Answer TWO out of the following THREE questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the TWO answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in TWO answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.
1. This question is about electron-positron annihilation to polarised photons. The spins, polarisation states and momenta are labelled as follows:
\[ e^{-}(p, s) + e^{+}(p', s') \rightarrow \gamma(k, \epsilon) + \gamma(k', \epsilon') \]

(i) Draw the two Feynman diagrams which contribute to the annihilation process at leading order and use the momentum-space Feynman rules to give expressions for the corresponding amplitudes \( iM_a \) and \( iM_b \). Give a physical reason for whether or not there is a relative sign between them.

How will the total cross-section depend on the centre-of-mass energy, in the limit of very high energy?

The amplitude \( iM \) is related by the LSZ reduction formula to a particular QED correlation function. Which correlation function is it? [5 marks]

(ii) The polarisation \( \epsilon \) is related to the plane-wave solution for the electromagnetic potential
\[ A_{\mu} = \epsilon_{\mu} e^{-ik \cdot x} + \epsilon^*_{\mu} e^{ik \cdot x}. \]

What constraint on \( \epsilon \) arises from the Lorenz gauge condition \( \partial_{\mu} A_{\mu} = 0 \)? Show that, under the residual gauge transformations that preserve the Lorenz gauge, one has
\[ \epsilon_{\mu} \mapsto \epsilon_{\mu} + \alpha k_{\mu}. \]

Hence show that, given a time-like vector \( q \), one can always choose a gauge such that \( q \cdot \epsilon = 0 \). [3 marks]

(iii) Suppose we fix a gauge where \( p \cdot \epsilon = p \cdot \epsilon' = 0 \). Show that
\[ iM = -ie^{2} \bar{v}_{s} \gamma'_{\mu} k_{\mu} u_{s} - ie^{2} \bar{v}_{s} \gamma'_{\mu} k'_{\mu} u_{s}. \]

(You may use \( \not{p} + \not{q} = 2(p \cdot q)\mathbb{1} \) and \( (\not{p} - m)u_{s}(p) = 0 \) without proof.) [3 marks]

(iv) Using the spin sum relations
\[ \sum_{s} \bar{u}^{s}(p) u^{s}(p) = \not{p} + m, \quad \sum_{s} \bar{v}^{s}(p) \not{v}^{s}(p) = \not{p} - m, \]

show that the quantity \( X \) that appears in the cross-section, assuming the electron and positron are unpolarised but retaining the photon polarisations, is given by
\[ X = e^{4} \left[ \frac{A}{(p \cdot k)^{2}} + \frac{B}{(p \cdot k')^{2}} + \frac{C + C^{*}}{(p \cdot k)(p \cdot k')} \right], \]

and give expressions for \( A, B \) and \( C \) in terms of traces of gamma matrices. [3 marks]

(v) Assuming the normalisation \( \epsilon^{2} = \epsilon'^{2} = -1 \), show that
\[ A = 8(p \cdot k) \left[ 2(k \cdot \epsilon')^{2} + (p' \cdot k) \right], \]

and hence use a simple argument to give an expression for \( B \). (The relation \( \not{p} \not{q} + \not{q} \not{p} = 2(p \cdot q)\mathbb{1} \) and any other gamma-matrix identities you use may be quoted without proof.) [3 marks]
(vi) Given that, at order $e^4$,

$$X_{e^- e^+ \rightarrow \gamma \gamma} = \frac{e^4}{2} \left[ \frac{p \cdot k'}{p \cdot k} + \frac{p \cdot k'}{p \cdot k} - 4(\epsilon \cdot \epsilon')^2 + 2 \right].$$

derive an expression for the corresponding Compton scattering quantity $X_{e^- \gamma \rightarrow e^- \gamma}$. (Label clearly which momenta refer to which particles in your answer.)

[3 marks]

[Total 20 marks]
2. The time evolution of operators in the interaction and Heisenberg picture are given by
\[ \partial_t \hat{O}_I = i [H_{0,I}, \hat{O}_I], \quad \partial_t \hat{O}_H = i [H_H, \hat{O}_H], \tag{2.1} \]
where in the interaction picture, \( \hat{O}_I \) is the operator and \( H_{0,I} \) is the free Hamiltonian, while in the Heisenberg picture \( \hat{O}_H \) is the operator and \( H_H \) is the total Hamiltonian. The two pictures are related by an operator \( U(t, t_0) \), such that
\[ \hat{O}_H(t_0) = \hat{O}_I(t_0), \quad \hat{O}_I(t) = U(t, t_0) \hat{O}_H(t) U(t, t_0)^{-1}. \]

(i) Write down the full Lagrangian density for “phi-fourth” theory and identify the free and interaction Hamiltonians.
Comment briefly on how the interaction picture is used in perturbation theory. [3 marks]

(ii) Write an expression for \( \hat{O}_H(t) \) in terms of \( \hat{O}_H(t_0) \) and \( H_H \), and one for \( \hat{O}_I(t) \) in terms of \( \hat{O}_I(t_0) \) and \( H_{0,I} \). Show that they satisfy (2.1). Hence show
\[ U(t, t_0) = e^{iH_{0,I}(t-t_0)} e^{-iH_H(t-t_0)}. \]
Defining \( U(t_1, t_2) = U(t_1, t_0) U(t_2, t_0)^{-1} \), show that
\[ U(t_1, t_2) U(t_2, t_3) = U(t_1, t_3), \]
\[ U(t_1, t_2)^{-1} = U(t_1, t_2)^\dagger = U(t_2, t_1) \]
[6 marks]

(iii) Using the results of part (ii) show that \( U(t, t') \) satisfies
\[ \partial_t U(t, t_0) = -i H_{int,I}(t) U(t, t_0), \]
where \( H_{int,I} \) is the interaction Hamiltonian in the interaction picture.
By considering a perturbative solution for \( U(t, t_0) \) as a sum of terms ordered in increasing powers of \( H_{int,I} \), show that
\[ U(t, t_0) = T \exp \left( -i \int_{t_0}^t dt' H_{int,I}(t') \right). \]
(You may use without proof that
\[ \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n f(t_1, t_2, \ldots, t_n) \]
\[ = \frac{1}{n!} \int_{t_0}^t \ldots \int_{t_0}^t dt_1 \ldots dt_n f(t_1, \ldots, t_n). \]
for any totally symmetric function \( f(t_1, \ldots, t_n) \).) [6 marks]
(iv) You are given that

$$|\Omega\rangle = \lim_{T' \to \infty} (1 - i\epsilon) U(t_0, -T') |0\rangle e^{-i E_{\Omega} (T' + t_0)} \langle \Omega | 0\rangle ,$$

$$\langle \Omega | = \lim_{T' \to \infty} (1 - i\epsilon) \langle 0 | U(T', t_0) e^{-i E_{\Omega} (T' - t_0)} | \Omega \rangle .$$

where $E_{\Omega}$ is the energy of the ground state $|\Omega\rangle$. Show that the Heisenberg picture two-point scalar field correlation function can be written in terms of interaction picture fields as

$$\langle \Omega | T \phi_i(x) \phi_i(y) | \Omega \rangle = \lim_{T' \to \infty} (1 - i\epsilon) \frac{\langle 0 | T \phi_i(x) \phi_i(y) \exp \left[ -i \int_{-T'}^{T'} dt' H_{\text{int},i}(t') \right] | 0 \rangle}{\langle 0 | T \exp \left[ -i \int_{-T'}^{T'} dt' H_{\text{int},i}(t') \right] | 0 \rangle} .$$

[5 marks]

[Total 20 marks]
3. This question is about two-point functions in QED. Let $e_0$ denote the bare electric charge and $m_0$ denote the bare electron mass. We define

$$\int d^4x \, d^4y \, e^{ip \cdot x} e^{ip' \cdot y} \langle \Omega | T \psi(x) \bar{\psi}(y) | \Omega \rangle = (2\pi)^4 \delta^4(p + p') G_F(p),$$

and

$$\int d^4x \, d^4y \, e^{ip \cdot x} e^{ip' \cdot y} \langle \Omega | T A_\mu(x) A_\nu(y) | \Omega \rangle = (2\pi)^4 \delta^4(p + p') G_{\mu\nu}(p),$$

and

$$S_F(x - y; m) = \int \frac{d^4p}{(2\pi)^3} \frac{1}{2E_p} \theta(x^0 - y^0) e^{-ip \cdot (x - y)} (\not{p} + m)$$

$$+ \int \frac{d^4p}{(2\pi)^3} \frac{1}{2E_p} \theta(y^0 - x^0) e^{ip \cdot (y - x)} (\not{p} - m),$$

where $p^0 = E_p = \sqrt{|p|^2 + m^2}$ and $\theta(t) = 1$ if $t > 0$ and $\theta(t) = 0$ if $t < 0$.

(i) Consider first an interacting scalar theory. Assume that there is a unique ground state $|\Omega\rangle$ and a set of one-particle states $|p\rangle$ with invariant mass $m$.

Let $\hat{\rho}^\mu$ be the four-momentum operator. What are the eigenvalues of $\hat{\rho}^\mu$ acting on $|p\rangle$? Sketch the generic form for the spectrum of states as a function of total energy and total three-momentum.

How does one expect the Hilbert space of the interacting theory to differ from that of the free theory? What about the expression for the field operator $\phi(x)$ in the Heisenberg picture and the invariant mass $m$ of the one-particle states? [4 marks]

(ii) Under a translation $x^\mu \mapsto x^\mu + a^\mu$ the scalar field transforms as

$$\phi(x) \mapsto \phi'(x) = \phi(x - a) = e^{-i\not{a} \cdot \hat{\rho}} \phi(x) e^{i\not{a} \cdot \hat{\rho}}$$

Hence show that one can write

$$\langle \Omega | \phi(x) | p \rangle = \sqrt{Z} e^{-ip \cdot x}$$

and argue that $\sqrt{Z}$ is a constant. [2 marks]

(iii) A similar argument with single electron and positron states in QED gives

$$\langle \Omega | \psi(x) | -, p, s \rangle = \sqrt{Z_2} u_s(p) e^{-ip \cdot x}, \quad \langle \Omega | \bar{\psi}(x) | -, p, s \rangle = 0,$$

$$\langle \Omega | \bar{\psi}(x) | +, p, s \rangle = \sqrt{Z_2} \bar{v}_s(p) e^{-ip \cdot x}, \quad \langle \Omega | \psi(x) | +, p, s \rangle = 0.$$

Using the complete set of states

$$1 = |\Omega\rangle \langle \Omega| + \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[ |-, p, s\rangle \langle -, p, s| + |+, p, s\rangle \langle +, p, s| \right]$$

$$+ \ldots$$

(This question continues on the next page . . . )
where the dots represent contributions from multi-particle states, show that the interacting two-point function can be written as

\[ \langle \Omega | T \psi(x) \bar{\psi}(y) | \Omega \rangle = Z_2 S_F(x - y; m) + \ldots, \]

where again the dots represent contributions from multi-particle states. (You may assume the completeness relations \( \sum_s u^s(p) \bar{u}^s(p) = \not{p} + m \) and \( \sum_s v^s(p) \bar{v}^s(p) = \not{p} - m. \))

(iv) In terms of the free photon propagator \( D_{\mu\nu}(p) = -i\eta_{\mu\nu}/(p^2 + i\epsilon) \) one has

\[ G_{\mu\nu}(p) = D_{\mu\nu}(p) + i e^2_0 D_{\mu\alpha}(p) \Pi^{\alpha\beta}(p) D_{\beta\nu}(p) + O(e^4_0). \]

Draw the relevant Feynman diagram and use the Feynman rules to write down the expression for the photon self-energy \( \Pi^{\alpha\beta}(p) \).

Argue that \( \Pi^{\alpha\beta}(p) \) is potentially divergent.

(v) Suppose the regulated the photon self-energy can be written as

\[ \Pi^{\alpha\beta}(p, \Lambda) = -\eta^{\alpha\beta} [A_0(\Lambda) + p^2 A_1(\Lambda) + p^2 \Pi_c(p^2, \Lambda)]. \]

How do \( A_0(\Lambda) \) and \( A_1(\Lambda) \) depend on \( \Lambda \)? Explain how the divergent terms can be absorbed into a renormalisation of the free photon propagator \( D_{\mu\nu}(p) \). Give a physical argument why \( A_0(\Lambda) \) should vanish for any reasonable regularisation procedure.

(vi) The electron self-energy \( \Sigma(p) \) is defined as

\[ G_F(p) = S_F(p) + S_F(p) \cdot i e^2_0 \Sigma(p) \cdot S_F(p) + O(e^4_0), \]

where \( S_F(p) = i/(\not{p} - m_0 - i\epsilon) \). The regulated self-energy can be expanded as

\[ \Sigma(p, \Lambda) = \Sigma_0(\Lambda) + \Sigma_1(\Lambda)(\not{p} - m_0) + \Sigma_c(p^2, \Lambda)(\not{p} - m_0), \]

where \( \Sigma_0, \Sigma_1 \) and \( \Sigma_c \) are scalars. How do \( \Sigma_0(\Lambda) \) and \( \Sigma_1(\Lambda) \) depend on \( \Lambda \)? Explain how the divergent terms can be absorbed into a renormalization of the free propagator \( S_F(p) \).