \(g\) and \(h\) is a linear map: \(\varphi : g \rightarrow h\) such that

\[\varphi([X,Y]) = [\varphi(X), \varphi(Y)] \quad \forall X, Y \in g\]

And if \(\varphi\) is bijective then it is an **isomorphism** and we write \(g \cong h\).

**Examples above**

- \(su(2) \cong so(3)\) \(\varphi : 2\mathbb{R}i \rightarrow \mathbb{R}i\) isomorphism Lie algebra!
- \(SU(2) \cong SO(3)\) Lie groups are not isomorphic!

An important result (not proven here!) is that

**Theorem**

Every abelian Lie subalgebra is a subalgebra isomorphic to the Lie algebra of some Lie group.

We can also define **subalgebra**:

**Definition**

A sub-\(g\)-Lie sub-algebra is a subspace \(h \subset g\) such that

\([X,Y] \in h \quad \forall X, Y \in h\)

and the direct sum of Lie algebras.
we have

**Definition**

If $g_1$ and $g_2$ are Lie algebras, the direct sum $g_1 \oplus g_2$ has a Lie algebra structure with the bracket

$$\left[ (x_1, x_2), (y_1, y_2) \right] = (x_1 y_2 - y_1 x_2, x_1 y_3 - y_1 x_3)$$

for $x, y \in g_1, x_2, y_2 \in g_2$.

We also have representations of Lie algebras: we have

$$\text{gl}(n, \mathbb{C}) = \text{Lie algebra for } \text{GL}(n, \mathbb{C})$$

as $n \times n$ complex matrices.

Thus

**Definition**

A (complex, finite-dimensional) representation $\rho$ of a Lie algebra $g$ is a homomorphism $\rho : g \to \text{gl}(n, \mathbb{C})$ that is

$$\rho (x, y) = [\rho (x), \rho (y)] = \rho (x)\rho (y) - \rho (y)\rho (x) \quad \forall \, x, y \in g.$$

Just as before we have:

- left $g$-module in a vector space $V$ over $\mathbb{C}$ (or $\mathbb{R}$) with a $g$-product $g \times V \rightarrow V$ defined as $a \cdot v \in V$ such that
  1) $[a, b] \cdot v = a(bv) - b(au)$
  2) $a(\lambda v + \lambda w) = \lambda av + \lambda aw$

Choosing a basis for $V$ fixes a representation.

- equivalent representations $\rho' = T^{-1}\rho T$, faithful, trivial.
- reducibility, decomposable
- direct sum and tensor products.

Note that:

* If $G = G_1 \times G_2$ then $g = g_1 \oplus g_2$. 
Suppose we have a basis $T_i$ for $g$ so any element $X \in g$ can be written $X = \sum a_i T_i$

then

$$[T_i, T_j] = C_{ijk} T_k$$

and

- $[X, Y] = -[Y, X] \Rightarrow C_{ijk} = -C_{jik}$
- Jacobian $\Rightarrow \frac{\delta}{\delta x^i} C_{jkm} C_{lkm} + C_{jkm} C_{lkm} + C_{jkm} C_{lkm} = 0$

3.2 **Baker–Campbell–Hausdorff Formula and Lie theory**

The central result of Lie theory is the following theorem about homomorphisms:

**Theorem**

Let $G$ and $H$ be matrix Lie groups with Lie algebras $g$ and $h$.

Let $\phi : g \rightarrow h$ be a Lie algebra homomorphism. If $G$ is *simply connected*, then there exists a unique Lie group homomorphism $\Phi : G \rightarrow H$ such that

$$\Phi(e^X) = e^{\phi(X)} \quad \text{for all } X \in g.$$ 

As a corollary:

**Corollary**

Let $G$ and $H$ be simply connected Lie groups with Lie algebras $g$ and $h$. If $g$ is isomorphic to $h$, then $G$ is isomorphic to $H$.

Essentially the "category" of simply-connected Lie groups (up to isomorphism) is the same as the "category" of Lie algebras — if we understand the Lie algebra, then we understand the Lie group.

To understand the theorem we need two elements:
we have:

1. Baker-Campbell-Hausdorff formula

   We have $e^x e^y = e^{z}$ where, provided $x, y$ not "too large"

   
   \[ z = x + y + [x, y] + \frac{1}{2} [x, [x, y]] + \cdots \]

   where all terms can be written on terms of the lie bracket.

2. topological notions of simple-connectedness

In essence:

1. imputes that (at least near the identity) the group product
   is encoded in the lie algebra

2. gives the topological criteria for when their local structure
   completely determines the lie group, without any "redundancy"

Note that we also have the general of the theorem, which is easy to be prove.

**Theorem**

Let $G$ and $H$ be matrix lie groups with lie algebras $g$ and $h$. Suppose $\phi : G \to H$ is a lie group homomorphism. Then,

\[
\phi(x) = \left. \frac{d}{dt} \phi(e^{tx}) \right|_{t=0} \quad \forall x \in g
\]

defines a unique lie algebra homomorphism

For the second component of the theorem, we need a notion of connectedness and simple-connectedness. We start with the notion of connectedness:

**Definition**

A lie group $G$, viewed as a manifold, is connected if for any two points $A$ and $B$ in $G$, there is a continuous path $A(t)$; $a \leq t \leq b$ in $G$ with

\[
A(a) = A \quad A(b) = B
\]
we have the prime

connected

not connected.

(Note that is what in topology is called "path connected". There is a
weaker notion of "connected" for manifolds to two notions are
equivalent.) For matrix groups:

A(2) describes a continuum family of matrices.

Note:

\[ O(2) : M = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad \text{or} \quad \tilde{M} = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \]

rotation

not connected.

reflect

some is true for \( O(n) \).

\[ O(3,1) : 4 \text{ different components:} \]

\[ O^{+}(3,1) \]

orthochronous.

in terms of product groups:

\[ O(2) \simeq SO(2) \times \mathbb{Z}_2 \]

\[ O(3,1) \simeq SO^{+}(3,1) \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \]

This is generally true of disconnected groups like groups:

\[ T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{true reversion} \]

\[ P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{passing} \]
Now:
* \( G_0 \subset G \) is the "connected component of the identity" if a normal subgroup \( G_0 \subset G \)
* \( \Gamma = G/G_0 \) is a discrete group

Next we have:

**Definition**
A Lie group \( G \), viewed as a manifold, is **simply connected** if it is connected and every loop in \( G \) can be shrunk continuously to a point.

We have:

- annulus
- torus
- sphere
- simply connected.

More formally:
* loop: \( A(t) \) \( 0 \leq t \leq 1 \) lying in \( G \) with \( A(0) = A(1) \)

\[ A(0) = A(1) \]

* continuous function \( A(s,t) \) \( 0 \leq s, t \leq 1 \)
  - \( A(s,0) = A(s,1) \) for all \( s \)
  - \( A(0,t) = A(t) \)
  - \( A(1,t) = A(1,0) \) for all \( t \)

We have a key example:

\[ SU(2) = \{ (a -i b, b + i a) : a^2 + b^2 = 1 \} \subset S^3 \]

\[ SO(3) = S^3/\mathbb{Z}_2 \quad \text{we identify} \quad (a, b) \leftrightarrow (-a, -b) \]
- $S^3$: simply connected

- $S^3/\mathbb{Z}_2$: not simply connected:

  \[
  \begin{array}{c}
  (a, b) \\
  \end{array}
  \]

  - closed path in $S^3/\mathbb{Z}_2$
  - cannot shrink to a point.

We see an example of what class of Lie groups we associate to a given Lie algebra:

- $\text{su}(2) \cong \text{so}(3)$ but $\text{SU}(2) \neq \mathbb{SO}(3)$

- we naturally associate the simply connected group $\text{SU}(2)$ to the Lie algebra, not $\mathbb{SO}(3)$.

For another example:

- $U(1) = S^1$: not simply connected

  \[
  \begin{array}{c}
  \end{array}
  \]

  loop \& cannot deformed to

  \[
  \begin{array}{c}
  \end{array}
  \]

  \[
  a(t) = \{ x \in \mathbb{C} \mid x_1x_3 = 0 \}
  \]

  The simply connected group is:

  - $\mathbb{R}$ under addition: simply connected

  \[
  S^1 = \mathbb{R}/\mathbb{Z} \quad (x + 2\pi n = x)
  \]

In general one has:

- $\text{SO}(n)$: not simply connected

- $\text{SU}(n)$: simply connected

- $\text{Sp}(n)$: simply connected.

This leads to:

- $\text{Spin}(n) = \text{simply connected group for $\text{so}(n)$ Lie algebra}$

  \[
  (\text{SO}(n) = \text{Spin}(n)/\mathbb{Z}_2, \quad \text{Spin}(3) = \text{SU}(2))
  \]
In fact there are several accidental isomorphisms:

- \( \text{Spin}(3) \cong \text{SU}(2) \)
- \( \text{Spin}^+(3,1) \cong \text{SL}(2, \mathbb{C}) \)
- \( \text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2) \)
- \( \text{Spin}(5) \cong \text{Sp}(4) \)
- \( \text{Spin}(6) \cong \text{SU}(4) \)

Now we can define the universal cover:

**Definition.**

Let \( G \) be a connected Lie group. The **universal cover** \( \tilde{G} \) of \( G \) is a simply connected Lie group \( \tilde{G} \) surjective with a Lie group homomorphism \( \varphi : \tilde{G} \rightarrow G \) such that the associated Lie algebra homomorphism \( \varphi_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{g} \) is an **isomorphism**.

**Theorem.**

The universal cover always exists — through the universal cover of a matrix group is not necessarily a matrix Lie group.

For our example, we have:

\( \tilde{\text{SO}}(3) \cong \text{SU}(2) \)

Note that once \( \varphi \) is given \( \tilde{\varphi} : \tilde{G} \rightarrow G \) one can define:

\( \text{ker} \tilde{\varphi} = \{ a \in \tilde{G} : \tilde{\varphi}(a) = \text{identity} \} \)

by defining:

- \( \text{ker} \tilde{\varphi} \) is a normal, abelian, discrete subgroup of \( \tilde{G} \)
- called the “fundamental group” \( \pi_1(G) \) of \( G \).

(Note actually the fundamental group can be defined for any topologically connected topological space.) One has the quotient:

\( G = \tilde{G} / \text{ker} \tilde{\varphi} \)

In our example, \( \tilde{\text{SU}}(2) \rightarrow \text{SO}(3) \) since \( M \) and \( -M \) are identified.

\( \text{ker} \tilde{\varphi} = \{ \pm 1, -1 \pm i \} = \mathbb{Z}_2 \)

So \( \text{SO}(3) = \text{SU}(2) / \mathbb{Z}_2 \).
3.3. Lie group and Lie algebra representation

Recall that

- Lie group: \( \rho : G \rightarrow GL(n, \mathbb{C}) \) homomorphism.
- Lie algebra: \( \hat{\rho} : g \rightarrow g(\mathbb{C}) \) homomorphism.

By Lie theory we have:

Theorem
Far simply-connected Lie groups there is a one-to-one correspondence between representations of \( G \) and representations of \( g \).

However,

+ For non-simply-connected Lie group \( G \), we can have reps. of \( G \) with no corresponding rep. of \( G \).

Essentially the correspondence is:

\[
\rho(e^{t \xi}) = 1 + t\hat{\rho}(\xi) + \ldots = e^{t\hat{\rho}(\xi)}
\]

Note that for tensor products:

\[
\rho_1(e^{t \xi}) \otimes \rho_2(e^{s \eta}) = (1 + t\hat{\rho}_1(\xi) + \ldots) \otimes (1 + s\hat{\rho}_2(\eta) + \ldots)
\]

\[
= 1 \otimes 1 + t(\hat{\rho}_1(\xi) \otimes 1 + 1 \otimes \hat{\rho}_2(\eta)) + \ldots
\]

Hence we have:

Definition:
Given two \( g \)-modules \( V \) and \( W \) then the tensor product \( g \)-module \( V \otimes W \) is given by

\[
\hat{\rho}(v \otimes w) = \hat{\rho}(v) \otimes \hat{\rho}(w) + \lambda v \otimes \xi w \quad \forall \xi \in g, v \in V, w \in W
\]

Or as given a harm:

\[
(\rho, \otimes \rho_2)(x) = \rho_1(x) \otimes 1 + 1 \otimes \rho_2(x)
\]

We can use this form for \( SO(3) \) and \( SU(2) \).
Recall,
\[ SU(2) = \left\{ M = \begin{pmatrix} a_{1} & b_{2} \\ b_{1} & a_{2} \end{pmatrix} : a_{1}a_{2} + b_{1}b_{2} = 1 \right\} \]
\[ SO(3) = SU(2) \otimes \mathbb{Z} \]
\[ M \leftrightarrow -M \]
\[ so(3) = su(2) = \left\{ \begin{pmatrix} i\sigma_{3} & xi + y\sigma_{1} \\ -x\sigma_{1} - y\sigma_{3} \end{pmatrix} \right\} \]

Then we have:

* 1. \( \hat{\rho}_{2}(M) = \begin{pmatrix} a_{1} & b_{2} \\ b_{1} & a_{2} \end{pmatrix} \)

\[ \rho_{2}(M) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} i\sigma_{3} & x_{1} + y\sigma_{1} \\ -x_{1}\sigma_{1} - y\sigma_{3} \end{pmatrix} + ... \]

* 3. \( \hat{\rho}_{2}(X) = ... \)

but \( \rho_{2}(-M) \neq \rho_{2}(M) \) so:

\[ \rho_{2} : \text{reg \& SU(2) \& not \& SO(3)} \]

\[ \hat{\rho}_{2} : \text{reg \& SU(2) \& SO(3)} \]

\[ \hat{\rho}_{2} = \rho_{2}(M) \]

\[ \omega^{\hat{\rho}_{2}} = \omega^{\rho_{2}} \rightarrow \rho_{2}(M)^{2} = \rho_{2}(M)^{3} = \omega^{k_{l}} \]

\[ \rho_{2}(M)^{i} \rho_{2}(M)^{j} = \rho_{2}(M)^{i+j} \]

\[ \rho_{2}(-M)^{i} \rho_{2}(-M)^{j} = \rho_{2}(M)^{i+j} \]

so:

\[ \rho_{3} : \text{reg \& SU(2) \& SO(3)} \]

\[ \hat{\rho}_{3} : \text{reg \& SU(2) \& SO(3)} \]

We have:

\[ \hat{\rho}_{n} = \begin{cases} \text{SU(2) \& SO(3) \& reg \& for \& n \& odd} \\ \text{SU(1) \& only \& for \& n \& even} \end{cases} \]

Finally, it is useful to derive the complexified real of a Lie algebra, and similarly its "real form."
we have.

deform.

Let \( g \) be a finite-dimensional real Lie algebra. We define

its complexification \( g_{\mathbb{C}} \) as the complex Lie algebra

\[
g_{\mathbb{C}} = \{ aX_1 + ibX_2 : X_1, X_2 \in g \}
\]

such that

\[
\begin{align*}
\iota(X_1 + iX_2) &= -X_2 + iX_1 \\
[X_1, X_2] &= [X_1, Y_1] + \iota([X_1, Y_2] + [X_2, Y_1])
\end{align*}
\]

we have: \((u=\mathfrak{u}, v=\mathfrak{u})\)

\[
\begin{align*}
\text{su}(n, \mathbb{R}) \subset & \text{su}(p, q)_{\mathbb{C}} = \text{su}^*(p, q)_{\mathbb{C}} = \text{sl}(n, \mathbb{C}) \\
\text{so}(p, q)_{\mathbb{C}} &= \text{so}^*(p, q)_{\mathbb{C}} = \text{so}(n, \mathbb{C}) \\
\text{sp}(p, q)_{\mathbb{C}} &= \text{sp}(2n, \mathbb{R})_{\mathbb{C}} = \text{sp}(2n, \mathbb{C})
\end{align*}
\]

Example

\[
\begin{align*}
\text{su}(n) &= \{ X \in \mathbb{C}^{n \times n} : X^t = -X, \ iX = 0 \} \\
\text{sl}(n, \mathbb{C}) &= \{ Y \in \mathbb{C}^{n \times n} : iY = 0 \}
\end{align*}
\]

but quas:

\[
\text{give } Y \in \text{sl}(n, \mathbb{C}) : Y = X + iX' \quad X^{\pm} = -X, \quad X'^{\pm} = -X'
\]

which a part \( \text{su}(n, \mathbb{C}) \).

Now, methylly, we have:

Properties

Let \( g \) be a real Lie algebra and \( g_{\mathbb{C}} \) its complexification.

Then every finite-dimensional complex representation \( \beta \) of \( g_{\mathbb{C}} \) has a unique extension to a complex-linear representation \( \beta \) of \( g \) by

\[
\beta(X + iY) = \rho(X) + i\rho(Y)
\]

and \( \beta \) is irreducible as a rep. of \( g_{\mathbb{C}} \) if and only if it is an rep. of \( g \).
For a simply-connected Lie group \( G \) we then have

1. Finite-dim. reps. of \( G \xrightarrow{1:1} \) Finite-dim. reps. of \( g \)

and if \( G \) is compact:

2. Use finite-dim. unitary reps. of \( G \xrightarrow{1:1} \) Finite-dim. reps. of \( g \)

Finally, we need the precise definition of the adjoint rep: we recall.

If \( X \in g \) then \( MXM^{-1} \in g \) for some Lie algebra.

\[ e \in MXM^{-1} = M e x M^{-1} \in G \quad \forall M \in G \]

then:

**Definition**

If we view \( g \) as a vector space \( V \), it forms a \( \mathbb{C} \)-module

via the way: \( G \times V \to V \)

\[ M \cdot x = MXM^{-1} \quad \forall M \in G, x \in g \]

This is the "adjoint representation."

Sometimes, we will call the homomorphism \( \text{Ad} \).

\[ \text{Ad}: G \to GL(g) \quad \text{homomorphism} \]

\[ M \mapsto \text{Ad}_M \quad \text{Ad}_M : x \mapsto MXM^{-1} \]

We also have the corresponding Lie algebra rep.

**Definition**

If we view \( g \) as a vector space \( V \), it forms a \( \mathbb{C} \)-module

via the way: \( g \times V \to V \)

\[ X \cdot y = [x, y] \quad \forall x \in g, \forall y \in V \in g \]

which is the Lie algebra adjoint rep.

We see that \( g \) itself forms a \( g \)-module for itself!

To check we have a homomorphism:

\[ M_1 \cdot (M_2 \cdot X) = (M_1 M_2) \cdot X = (M_1 M_2) C (M_1 M_2)^{-1} \]

\[ = (CM_2) \cdot X \]