and

$$X_1 \cdot (X_2 \cdot v) - X_2 \cdot (X_1 \cdot v)$$
$$= [X_1, [X_2, v]] - [X_2, [X_1, v]]$$
$$= [[X_1, X_2], v] \quad \text{by Jacobi identity.}$$
$$= [X_1, X_2] \cdot v$$

We also see that the adjoint rep is real since $\mathfrak{g}$ is a real vector space.

3.4 Semi-simple Lie algebras and roots

Recall we defined.

Def: A group G is simple if it has no proper normal subgroups. It is semi-simple if all proper normal subgroups are non-Abelian.

For the corresponding Lie algebras:

* normal subgroup $\longrightarrow$ "ideal" $h \subset \mathfrak{g}$ such that
$$[X, H] \in h \quad \text{for all } X \in \mathfrak{g},\ H \in h.$$
* simple $\longrightarrow$ only ideals are $\mathfrak{g}$ itself and 0.
* semi-simple $\longrightarrow$ $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_n$ $\mathfrak{g}_i$ is simple.

Theorem

* $\mathfrak{g}$ is semi-simple if and only if $\mathfrak{g}_{\mathbb{C}}$ is semi-simple.

Remarkable result: (Killing and Cartan (PhD!))

* the ~~semi-simple~~ complex semi-simple Lie algebras can be completely classified, along with their finite dimensional irreps.

Important:

Theorem

~~A compact, simply-connected Lie group is the~~

We have:

Theorem

A complex Lie algebra is semi-simple if and only if it is isomorphic to the complexification of a compact, simply-connected Lie group.

so

* classifying semi-simple Lie algebras, classifies compact simply-connected Lie groups.

How do we classify them? The key is to view $\mathfrak{g}_{\mathbb{C}}$ as built out of $\mathfrak{sl}(2,\mathbb{C})$ blocks – the Lie algebra is defined by how these blocks fit together.

Let's start with $\mathfrak{sl}(2,\mathbb{C})$

$$\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2)_{\mathbb{C}} = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} : a,b,c \in \mathbb{C} \right\}$$

Consider the Abelian sub-algebra

$$h = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right\} \subset \mathfrak{sl}(2,\mathbb{C})$$

Recall that, viewed as the adjoint representation of itself is a $\mathfrak{g}$-module. Thus $\mathfrak{sl}(2,\mathbb{C})$ is an h-module, and we can decompose it into invariant subspaces:

$$\mathfrak{sl}(2,\mathbb{C}) = h \oplus g_2 \oplus g_{-2}$$

where:

$$g_2 = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right\} \qquad g_{-2} = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \right\}$$

NdE:

$$\left[\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right] = 2\lambda \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \right] = -2\lambda \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$$

we say:

- 2, -2 are roots of $sl(2,\mathbb{C})$

they give the charge of the g_2, g_{-2} representations of h

Note we can choose a basis: (cf. spin)

$h = \{\lambda J_3\}$ $\quad J_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$g_2 = \{a J_+\}$ $\quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$g_{-2} = \{b J_-\}$ $\quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

and

$[J_3, J_+] = 2J_+$ $\qquad [J_3, J_-] = -2J_-$ $\qquad [J_+, J_-] = J_3$

Now consider a general complex semi-simple Lie algebra. One defines

Definition

A Cartan sub-algebra of g is a complex subspace $h \subset g$ such that:

1. $[H_1, H_2] = 0 \quad \forall H_1, H_2 \in h$ (Abelian)
2. if $[H, X] = 0 \quad \forall H \in h$ then $X \in h$ (maximal).

The rank of g is the dimension of h.

It is a theorem that every complex semi-simple Lie algebra has a Cartan sub-algebra. It is not unique but there is always an automorphism $\phi: g \to g$ such that $\phi(h') = h$ for any two Cartan sub-algebras — ie it is unique up to equivalence.

Then one defines:

Definition

A root of g is a linear function $\alpha: h \to \mathbb{C}$ such there exists a non-zero $X \in g$ such that

$$[H, X] = \alpha(H) X \qquad \alpha \in h^* \text{ (dual space)}$$

for all $H \in h$. The root space $g_\alpha = \{X \in g : [H, X] = \alpha(H) X \ \forall H \in h\}$

If we take a basis $H = \lambda^i H_i$ then:

Abelian.

$$[H, X] = \alpha_i \cdot X \qquad \text{root } \alpha_i \text{ is set of charges for } H_i$$

$$\alpha(H) = \alpha_i \lambda^i$$

and each g_α forms an invariant subspace for h, hence:

$$g = h \oplus \bigoplus_{\alpha \in R} g_\alpha \qquad R = \text{set of all roots.}$$

Since g is finite dimensional there can only be a finite number of roots. One finds.

- if α is a root then $-\alpha$ is a root.
- ~~each g_α is one-dimensional~~ the roots span h^*
- for each α one can find $J^\alpha_+ \in g_\alpha$, $J^\alpha_- \in g_{-\alpha}$, $J^\alpha_3 \in H$

$$[J^\alpha_3, J^\alpha_+] = 2J^\alpha_+$$

$$[J^\alpha_3, J^\alpha_-] = -2J^\alpha_- \qquad su(2,\mathbb{C}) \subset g$$

$$[J^\alpha_+, J^\alpha_-] = J^\alpha_3$$

so we see each root defines a separate $su(2,\mathbb{C})$ subalgebra.

The basic result is then:

* classifying the possible roots classifies the algebra g

example: ~~$su(3)$~~ $su(3,\mathbb{C}) \cong su(3)_a$

$$h = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} : \lambda_1 + \lambda_2 + \lambda_3 = 0 \right\} \qquad \text{rank} = 2$$

$$g_{\alpha_1} = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \qquad [H, X] = (\lambda_1 - \lambda_2) X \qquad \alpha_1 = (1, -1, 0)$$

$$g_{-\alpha_1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \qquad [H, X] = (\lambda_2 - \lambda_1) X \qquad -\alpha_1 = (-1, 1, 0)$$

$$J^{\alpha_1}_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad J^{\alpha_1}_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad J^{\alpha_1}_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then:

$$g_{\alpha_2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \right\} \qquad [H, X] = (\lambda_2 - \lambda_3)X \qquad \alpha_2 = (0, 1, -1)$$

$$g_{-\alpha_2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a & 0 \end{pmatrix} \right\} \qquad [H, X] = (\lambda_3 - \lambda_2)X \qquad -\alpha_2 = (0, -1, 1)$$

$$J_3^{\alpha_2} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \quad \text{etc.}$$

$$g_{\alpha_3} = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \qquad [H, X] = (\lambda_1 - \lambda_3)X \qquad \alpha_3 = (1, 0, -1)$$

$$g_{-\alpha_3} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \right\} \qquad [H, X] = (\lambda_3 - \lambda_1)X \qquad -\alpha_3 = (-1, 0, 1)$$

$$J_3^{\alpha_3} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \quad \text{etc.}$$

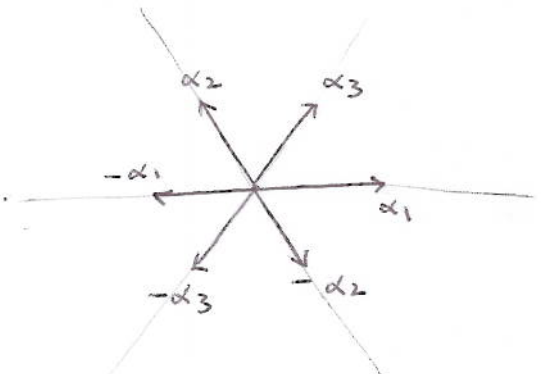
note:

$$\alpha_3 = \alpha_1 + \alpha_2$$

If we take a ~~weird~~ standard metric

$$|\alpha_1| = |\alpha_2| = |\alpha_3| = \sqrt{2} \qquad \alpha_1 \cdot \alpha_2 = -1 \qquad \alpha_1 \cdot \alpha_3 = 1 \qquad \alpha_2 \cdot \alpha_3 = 1$$

and we can plot the vectors (lie in a plane since $\lambda_1 + \lambda_2 + \lambda_3 = 0$)



"root lattice"

- 3 different ~~SU(4)~~ $SL(2, \mathbb{C})$

example: $so(5)_{\mathbb{C}} \cong sp(2)_{\mathbb{C}} \cong sp(4, \mathbb{C})$

$$sp(4, \mathbb{C}) = \left\{ M \in sl(4, \mathbb{C}) : M^T \Omega + \Omega M = 0 \text{ or } M = \left(\begin{array}{cc|cc} \lambda_1 & a & c & d \\ b & -\lambda_1 & e & f \\ \hline -f & d & \lambda_2 & a' \\ e & -c & b' & -\lambda_1 \end{array} \right) \right\}$$

$$\Omega = \left(\begin{array}{cc|cc} -1 & & 1 & \\ & -1 & & 1 \\ \hline 1 & & & \\ & 1 & & \end{array} \right)$$

$$h = \left\{ \begin{pmatrix} \lambda_1 & & & \\ & -\lambda_1 & & \\ & & \lambda_2 & \\ & & & -\lambda_2 \end{pmatrix} \right\} \qquad \text{rank} = 2$$

$$g_{\alpha_1} \ni \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & 0 \\ \hline & & 0 \end{array}\right) \qquad [H, X] = 2\lambda_1 X \qquad \alpha_1 = (2, 0)$$

$$g_{\alpha_2} \ni \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & -1 & 0 \end{array}\right) \qquad [H, X] = (\lambda_1 - \lambda_2) X \qquad \alpha_2 = (1, -1)$$

$$g_{\alpha_3} \ni \left(\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad [H, X] = (\lambda_1 + \lambda_2) X \qquad \alpha_3 = (1, 1)$$

$$g_{-\alpha_2} \ni \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline -1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \qquad [H, X] = (-\lambda_1 + \lambda_2) X \qquad -\alpha_2 = (-1, 1)$$

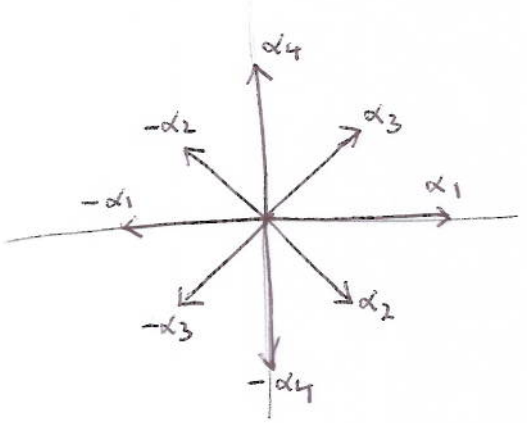
$$\vdots$$

$$g_{\alpha_4} \ni \left(\begin{array}{cc|c} 0 & 0 & \\ 0 & 1 & 0 \\ \hline & & 0 \end{array}\right) \qquad [H, X] = 2\lambda_2 X \qquad \alpha_4 = (0, 2)$$

Note:

$|\alpha_1| = |\alpha_4| = 2 \qquad |\alpha_2| = |\alpha_3| = \sqrt{2}$

$\alpha_1 \cdot \alpha_2 = 2$ etc



$$\alpha_3 = \alpha_1 - \alpha_2$$

$$\alpha_4 = \alpha_1 - 2\alpha_2$$

3.5 Classification and Dynkin diagrams

Given a Cartan sub-algebra $h \subset g$ we have

$$g = h \oplus \bigoplus_{\alpha \in R} g_\alpha \qquad R = \{\alpha_1, \ldots, \alpha_n\} \text{ is set of all roots.}$$

where the roots $\alpha \in h^*$ are defined by

$$[H, X] = \alpha(H) X \qquad \text{for all } H \in h \text{ and } X \in g_\alpha$$

The ~~roots~~ g_α are h-modules and since h is Abelian, they are all are dimensional. Hence: (recall $\dim h$ = rank)

- number of roots = $\dim g - \dim h$

Each root defines an $sl(2,\mathbb{C})$ subalgebra: $s_\alpha \subset g$

$$J_3^\alpha \in h \qquad J_+^\alpha \in g_\alpha \qquad J_-^\alpha \in g_{-\alpha}$$

Thus for each root we also have a decomposition of g in to s_α-modules.

$$g = s_\alpha \oplus \bigoplus_r s_r^\alpha$$

↖ $sl(2,\mathbb{C})$ modules ie $\square$, $\square\square$, $\square\square\square$, etc.

The fact that we can make this decomposition for any of the roots is very powerful.

The other powerful ingredient is that for semi-simple Lie groups there is a bilinear form (inner product) on g

Thm

If g is semi-simple then there exists an invariant, non-degenerate inner product $\langle , \rangle : g \times g \to \mathbb{C}$, that is

1. $\langle [X,Y], Z \rangle + \langle Y, [X,Z] \rangle = 0$ (invariant)
2. if $\langle X, Y \rangle = 0$ for all Y then $X = 0$ (non-degenerate)

If g is simple then $\langle , \rangle$ is unique up to scale.

If we introduce a basis $X = X^i T_i$ then

$$\langle X, X \rangle = X^i X^j \langle T_i, T_j \rangle = g_{ij} X^i X^j$$

↖ inner product matrix.

The inner product can be explicitly constructed by:

* Killing form : let $\hat{\rho}_{adj}(X)^i{}_j$ be the adjoint representation of g

then

$$\langle X, Y\rangle = \operatorname{tr} \hat{\rho}_{adj}(X)\,\hat{\rho}_{adj}(Y) = \hat{\rho}_{adj}(X)^i{}_j\,\hat{\rho}_{adj}(Y)^j{}_i$$

For matrix groups one can also take the trace in the defining representation $\hat{\rho}_{def}$

$$\langle X, Y\rangle = \operatorname{tr} \hat{\rho}_{def}(X)\,\hat{\rho}_{def}(Y) = \operatorname{tr} XY$$

There is an important lemma:

Lemma

The restriction of the inner product $\langle\ ,\ \rangle$ to the real Cartan sub-algebra $h_{\mathbb{R}}$ is positive definite. (i.e. $\langle H, H\rangle > 0$ for all $H \in h_{\mathbb{R}}$)

In our examples:

* for $SL(3,\mathbb{C})$:

$$\langle H, H\rangle = \operatorname{tr} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 > 0 \qquad \lambda_1+\lambda_2+\lambda_3 = 0$$

or in the basis:

$$H = h_1 \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} + h_2 \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} = \begin{pmatrix} h_1 & & \\ & h_2 - h_1 & \\ & & -h_2 \end{pmatrix}$$

we have:

$$\langle H, H\rangle = h_1^2 + (h_2 - h_1)^2 + (-h_2)^2 = 2h_1^2 - 2h_1h_2 + 2h_2^2$$

$$= g_{ij}\, h^i h^j \qquad g_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

* for $sp(4,\mathbb{C})$:

$$\langle H, H\rangle = \operatorname{tr} \begin{pmatrix} h_1 & & & \\ & -h_1 & & \\ & & h_2 & \\ & & & -h_2 \end{pmatrix}^2 = 2h_1^2 + 2h_2^2 = g_{ij}\, h^i h^j$$

$$g_{ij} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

There is then also an inner product on h^* given by the inverse metric

so for our example:

* for $SL(3,\mathbb{C})$: $\quad g^{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{ij}$

* for $Sp(4,\mathbb{C})$: $\quad g^{ij} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{ij}$

we denote the inner product on h^* by the same notation $\langle\ ,\ \rangle$.

Using these ingredients we can prove:

(1) If $X \in g_\alpha$ and $Y \in g_\beta$ then

$[X,Y]$ is a non-zero element of $g_{\alpha+\beta}$ if $\alpha+\beta \in R$

and otherwise $[X,Y] = 0$

(2) The set of roots R spans h^* (ie every element of h^* can be written as a linear combination of roots)

(3) If $\alpha \in R$ then $n\alpha \in R$ only for $n = \pm 1$

This information is sufficient to show that

* given the set of roots R we can reconstruct the whole Lie algebra g (ie we know all the commutators)

What possible sets of roots R are allowed? It is useful to define

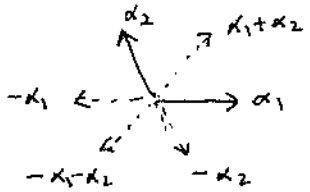
Def

A set of fundamental roots $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset R$ where $r = \text{rank } g$ is a set of roots such that

1. $\alpha_1, \dots, \alpha_r$ form a basis for h^*
2. any root α can be written as $\alpha = n^i \alpha_i$ with n^i all ~~positive~~ positive or all negative integers ($n_i \geq 0$ or $n_i \leq 0$)

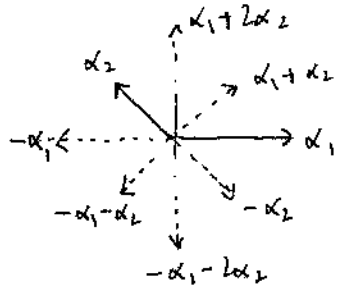
In our examples

* $sl(3,\mathbb{C})$



$\Pi = \{\alpha_1, \alpha_2\}$

* $sp(4,\mathbb{C})$



$\Pi = \{\alpha_1, \alpha_2\}$

We also have:

Thm

Let $\alpha \in R$ and $s_\alpha : h^* \to h^*$ be the reflection in the plane orthogonal to α, that is:

$$s_\alpha(\lambda) = \lambda - \frac{2\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \cdot \alpha$$

then if $\beta \in R$ then $s_\alpha(\beta) \in R$. In other words s_α is an automorphism of the set of roots R.

and we define:

Def

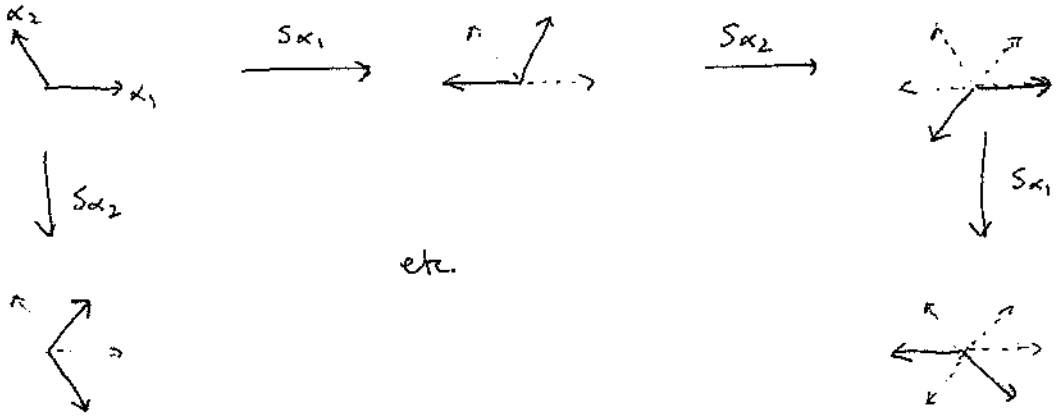
The Weyl group is the finite group generated by s_α for all $\alpha \in R$.

Then

* the set of Fundamental roots Π is unique up to the action by ~~the Weyl group~~ an element of the Weyl group.
* the set of all roots R can be generated from Π by acting with all elements of the Weyl group.

for example.

* $sl(3, \mathbb{C})$



A key quantity is ~~these~~ to construct the root diagram is then:

Def.

Given a set of fundamental roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ we define:

$$\text{Cartan matrix} \quad A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} \qquad \text{for } \alpha_i, \alpha_j \in \Pi$$

For example:

* $su(3,\mathbb{C})$ $\quad \langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = 1$ (choose normalisation of $\langle , \rangle$)

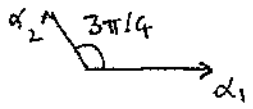
$\langle \alpha_1, \alpha_2 \rangle = \cos(2\pi/3) = -\frac{1}{2}$

so:

$$A_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

* $sp(4,\mathbb{C})$:

$\langle \alpha_1, \alpha_1 \rangle = 2 \quad \langle \alpha_2, \alpha_2 \rangle = 1$



$\langle \alpha_1, \alpha_2 \rangle = \sqrt{2} \cdot \cos(3\pi/4) = -1$

so:

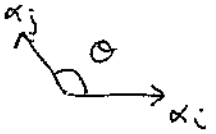
$$A_{ij} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

Now we note that we can show:

1) $A_{ij} \in \mathbb{Z}$ $\quad$ 2) $A_{ij} \leq 0$ if $i \neq j$

but

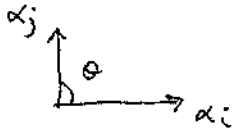
$$A_{ij} \cdot A_{ji} = 4 \frac{\langle \alpha_i, \alpha_j \rangle^2}{\langle \alpha_i, \alpha_i \rangle \langle \alpha_j, \alpha_j \rangle} = 4\cos^2\theta_{ij} \in \mathbb{Z}$$



For two roots we have 4 possibilities:

(a) $\langle \alpha_i, \alpha_j \rangle = 0 \quad \theta = \frac{1}{2}\pi$ $\quad$ any lengths.

$A_{ij} = A_{ji} = 0$

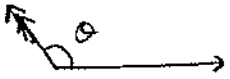


(b) $A_{ij} = A_{ji} = -1 \quad \theta = \frac{2}{3}\pi$ $\quad$ equal lengths.

$\langle \alpha_i, \alpha_i \rangle = \langle \alpha_j, \alpha_j \rangle$

(c) $A_{ij} = -1, \; A_{ji} = -2 \quad \theta = \frac{3}{4}\pi$ $\quad$ α_i longer than α_j

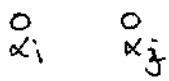
$\langle \alpha_i, \alpha_i \rangle = 2 \langle \alpha_j, \alpha_j \rangle$

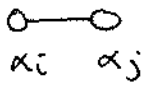


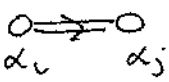
(d) $A_{ij} = -1, \; A_{ji} = -3 \quad \theta = \frac{5}{6}\pi$ $\quad$ α_i longer than α_j

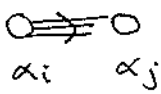
$\langle \alpha_i, \alpha_i \rangle = 3 \langle \alpha_j, \alpha_j \rangle$

We denote these by "Dynkin diagrams"

(a) 

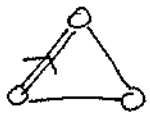
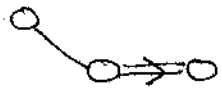
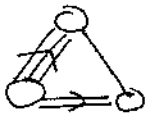
(b) 

(c) 

(d) 

Thus we can encode the Cartan matrix by a diagram: ~~eg~~

eg: rank = 3 : $\Rightarrow$ 3 nodes. (one for each α_i)

 or  or  etc.

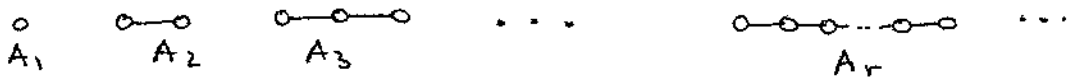
The requirement that the inner product $\langle , \rangle$ is positive definite on h_R is very restrictive. Only certain Cartan matrices are possible: Listing the allowed Cartan matrices classifies the algebras.

Recall if g is semi-simple then it decomposes as

$$g = g_1 \oplus g_2 \oplus \cdots \oplus g_s \qquad \text{where } g_i \text{ are simple.}$$

For the simple Lie algebras, the Cartan matrix of rank r must be one of the following:

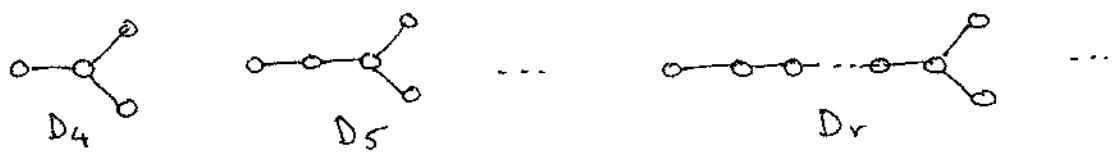
- A_r series



- B_r series

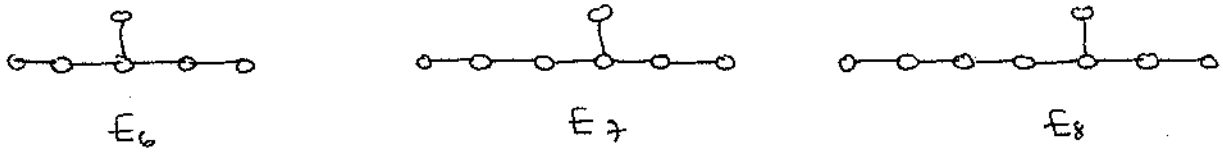
- C_r series

- D_r series



and

exceptional cases



E_6 E_7 E_8

F_4 G_2

The first four series correspond to the classical Lie groups

Lie algebra	compact, simply connected Lie group
$A_r \simeq sl(r+1, \mathbb{C})$	$SU(r+1)$
$B_r \simeq so(2r+1, \mathbb{C})$	$Spin(2r+1)$
$C_r \simeq sp(2r, \mathbb{C})$	$Sp(r)$
$D_r \simeq so(2r, \mathbb{C})$	$Spin(2r)$

for the exceptional groups we have:groups

E_6 : rank = 6 , dimension = 78

E_7 : rank = 7 , dimension = 133

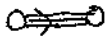
E_8 : rank = 8 , dimension = 248

F_4 : rank = 4 , dimension = 52

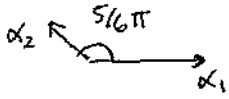
G_2 : rank = 2 , dimension = 14.

In all cases the groups can be detected using some property of the octonians.

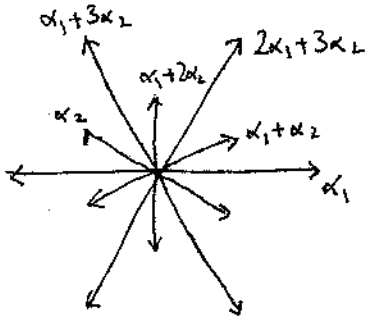
Let's construct some explicit examples:

* $g_{2,\mathbb{C}}$  $\alpha_1 \quad \alpha_2$ $\qquad A_{ij} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

$\langle \alpha_1, \alpha_1 \rangle = 1 \qquad \langle \alpha_2, \alpha_2 \rangle = \frac{1}{3} \qquad \langle \alpha_1, \alpha_2 \rangle = -\frac{1}{2}$



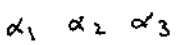
Using the Weyl group we get:



- 12 roots
- $\dim h = 2$

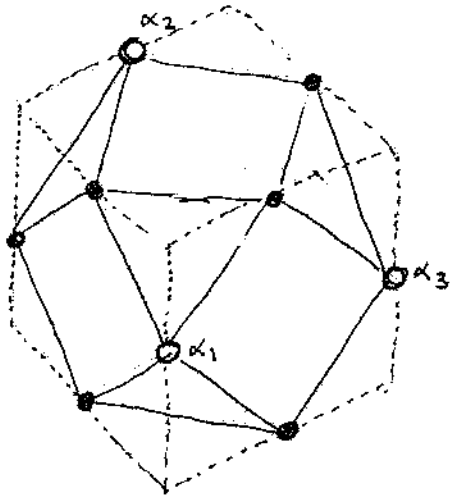
$\rightarrow \dim g_{2,\mathbb{C}} = 14$

If we go to rank 3:

* $sl(4,\mathbb{C})$ 

$A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ choose normalisation...

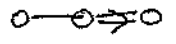
$\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = \langle \alpha_3, \alpha_3 \rangle = 2$ $\quad \langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_3 \rangle = -1$



$\alpha_1 = (1,-1,0)$ 12 roots

$\alpha_2 = (-1,0,1)$

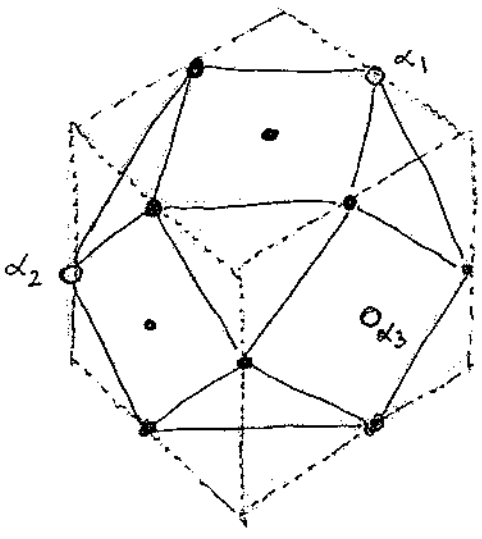
$\alpha_3 = (1,1,0)$

* $so(7,\mathbb{C})$ 

$A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$

choose normalisation $\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = 2 \quad \langle \alpha_3, \alpha_3 \rangle = 1$

$\langle \alpha_1, \alpha_2 \rangle = -1 \quad \langle \alpha_2, \alpha_3 \rangle = -1$



$\alpha_1 = (0,1,1)$ 18 roots.

$\alpha_2 = (-1,-1,0)$

$\alpha_3 = (1,0,0)$

3.6 Irreducible representations of ~~semi~~-simple Lie algebras

Recall that the ~~key~~ tool to classifying the semi-simple Lie algebras was to decompose g as an h-module:

$$g = h \oplus \bigoplus_{\alpha} g_\alpha \qquad [H, X] = \alpha(H)X \qquad H \in h,\ X \in g_\alpha$$

α ← roots.

But we can similarly decompose any g-module. Given a g-module V it will decompose into ~~one~~ one-dimensional h-modules:

$$V = \bigoplus_{\omega} V_\omega \qquad H \cdot v = \omega(H) v \qquad H \in h,\ v \in V_\omega$$

ω ← "weights of g-module V"

Note that:

- a given module V may have multiple V_ω with the same weight.
- the idea is to describe the allowed weights of the finite-dimensional irreducible g-modules.

~~As before we can take $sl(2,\mathbb{C})$ as our guide~~. ~~As~~ Recall that for a given ~~weight~~ root α: we have: $g_\alpha = \{a J^+_\alpha\}$ $\quad g_{-\alpha} = \{b J^-_\alpha\}$

Note that: if $v \in V_\omega$ then

$$\begin{aligned} H \cdot (J^\alpha_\pm \cdot v) &= J^\alpha_\pm \cdot H \cdot v + [H, J^\alpha_\pm] \cdot v \\ &= \omega(H)\, J^\alpha_\pm \cdot v \pm \alpha(H)\, J^\alpha_\pm \cdot v \\ &= (\omega(H) \pm \alpha(H))\, J^\alpha_\pm \cdot v \end{aligned}$$

so:

$$J^\alpha_\pm \cdot v \in V_{\omega \pm \alpha} \qquad (\text{or } J^\alpha_\pm \cdot v = 0 \text{ if } \omega \pm \alpha \text{ is not a weight})$$

As before, we can take $sl(2,\mathbb{C})$ as our basic example. We have:

$$h = \{\lambda J_3\} \qquad J_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then:

* trivial rep V: $\quad J_3 \cdot v = 0 \qquad \omega(J_3) = 0 \qquad V = V_0$

Next

* defining rep

$$(J_3 \cdot v)^i = \hat{\rho}(J_3)^i{}_j v^j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

and we have eigenspaces:

$$V_1 = \left\{ a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \qquad V_{-1} = \left\{ b\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

so:

$$V = V_1 \oplus V_{-1} \qquad \omega(J_3) = \pm 1$$

Recall also:

$$\kappa(J_3) = 2 \qquad J_+\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \qquad J_+\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$J_-\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad J_-\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

so J_+ and J_- are "raising and lowering ops"

* triplet (symmetric φ^{ij})

$$(J_3 \cdot_{\text{sym}} \varphi)^{ij} = \hat{\rho}(J_3)^i{}_k \varphi^{kj} + \hat{\rho}(J_3)^j{}_k \varphi^{ik}$$

and we have eigenspaces:

$$V_2 = \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \qquad V_0 = \left\{ b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \qquad V_{-2} = \left\{ c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

so

$$V = V_2 \oplus V_0 \oplus V_{-2} \qquad \omega(J_3) = 2, 0, -2$$

In general: for the n-dimensional rep:

$$V = V_{n-1} \oplus V_{n-3} \oplus \cdots \oplus V_{-(n-3)} \oplus V_{-(n-1)}$$

$$\omega(J_3) = n-1, n-3, \ldots, -(n-3), -(n-1)$$

and we say:

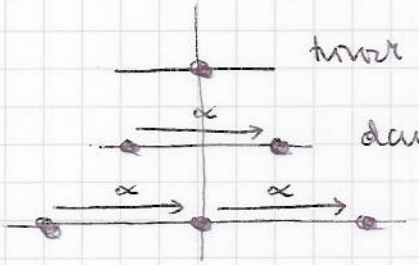
* V_{n-1} is the "highest weight" multiplet of the rep

so: If $\nexists\, v \in V_{n-1}$ then $J_+ \cdot v = 0$

we can plot these along with the roots ($[J_3, J_\pm] = \pm 2 J_\pm$)

"weight lattice" $\omega(J_3) \in \mathbb{Z}$

and we have



triplet (highest weight = 2)

etc.

etc.

Note:

- rep is defined by giving the highest weight

Now we consider the same ideas for general semi-simple g-module V

Since V decomposes as $SU(2,\mathbb{C})$ modules for $\{J_+^\alpha, J_-^\alpha, J_3^\alpha\}$

we have: $V = \bigoplus_\omega V_\omega$ such that

$$\omega(J_3^\alpha) \in \mathbb{Z}$$

If we take: $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ fundamental roots, then

define:

$$\omega_i(J_3^{\alpha_j}) = \delta_{ij} \qquad \text{fundamental weight } \omega_i$$

$$\Gamma = \{ n^i \omega_i : n^i \in \mathbb{Z} \} \qquad \text{weight lattice}$$

Recall.

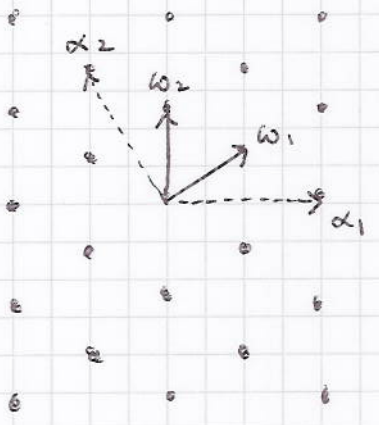
$$\alpha_i(J_3^{\alpha_j}) = \frac{2\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_j, \alpha_j \rangle} = A_{ji} \in \mathbb{Z} \quad \text{(Cartan matrix element)}$$

so:

- $\alpha_i = \sum_j A_{ji}\, \omega_j$
- hence the roots are elements of the weight lattice

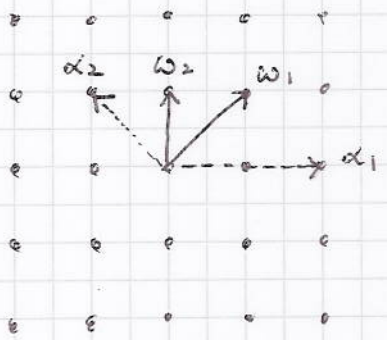
As an example: $sl(3,\mathbb{C})$: $A_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ so

$\alpha_1 = 2\omega_1 - \omega_2 \qquad \alpha_2 = -\omega_1 + 2\omega_2$



or $sp(4,\mathbb{C})$: $A_{ij} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

$\alpha_1 = 2\omega_1 - 2\omega_2 \qquad \alpha_2 = -\omega_1 + 2\omega_2$



To define the reps we need first the notion of "highest weight". We define,

* If $\lambda = \sum_i n_i \omega_i$ where n_i are non-negative integers then

λ is a dominant integral weight.

The key theorem is then:

Thm

Every finite-dimensional irreducible representation of g corresponds to some dominant integral weight λ. Furthermore in the decomposition of the module

$$V = \bigoplus_{\omega} V_\omega$$

each weight ω has the form:

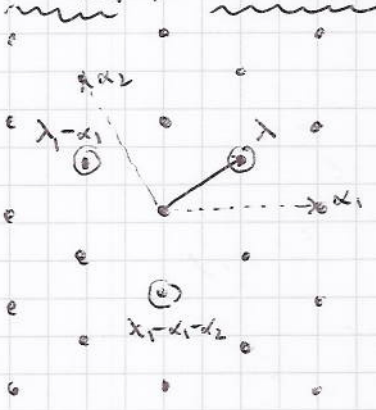
$\omega = \lambda - \sum m_i \alpha_i$ $\quad m_i$ are non-negative integers

so λ is the "highest weight".

The allowed weights in the decomposition follow from thinking about the $su(2,\mathbb{C})$ representation defined by each root α_i.

This is easiest to see by example: $su(3,\mathbb{C})$:

* $\lambda = \omega_1$, $\alpha_1 = 2\omega_1 - \omega_2$, $\alpha_2 = -\omega_1 + 2\omega_2$



$\lambda = \omega_1$

$\lambda - \alpha_1 = -\omega_1 + \omega_2$

$\lambda - \alpha_1 - \alpha_2 = -\omega_2$

so: for $J_3^{\alpha_1}$:

$\begin{pmatrix} \lambda \\ \lambda - \alpha_1 \end{pmatrix}$: doublet $\quad \lambda - \alpha_1 - \alpha_2$ singlet

and for $J_3^{\alpha_2}$

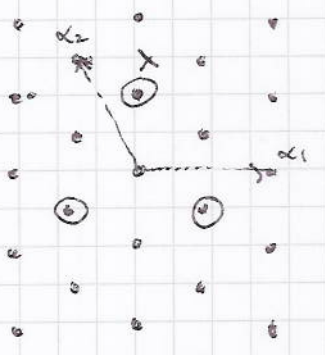
λ : singlet $\quad \begin{pmatrix} \lambda - \alpha_1 \\ \lambda_1 - \alpha_1 - \alpha_2 \end{pmatrix}$ doublet.

Note that $\lambda - \alpha_2 = 2\omega_1 - 2\omega_2$, but we can complete the $su(2,\mathbb{C})$ reps for α_1 and α_2 with charge 2 states so we see this to zero: we get:

* triplet $\quad \underline{3} = V_{\omega_1} \oplus V_{-\omega_1+\omega_2} \oplus V_{-\omega_2}$ (defining rep)

We are here.

* $\lambda = \omega_2$



$\lambda = \omega_2$

$\lambda - \alpha_2 = \omega_1 - \omega_2$

$\lambda - \alpha_1 - \alpha_2 = -\omega_1$

* dual triplet $\quad \overline{3} = V_{\omega_2} \oplus V_{\omega_1 - \omega_2} \oplus V_{-\omega_1}$

More generally one finds:

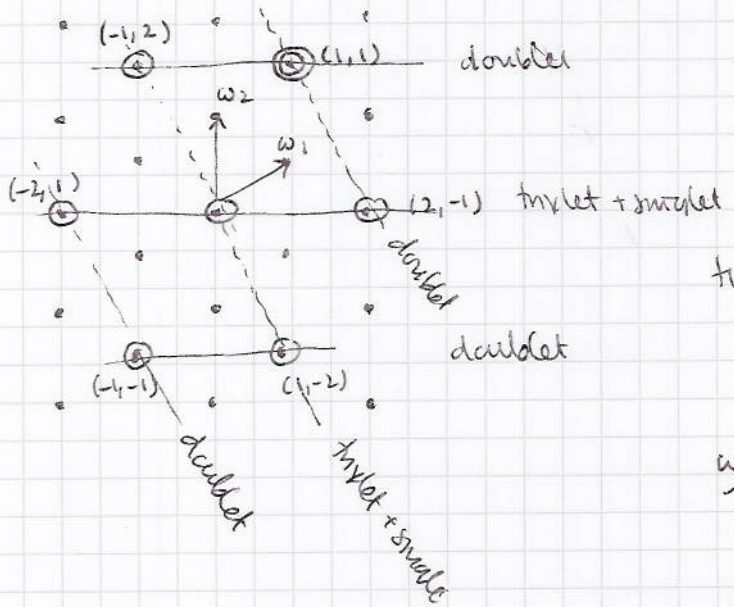
* weight diagram is symmetric under action of Weyl group

In fact:

* to construct set of weights:
  - all contained within "convex hull" of orbit of λ under the Weyl group
  - each weight can be written as $\omega = \lambda - \sum_i n_i \alpha_i \quad n_i \geq 0$

As another example:

* $\lambda = \omega_1 + \omega_2$



$$V = V_{\omega_1+\omega_2} \oplus V_{-\omega_1+2\omega_2} \oplus V_{2\omega_1-\omega_2} \oplus V_0 \oplus V_{-2\omega_1+\omega_2} \oplus V_{-\omega_1-\omega_2} \oplus V_{\omega_1-2\omega_2}$$

thus is:

$\underline{8} \otimes \underline{8}$: adjoint rep.

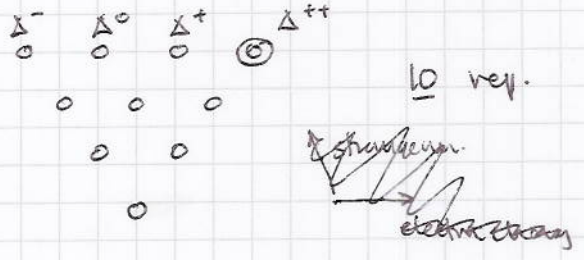
weight diagram = roots + center

V_0 is 2-dimens

We see:

* the weight diagram is exactly what we drew for the baryon and meson in the quark model

For example:



However we would also like to know:

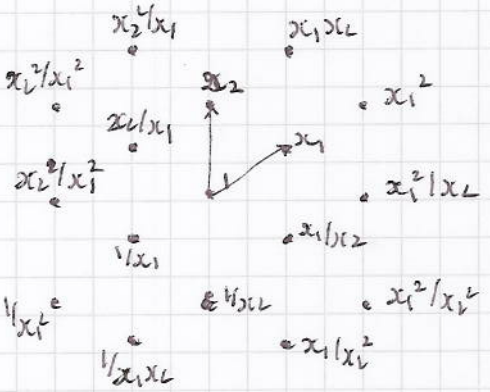
① what is degeneracy of V_ω in V?

② what is dimension of irrep V for λ?

Useful concept is "character" of irrep: Start by introducing a polynomial for each point in lattice:

- if $\omega = \sum_i n_i \omega_i$ then $e(\omega) = x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$

for example.



then:

$$\mathrm{char}(V) = \sum_{\omega \in \Gamma} (\dim V_\omega)\, e(\omega)$$

so: $(x_1 = x, x_2 = y)$

$[1,0] \sim \underline{3}$ $\quad$ $\mathrm{char}(V) = x + y/x + 1/y$

$[0,1] \sim \overline{\underline{3}}$ $\quad$ $\mathrm{char}(V) = y + x/y + 1/x$

dim 2.

$[1,1] \sim \underline{8}$ $\quad$ $\mathrm{char}(V) = xy + y^2/x + x^2/y + 2 + y/x^2 + x/y^2 + 1/xy$

It has the property that

* $\mathrm{char}(V_1 \oplus V_2) = \mathrm{char}(V_1) + \mathrm{char}(V_2)$
* $\mathrm{char}(V_1 \otimes V_2) = \mathrm{char}(V_1)\,\mathrm{char}(V_2)$

This is useful for decomposition.

we have:

$$\text{char}(V_{1,0})\ \text{char}(V_{0,1})$$
$$= (x + y/x + 1/y)(y + x/y + 1/x)$$
$$= xy + y^2/x + x^2/y + 3 + y/x^2 + x/y^2 + 1/xy$$
$$= \text{char}(V_{1,1}) + \text{char}(V_{0,0})$$

so

$$\underline{3} \otimes \underline{\bar{3}} = \underline{8} \oplus \underline{1}$$

There is a remarkable formula: Weyl's character formula

$$\text{char}(V_\lambda) = \frac{\sum_{s \in W} (\det s)\, e(s(\lambda+\rho))}{\sum_{s \in W} (\det s)\, e(s(\rho))}$$

where:

$$\rho = \frac{1}{2}\sum_{R^+} \alpha = \sum_i \omega_i \qquad s \in \text{Weyl group, } W \qquad \det s = \begin{cases} +1 & \text{if rotation} \\ -1 & \text{if reflection} \end{cases}$$

(positive roots)

Note also: $\text{char}(V_\lambda)|_{x_i = 1} = \dim V_\lambda$. One finds

$$\dim V_\lambda = \frac{\prod_{\alpha \in R^+} \langle \alpha, \lambda+\rho \rangle}{\prod_{\alpha \in R^+} \langle \alpha, \rho \rangle}$$

For example: $SU(3, \mathbb{C})$.

$$R^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\} = \{2\omega_1 - \omega_2,\ 2\omega_2 - \omega_1,\ \omega_1 + \omega_2\}$$

$$\rho = \tfrac{1}{2}[\alpha_1 + \alpha_2 + (\alpha_1+\alpha_2)] = \alpha_1 + \alpha_2 = \omega_1 + \omega_2$$

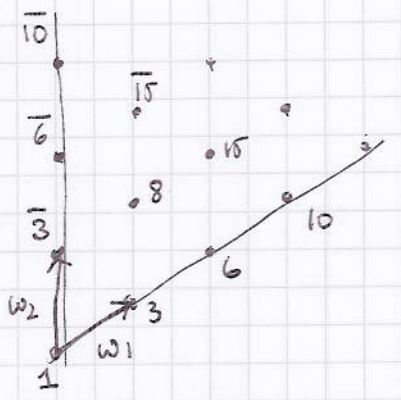
$$\lambda = n_1\omega_1 + n_2\omega_2$$

$$\langle \omega_1, \omega_1 \rangle = \langle \omega_2, \omega_2 \rangle = 2\langle \omega_1, \omega_2 \rangle$$

so:

$$\dim V_{n_1,n_2} = \tfrac{1}{2}(n_1+1)(n_2+1)(n_1+n_2+2)$$

highest weights:



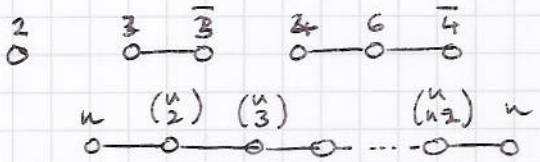
It is also useful to define:

* fundamental representation: V_{ω_i}, $\lambda = \omega_i$ for each i, one for each node of Dynkin diagram
* all irreps are subspaces of tensor products of fund. rep.

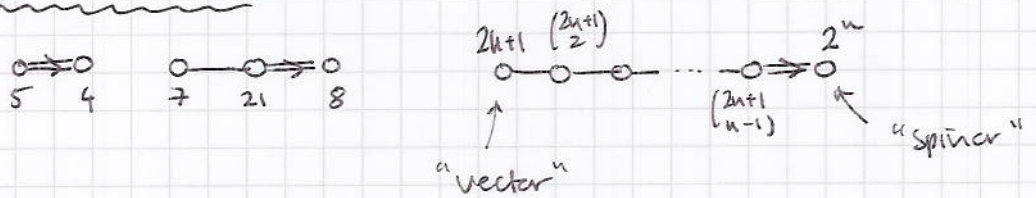
We have:

$A_{n-1} \cong su(n)_{\mathbb{C}}$

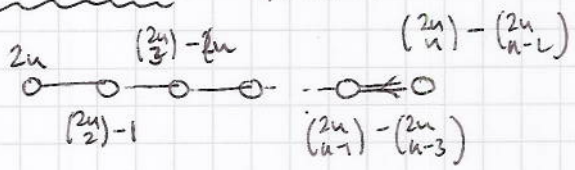
$$\binom{p}{q} = \frac{p!}{(p-q)!\,q!}$$



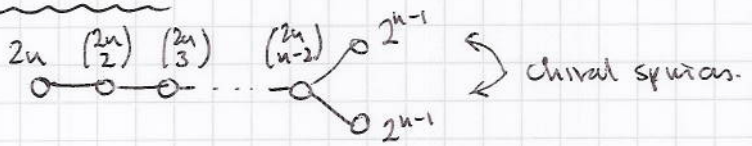
$B_n \cong so(2n+1)_{\mathbb{C}}$



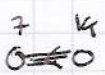
$C_n \cong sp(n)_{\mathbb{C}} \cong sp(2n, \mathbb{C})$



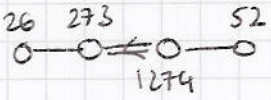
$D_n \cong so(2n)_{\mathbb{C}}$



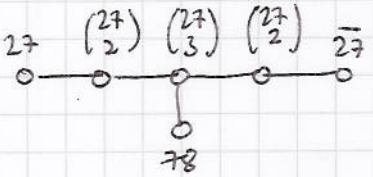
G_2



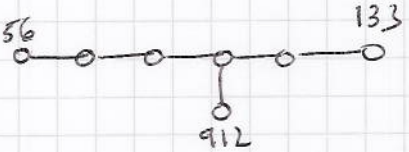
F_4



E_6



E_7



E_8

There are a number of topics we haven't covered:

* Clifford algebras and construction of Spin(n)
* tensors and reps:

  $sl(2,\mathbb{C})$: ϕ, ϕ^i, $\phi^{(ij)}$, $\phi^{(ij)k}$, ...

  in general:

  $sl(n,\mathbb{C})$: $\phi^{i_1 \cdots i_m}$ symmetrized / antisymmetrized.

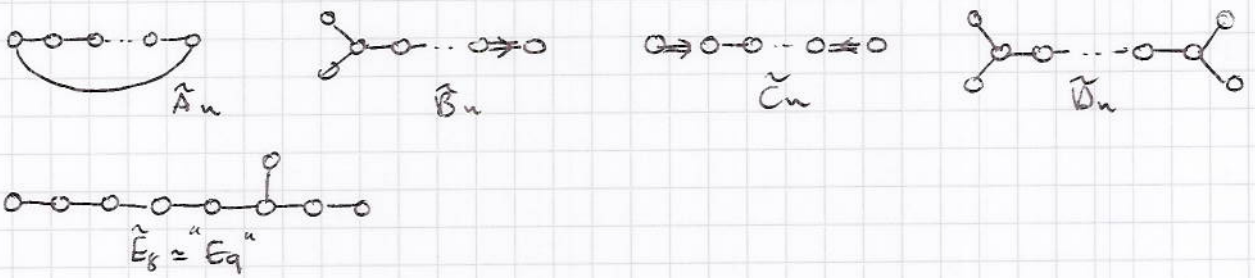
  "Young tableau"

  with similar rules for $so(n)_{\mathbb{C}}$, $sp(n)_{\mathbb{C}}$
* reps of Heisenberg grp — unique for QM!!

Finally 2 further interesting points: Lie grps and algs help us classify the simple finite grps (completed in 1981!)

* cyclic grp of prime order
* alternating grp of degree $n \geq 5$ (even perms. of n elements)
* finite simple analogue of automorphisms of simple Lie algebras
* one of 26 sporadic grps
  - largest is the MONSTER: $\sim 8 \times 10^{53}$ elements
  - smallest faithful rep: 196883

What about extending Dynkin diagrams? "affine Lie algs"



* infinite dimensional Lie algebras. "Kac-Moody algebras"
* actually appear in QFT / string theory / geometry
* MONSTER is related to $\hat{E}_8$