Answer TWO out of the following THREE questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the TWO answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in TWO answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.
1. This question is about quantizing the electromagnetic field $A_\mu$. The Lagrangian density is
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \]
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(i) What is the momentum density $\pi^\mu$ conjugate to the field $A_\mu$? What problem arises if one tries to use the equal time commutation relations (ETCR) to define the quantum theory? [3 marks]

(ii) Now consider a new Lagrangian density
\[ \mathcal{L}' = -\frac{1}{2} (\partial_\mu A_\nu)(\partial^\mu A^\nu). \]
Show that the Euler–Lagrange equations for $\mathcal{L}'$ are $\partial_\mu^2 A_\mu = 0$.
The Lorenz gauge is defined by the condition $\partial_\mu A_\mu = 0$. Show that, in this gauge, the two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ encode the same equations of motion.
(You may assume the form of the equations of motion for $\mathcal{L}$ without derivation.) [3 marks]

(iii) Consider the plane wave
\[ A_\mu = \epsilon_\mu e^{-ip \cdot x} + \epsilon_\mu^* e^{ip \cdot x}. \]
What are the constraints on the wave-vector $p^\mu$ and polarization $\epsilon_\mu$ for the plane wave to satisfy $\partial_\mu^2 A_\mu = 0$ and the Lorenz gauge condition?
Consider gauge transformations $A_\mu \mapsto A_\mu + \partial_\mu \Lambda$. Give the condition on $p^\mu$ for transformations of the form $\Lambda = \lambda e^{-ip \cdot x} + \lambda^* e^{ip \cdot x}$ to preserve the Lorenz gauge condition.
Show that, for the plane wave, under such a gauge transformation, one has
\[ \epsilon_\mu \mapsto \epsilon_\mu - i\lambda p_\mu. \] [2 marks]

(iv) Show that the ETCR for the Lagrangian density $\mathcal{L}'$ imply
\[ [A_\mu(t, x), A_\nu(t, y)] = -i \eta_{\mu\nu} \delta^{(3)}(x - y), \]
where $A_\mu = \partial_0 A_\mu$. Given the expansion
\[ A^\mu = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (c^\mu_p e^{-ip \cdot x} + c^{\mu\dagger}_p e^{ip \cdot x}), \]
where $E_p = |p|$, show that the ETCR is implied by the relations
\[ [c^\mu_p, c^{\nu\dagger}_q] = -(2\pi)^3 \eta^{\mu\nu} \delta^{(3)}(p - q) \]
and $[c^\mu_p, c^\nu_q] = [c^{\mu\dagger}_p, c^{\nu\dagger}_q] = 0$. [3 marks]
(v) Given a polarization vector $\epsilon_\mu$, we define the one-particle states

$$|p, \epsilon\rangle = \sqrt{2E_p} \epsilon_\mu \epsilon^{\mu\dagger} |0\rangle.$$  

How should the ground state $|0\rangle$ be defined?

Show that

$$\langle p', \epsilon'|p, \epsilon\rangle = -2E_p (\epsilon'^x \cdot \epsilon)(2\pi)^3 \delta^{(3)}(p - p').$$

Comment on the value of the norm of the state $|p, \epsilon\rangle$ when $\epsilon_\mu$ is timelike.

What if $\epsilon_\mu$ is null? [3 marks]

(vi) To impose the Lorenz gauge condition, we require that all physical states $|\text{phys}\rangle$ satisfy

$$ (\partial_\mu A^\mu)^{(+)}|\text{phys}\rangle = 0.$$  

where $(\partial_\mu A^\mu)^{(+)}$ denotes the part of the expansion of $\partial_\mu A^\mu$ that contains a $e^{-ip \cdot x}$ factor. Furthermore, the residual gauge symmetry $\epsilon_\mu \rightarrow \epsilon_\mu - i\lambda p_\mu$ implies that, physically, we should regard $|p, \epsilon - i\lambda p\rangle$ and $|p, \epsilon\rangle$ as defining the same one-particle state.

Show that a one-particle state $|p, \epsilon\rangle$ is physical if and only if $p \cdot \epsilon = 0$.

Taking $p^\mu = (\omega, 0, 0, \omega)$, argue that a basis for physical, physically distinct, one-particle states with momentum $p^\mu$ is given by the transverse polarization states

$$|p, \epsilon^1\rangle \text{ and } |p, \epsilon^2\rangle$$

where

$$\epsilon^1_\mu = (0, 1, 0, 0) \text{ and } \epsilon^2_\mu = (0, 0, 1, 0).$$  

[6 marks]

[Total 20 marks]
2. This question is about positron-positron scattering. The spins and momenta are labelled as follows

\[ e^+ (p_1, s_1) + e^+ (p_2, s_2) \rightarrow e^+ (p_1', s_1') + e^+ (p_2', s_2'). \]

In the centre-of-mass frame, for equal mass particles, the differential cross-section is given by

\[ \frac{d\sigma}{d\Omega} = \frac{X}{64\pi^2 E_{CM}^2}, \]

where \( E_{CM} \) is the total energy.

(i) Draw the two Feynman diagrams that contribute to the \( e^+e^+ \rightarrow e^+e^+ \) process at order \( e^2 \) and use the momentum-space Feynman rules to give expressions for the corresponding amplitudes \( iM_a \) and \( iM_b \). [4 marks]

(ii) As functions of the four momenta, what is the relation between the two amplitudes \( iM_a \) and \( iM_b \)? The amplitudes are related by the LSZ reduction formula to a particular QED correlation function. Which correlation function is it? [2 marks]

(iii) Consider the high-energy limit where the electron mass \( m \) can be taken to be effectively zero. Show that, in the unpolarized cross-section, \( X \) takes the form

\[ X = \frac{e^4}{4} \left[ A + \frac{B}{u^2} + \frac{C + C^*}{tu} \right], \]

where we are using the Mandelstam variables \( s = (p_1 + p_2)^2 \), \( t = (p_1' - p_1)^2 \) and \( u = (p_1' - p_2)^2 \). Using the spin-sum relations

\[ \sum_s u^s(p)\bar{u}^s(p) = \slashed{p} + m, \quad \sum_s v^s(p)\bar{v}^s(p) = \slashed{p} - m, \]

give \( A \), \( B \) and \( C \) in terms of traces of gamma matrices. [5 marks]

(iv) Show, in particular, that

\[ A = 8(s^2 + u^2) \]

(You may wish to use the identity \( \text{Tr} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = 4\eta^{\mu\nu} \eta^{\rho\sigma} - 4\eta^{\mu\rho} \eta^{\nu\sigma} + 4\eta^{\mu\sigma} \eta^{\nu\rho} \). This and any other relevant trace identities may be quoted without proof.) [3 marks]

(v) Given that

\[ X = 4e^4 \left[ \frac{s^2}{t^2} + \frac{s^2}{u^2} + 1 \right]. \]

and \( p_1 \cdot p_1' = |p_1||p_1'| \cos \theta \), show that, in the centre-of-mass frame,

\[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{E_{CM}^2} \left( \frac{1}{\sin^4 \frac{1}{2}\theta} + \frac{1}{\cos^4 \frac{1}{2}\theta} + 1 \right), \]

where \( \alpha = e^2/4\pi \). [3 marks]
(vi) Now consider $e^+e^- \rightarrow e^+e^-$ scattering. Draw the two Feynman diagrams that contribute to this process at order $e^2$.

By considering the relationship between these Feynman diagrams and those in part (i), give an expression for $X$ for $e^+e^- \rightarrow e^+e^-$ scattering in terms of $s$, $t$ and $u$.

[3 marks]

[Total 20 marks]
3. This question is about two-point functions in scalar field theory and QED. Let $e_0$ denote the bare electric charge and $m_0$ denote the bare electron mass.

(i) Consider an interacting scalar theory. Assume that there is a unique ground state $|\Omega\rangle$ and a set of one-particle states $|p\rangle$ with invariant mass $m$.

Let $\hat{P}^\mu$ be the four-momentum operator. What are the eigenvalues of $\hat{P}^\mu$ acting on $|p\rangle$? Sketch the generic form for the spectrum of states as a function of total energy and total three-momentum.

How does one expect the Hilbert space of the interacting theory to differ from that of the free theory? What about the expression for the field operator $\phi(x)$ in the Heisenberg picture and the invariant mass $m$ of the one-particle states? [4 marks]

(ii) Under a translation $x^\mu \mapsto x^\mu + a^\mu$ the scalar field transforms as

$$\phi(x) \mapsto \phi'(x) = e^{-ia^\mu \hat{P}_\mu} \phi(x) e^{ia^\mu \hat{P}_\mu}$$

Hence show that one can write

$$\langle \Omega | \phi(x) | p \rangle = \sqrt{Z} e^{-ip \cdot x}$$

and argue that $\sqrt{Z}$ is a constant. [3 marks]

(iii) Using the complete set of states

$$1 = |\Omega\rangle \langle \Omega| + \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |p\rangle \langle p| + \ldots$$

where the dots represent contributions from multi-particle states, show that the interacting two-point function can be written as

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = Z D_F(x-y; m) + \ldots,$$

where again the dots represent contributions from multi-particle states and

$$D_F(x-y; m) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[ \theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)} \right].$$

(Here $E_p = \sqrt{|p|^2 + m^2}$ and $\theta(t) = 1$ if $t > 0$ and $\theta(t) = 0$ if $t < 0$.)

What is assumed about $\langle \Omega | \phi(x) | \Omega \rangle$ in this derivation? [3 marks]

(iv) The Fourier transform of the photon two-point function can be written as

$$\int d^4x d^4y e^{ip \cdot x} e^{ip' \cdot y} \langle \Omega | T A_\mu(x) A_\nu(y) | \Omega \rangle = (2\pi)^4 \delta^{(4)}(p-p') G_{\mu\nu}(p).$$

In terms of the free photon propagator $D_{\mu\nu}(p) = -i\eta_{\mu\nu}/(p^2 + i\epsilon)$ one has

$$G_{\mu\nu}(p) = D_{\mu\nu}(p) + i e_0^2 D_{\mu\alpha}(p) \Pi^{\alpha\beta}(p) D_{\beta\nu}(p) + O(e_0^4).$$

Draw the relevant Feynman diagram and use the Feynman rules to write down the expression for the photon self-energy $\Pi^{\alpha\beta}(p)$.

Argue that $\Pi^{\alpha\beta}(p)$ is potentially divergent. [3 marks]

[This question continues on the next page . . . ]
(v) Suppose we regulate the photon self-energy to give a finite function
\[ \Pi^{\alpha\beta}(p, \Lambda) \xrightarrow{\Lambda \to \infty} \Pi^{\alpha\beta}(p) \] with a cut-off scale \( \Lambda \). You are given that
\[
\Pi^{\alpha\beta}(p, \Lambda) = -\eta^{\alpha\beta} \left[ A_0(\Lambda) + p^2 A_1(\Lambda) + p^2 \Pi_c(p^2, \Lambda) \right]
\]
where \( \Pi_c(p^2, \Lambda) \sim p^2 \) for small \( p^2 \) and is finite as \( \Lambda \to \infty \), while \( A_0(\Lambda) \) and \( A_1(\Lambda) \) are divergent.
How do \( A_0(\Lambda) \) and \( A_1(\Lambda) \) depend on \( \Lambda \)? Explain how the divergent terms can be absorbed into a renormalisation of the free photon propagator \( D_{\mu\nu}(p) \). Give a physical argument why \( A_0(\Lambda) \) should vanish for any reasonable regularisation procedure. [3 marks]

(vi) Let \( \Pi_c(p^2) \) be the limit of the finite part \( \Pi_c(p^2, \Lambda) \) as \( \Lambda \to \infty \). You are given that, for \( p = (0, \mathbf{p}) \) and \( p^2 \ll m^2 \), one has
\[
e_0^2 \Pi_c(p^2) = -\frac{\alpha |\mathbf{p}|^2}{15\pi m^2},
\]
where \( \alpha = e_0^2 / 4\pi \).
Using the renormalisation derived in part (v), write \( G_{\mu\nu}(p) \) at order \( e_0^2 \), and show that in the limit \( \Lambda \to \infty \) all cut-off dependence can be incorporated into an overall factor.
Given this form, and assuming that in this limit the \( \mathbf{p} \) dependence of \( G_{\mu\nu}(p) \) encodes the three-dimensional Fourier transform of the electric potential, what modification to Coulomb’s law does the \( \Pi_c(p^2) \) correction produce? Is this a measurable effect? [4 marks]

[Total 20 marks]