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Symmetry and Unification

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Chapter 2

Field Theory

The derivations in this book are based almost entirely on classical field theory. In many problems the classical description gives a good qualitative account of what the full quantum theory predicts and indeed the classical analysis often forms the starting point for the quantum analysis. Thus it is useful to sideline the complications of QFT so we can concentrate in this book on the symmetry aspects. So in the first section we give a brief description of classical field theory.

However the full theories are quantum so while we leave the details of QFT to other texts, we still need an outline of — a recipe for — the link between our classical results and the full quantum field theory. We will look at the link between this classical analysis and the quantum theory in section 2.2 below and make some additional comments as we go.

2.1 Classical Field Theory

What is a field theory? How do we calculate the behaviour of classical fields? How does a field relate to what we call particles. Let us try to answer these questions in this section starting with some examples.

Examples of Classical Fields

When we solve for the motion of a classical particle in classical mechanics, the answer is given as the position, $\boldsymbol{x}$ as a function of time t, or simply written as $\boldsymbol{x}(t)$. The classical particle is at one point in space at any one time. Classical fields on the other hand are things which spread through large amounts of space, and which can also vary in time. Thus to specify the field we must give its value (amplitude), say f, at every point in space and at every time. They are functions of space and time, $f(\boldsymbol{x}, t)$. Familiar examples of fields in the classical world are the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields.

The best known example of a classical field theory is therefore that of classical electromagnetism. There the physically observed evolution of electric and magnetic fields solutions satisfy Maxwell's equations which are

$$\nabla.\boldsymbol{E} = \rho, \quad \nabla \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla.\boldsymbol{B} = 0, \quad \nabla \wedge \boldsymbol{B} = \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{j} \tag{2.1.1}$$

where ρ is the electric charge density and $\boldsymbol{j}$ its current. It is possible to rewrite these equations in a manner more appropriate for a relativistic problem

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad \partial_\mu \mathcal{F}^{\mu\nu} = 0 \tag{2.1.2}$$

where

$$
\begin{aligned}
j^\mu &= (\rho, \boldsymbol{j}), && (2.1.3)\\
A^\mu &= (\phi, \boldsymbol{A}), \quad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi, \quad \boldsymbol{B} = \nabla \wedge \boldsymbol{A} && (2.1.4)\\
F^{\mu\nu} &:= \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} && (2.1.5)\\
\mathcal{F}^{\mu\nu} &:= \frac{1}{2}\epsilon^{\mu\nu\eta\lambda} F_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} && (2.1.6)
\end{aligned}
$$

The components of the four-vector A^μ are the electrostatic potential and the vector potential. The $F^{\mu\nu}$ is the **field-strength tensor** and $\mathcal{F}^{\mu\nu}$ is its **dual**. In this form the field for EM is now the function $A^\mu(\boldsymbol{x}, t)$ and not the familiar electric and magnetic fields. This form is more useful for particle physicists as the space-time transformation properties, in particular under general Lorentz transformations, are much simpler in this form. The field A^μ is a four-vector, that is it transforms like a vector under Lorentz transformations. For this reason it is often called a **vector field**. Such fields always correspond to particles with spin one though this connection requires a study of the group theory behind the Lorentz transformations. These fields also appear in what is called **local symmetry** or **gauge symmetry**, as we will see in chapter 6. Thus they are also called **gauge bosons**. While EM provides the best known example of classical fields, we will see later that it is not the simplest example.

Having noted the link with spin, from the list of fundamental particles given in section 1.3, its clear most of them are spin one-half fermions. The Dirac equation (see (8.1.1)) describes how such particles evolve, at least when there are no interactions. The spin one-half property can again be linked with non-trivial behaviour under Lorentz transformations.

However, many issues can be illustrated with the simplest examples of fields which are those representing spin zero particles. These are called **scalar fields** as they remain unchanged under Lorentz transformations. We will encounter two types of scalar field, real $\phi(x) \in \mathbb{R}$ and complex $\Phi(x) \in \mathbb{C}$ though the latter can always be reexpressed as two real fields $\Phi = (\phi_1 + i\phi_2)/(\sqrt{2})$. No fundamental scalar particles are yet known, but the undiscovered Higgs particle of the EW model is such a particle. The pions, π^+, π^0, π^-, are composite scalar particles (made from two up/down quarks) and there are many other spin-zero mesons. In a non-relativistic context, Cooper pairs, two electron bound states in superconductors, are also scalar particles. Thus scalar fields can be useful in the real world. However their mathematical simplicity means that they are the ubiquitous example in QFT texts, and indeed many examples in this book will use them. They do have one unique role that no other field can play and that is in **symmetry breaking**. The existence of superconducting and superfluid states are examples of this phenomena and we are sure that it is responsible for the mass of the $W^\pm$ and Z^0 gauge bosons too. Symmetry breaking is only possible with a scalar field (be it fundamental or otherwise) and one of the main goals of this book is to study this process. This is another reason why scalar fields will be central to our discussion. A typical equation of motion would be

$$(\Box + m^2)\phi(x) = -\lambda\,(\phi(x))^3 \qquad (2.1.7)$$

where the left-hand side is the **Klein-Gordon** equation for a non-interacting scalar particle. Parameters called **coupling constants** control the strength of interactions. Here λ on the right hand side is a coupling constant and this term describes interactions between the scalar particles.

Principle of least action

A classical theory, say for a mass on a spring moving in d-dimensions with position $\boldsymbol{x}(t) = (x_1(t), x_2(t), \ldots, x_d(t))$, will be able to give the allowed motion of the mass as functions of time $\boldsymbol{x}(t)$. To do this one needs equations of motion, such as derived from Newton's laws of motion, and boundary conditions, two in simple cases (say $\dot{\boldsymbol{x}}(0)$ and $\boldsymbol{x}(0)$). The equations of motion can also be derived from the **principle of least action**, where the action S is given in terms of either a Hamiltonian H or a Lagrangian L

$$S := \int dt\, H\left[\boldsymbol{p}(t), \boldsymbol{x}(t)\right] = \int dt\, L\left[\dot{\boldsymbol{x}}(t), \boldsymbol{x}(t)\right] \tag{2.1.8}$$

Here p is the momentum, the conjugate variable to position x

$$p_i(t) = \frac{\mathrm{d}L}{\mathrm{d}\dot{x}_i(t)} \tag{2.1.9}$$

For instance for a mass on a perfect spring we might have

$$L\left[\dot{\boldsymbol{x}}(t), x(t)\right] = \frac{1}{2m}(\dot{\boldsymbol{x}}(t))^2 - \frac{k}{2}(\boldsymbol{x}(t))^2, \quad H\left[\boldsymbol{p}(t), \boldsymbol{x}(t)\right] = \frac{1}{2m}(\boldsymbol{p}(t))^2 + \frac{k}{2}(\boldsymbol{x}(t))^2 \tag{2.1.10}$$

Note that the Hamiltonian is the total energy, kinetic plus potential energy and it is easiest to define this. The Lagrangian can then be defined through a Legendre transform as

$$L[\boldsymbol{x}, \dot{\boldsymbol{x}}] = \boldsymbol{p}.\dot{\boldsymbol{x}} - H[\boldsymbol{x}, \boldsymbol{p}] \tag{2.1.11}$$

The principle of least action states that the possible classical behaviours, $\bar{\boldsymbol{x}}(t)$, are functions which extremise the action. That is if we compare the action for $\bar{\boldsymbol{x}}(t)$ against a solution which is slightly different, $\bar{\boldsymbol{x}}(t) + \delta\bar{\boldsymbol{x}}(t)$, then the difference is not first order but second order in these small perturbations

$$\delta S = S[\bar{\boldsymbol{x}} + \delta\bar{\boldsymbol{x}}, \dot{\bar{\boldsymbol{x}}} + \delta\dot{\bar{\boldsymbol{x}}}] - S[\bar{\boldsymbol{x}}, \dot{\bar{\boldsymbol{x}}}] = O(\delta\bar{\boldsymbol{x}}^2), \tag{2.1.12}$$

$$\rightarrow \quad \frac{\partial S}{\partial x_i}\delta x_i + \frac{\partial S}{\partial \dot{x}_i}\delta \dot{x}_i = 0 \tag{2.1.13}$$

In the second form we are using **functional differentiation** not ordinary differentiation, as $x_i = x_i(t)$ is a function not a simple variable. Equation (2.1.13) is more normally seen written in terms of the Lagrangian

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\boldsymbol{x}}(t)} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{x}(t)} = 0. \tag{2.1.14}$$

These are the **Euler-Lagrange** equations. They are the **equations of motion** or **eom**. Solutions of these equations give all the possible classical behaviours for this system. So, for a classical theory, the equations of motion contain all the physics. The equations are exactly those which are obtained using Newton's laws and principle of least action is an alternative or equivalent starting point to Newton's laws. So if our system described a mass on a spring, the equations of motion will be exactly those equations obtained from Newton's laws of motion,

$$\ddot{\boldsymbol{x}} = -k\boldsymbol{x} \tag{2.1.15}$$

The behaviour of a classical system is *always* one solution to the equation of motion. There are many different solutions, typically of different energies, momentum etc. The equation of motion are satisfied by solutions $x(t)$ which describe the possible behaviour of the system.

Different initial conditions will select which of these the system actually follows. Since it is classical, then these are deterministic equations, i.e. once initial conditions are given, the behaviour is completely fixed[1].

This principle of least action is also at the heart of the path integral approach to QFT pioneered by Feynman. This emphasises the close link between classical and quantum analysis often (but not always) present.

Equations of Motion for Fields

Classical fields are functions of space and time. There may be d of them, f_i $(i = 1, 2, \ldots d)$, and assume they take real values $f_i(x) \in \mathbb{R}$. Their classical behaviour is usually given in terms of a **Lagrangian density** $\mathcal{L}$, which is simply related to the Lagrangian L and action S through

$$S = \int dt\, L, \quad L = \int d^3x\, \mathcal{L} \tag{2.1.16}$$

The equations of motion are again defined uniquely by the condition that the action S defined as

$$S \;=\; \int d^4x\, \mathcal{L}(f_i, \partial_\mu f_i) \tag{2.1.17}$$

should be extremised. Consider a small (infinitesimal) variation $\delta f_i(x)$ from a classical solution $\bar{f}_i(x)$ so

$$0 = \delta S \;=\; \int d^4x\, \delta\mathcal{L} = \int d^4x\, \left[\frac{\partial\mathcal{L}}{\partial f_i}\delta f_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu f_i)}\delta(\partial_\mu f_i)\right] \tag{2.1.18}$$

$$= \int d^4x\, \left[\left(\frac{\partial\mathcal{L}}{\partial f_i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu f_i)}\right)\right)\delta f_i + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu f_i)}\delta f_i\right)\right] \tag{2.1.19}$$

where this is to be evaluated at $f_i(x) = \bar{f}_i(x)$. The last term gives a contribution only on the boundary but we will assume such contributions are zero. Thus for action to be extremised for any and all variations δf_i we see that

$$\frac{\partial\mathcal{L}}{\partial f_i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu f_i)}\right) = 0 \quad \forall\, i \in 1, 2, \ldots, d. \tag{2.1.20}$$

These are again called the **Euler-Lagrange** field equations.

To make contact with the Hamiltonian, we define the canonical momentum $\pi_i(x)$ as before via

$$\pi_i(x) = \frac{\partial\mathcal{L}}{\partial(\dot{f}_i(x))} \tag{2.1.21}$$

Then the Lagrangian and Hamiltonian are related through the same Legendre transformation

$$\mathcal{H} = \pi_i \dot{f}_i - \mathcal{L}, \qquad H(f_i, \pi_i) := \int d^3x\, \mathcal{H} \tag{2.1.22}$$

Again it is the Hamiltonian and not the Lagrangian which is related to the total energy of the system of fields.

[1]We are leaving aside any practicabilities and any worries about classical chaos.

Particles and Degrees of Freedom

So far we have worked with a single generic field with many components $f_i(x)$. The discussion is so general that in principle different components could be describing different types of particle. However, it is much more natural to use a different symbol for every type of particle and so to work with a collection of different fields, say $A^\mu(x)(\mu = 0, 1, 2, 3)$ for photons, $\psi^\alpha(x)(\alpha = 0, 1, 2, 3)$ for a electrons/positrons, $\pi_i(x)(i = 1, 2, 3)$ for the three pions. In chapter 3 this split into different types of field will be done in a more precise definition manner based on symmetry considerations.

However, as these examples show, we will still encounter fields representing one type of particle yet which have several components. This is essential because sometimes what we call a single type of particle, a photon, an electron and which we represent with a single field, actually comes in several variants, known as **degrees of freedom**. There is one degree of freedom for every distinct way that a particle can transport energy and momentum along any one given direction in space.

In terms of the field theory each degree of freedom requires its own real field to describe it. Thus if a scalar particle or field has d degrees of freedom, then we normally use a d-component real vector of space-time variable functions, $\phi_i(x)(x = 1, 2, \ldots, d)^2$. For simple fields, the number of degrees of freedom is indeed going to be the same as the number of independent real functions of space-time needed to describe the field.

The pions illustrate several of these points. Perhaps we tend to think of the three pions as distinct particles, with different electric charges. We might be tempted to describe the presence of each type of pion using its own real field. However all three have closely related properties, in particular the same mass (this is relevant only if we ignore the small mass differences). It turns out to be advantageous to always use a single field with several components to describe closely related particles. Thus in the case of pions we will need a three-component real field to describe these three pions, one real field for each distinct way energy and momentum can be transferred through the system. It is really a matter of choice whether we want to think of the π^+, π^0, π^- as separate particles, or as three degrees of freedom of a single pion particle. The former is probably how pions are usually discussed while the latter reflects the way they are described in QFT .

In relativistic contexts, it is always best to use a field that also describes both the modes of one particle and the modes of the corresponding anti-particles. The pion example shows this as the ϕ^+/π_- particle/anti-particle pair are described together in the three component field. The electron is usually described by the same field as the positron, though we normally think of them as distinct particles. In this case each are spin one-half particle with two distinct modes, spin up or down. Thus the electron field has four degrees of freedom in relativistic QFT , and describes all the electron/positron modes.

In both the electron and pion examples, the number of degrees of freedom matched the number of components in the field vector used to describe them. The photon illustrates one last problem. For a given direction of travel (the direction of the Poynting vector and the direction of energy transfer) a photon in a vacuum has two modes corresponding to the two possible independent polarisations of light. Thus a photon in vacuo therefore has two degrees of freedom.[3] However, the relativistic field theory for the photon is given in terms of a four-component real-field, A^μ, which has simple Lorentz transformation properties. As the massless photon has only two physical modes (the two polarisations) this field has two redundant, that is unphysical, components. One could, in principle, work with a simple two-component field for the photon but the Lorentz transformation properties are much more complicated. In practice

[2] Note that it is common to be lazy and use x rather than x^μ, for example in the arguments of functions.

[3] Interestingly it can have one additional longitudinal polarisation in some materials and we will see that when symmetry is 'broken' the photon like fields - gauge fields, have mass and have three degrees of freedom.

it is preferred to have simple Lorentz properties but to complicate the relationship between the fields and the physical modes of physical particles. In the case of electromagnetism, the inclusion of unphysical modes in the fields is the issue of **gauge invariance**.

The grouping of the different physical modes in the system will be made precisely when we start to use group theory to study symmetry in chapter 3. However, as we start to look at more advanced topics, we will see that symmetry can be hidden and there may be ways of grouping particles together, describing them with single fields, that don't reflect their obvious properties but reflects deeper underlying truths. The simple relationships outlined here become more complicated. However, the most important point is that whatever way we choose to group the physical modes, there must always be one real function per degree of freedom. Thus whenever we decide to change our mathematical representation of the physics, when we choose to work in a second set of fields, we must always make sure that the new fields contain the original real functions used for the degrees of freedom, no more, no less. Changing the way we describe the problem mathematically must not alter the physical content. Likewise, while a change of field definitions may highlight different aspects of the physics, it must never be more than a shuffling of the physical modes.

One final remark is to note that one can use a single complex function to represent two independent real functions $f_1(x) + if_2(x)$. For instance the Klein-Gordon equation with mass parameter m can be an equation of motion for one real field representing a single particle of mass m, or it can be an equation of motion for a complex field. In the latter case we are describing two scalar modes, both of mass m, and further that these two modes indicate that there is a particl/anti-particle pair. We will look at this relationship more precisely in chapter 4.

Vacuum energy of the unbroken theory

For unbroken symmetry, $\boldsymbol{\phi}$ is function to use to describe the physics, rather than $\eta(x) = \phi(x)+a$ or $\eta(x) = sh(\phi(x))$ or some other redefinition which has the same mathematical information. The fact that $\boldsymbol{\phi}$ as given is a good choice is clear because its Lagrangian, (2.3.2), satisfies all the criteria set out above, e.g. its a simple polynomial of the fields and derivatives, etc. etc. The classical Lagrangian in terms of ϕ (2.3.7) can reflect the real quantum physics, meaning that a study of the classical theory can be a good guide to essential aspects of the full QFT .

However, there is one last criterion, namely that $\boldsymbol{\phi} = 0$ be a lowest energy solution for the theory, so that the field $\boldsymbol{\phi}$ represents fluctuations around the lowest energy or **vacuum** energy solution. This is the case of **unbroken symmetry** and in the case of (2.3.2) occurs if $m^2 > 0, \lambda > 0$. It corresponds to having a vacuum expectation value in the full QFT for all components of the $\boldsymbol{\phi}$ equal to zero $\langle 0|\boldsymbol{\phi}_i(x)|0\rangle = 0$ (and therefore constant in space and time). We call this the **vacuum** solution for $\boldsymbol{\phi}$. If we rewrote the Lagrangian density of a Hamiltonian density (the latter is roughly kinetic plus potential energy, the former is just the difference) it is then clear that zero field is also the lowest energy value classically too *provided* $m^2 > 0, \lambda > 0$. It therefore makes sense to think of $\boldsymbol{\phi}$ being small and representing small quantum fluctuations about this vacuum solution. Then, provided λ is small, terms $O(\phi^3)$ or $O(\phi^4)$ etc are going to be very small and won't alter the qualitative picture given by the larger $O(\boldsymbol{\phi}^2)$ terms, *provided* the **coupling constant** λ is small too.

Spin models and field theories

Another useful analogy starts with quantum spin models of the type often discussed in quantum mechanics courses. A typical example, used as a simple model of ferromagnetism in iron and other materials, considers the spin of a single electron at each lattice site of a material. Classically the spin at lattice point a might be represented as a vector $\boldsymbol{S}(a)$ of fixed length

$|\boldsymbol{S}| = 1/2$ but free to point in any direction in space. Each electron a interacts with its nearest neighbours, $b \in \mathrm{nn}_a$, with the potential energy proportional to vector product of the spins, so that the Hamiltonian takes the form

$$H = \sum_a \sum_{b \in \mathrm{nn}_a} g S_j(a) S_j b, \quad j = 1, 2, \ldots d \tag{2.1.23}$$

One can consider variations of this model. Perhaps the spin is confined in one direction $d = 1$ so only one component S_1 is involved. This model is the Ising model. Perhaps we add some simple quantum mechanics and allow the size of the spin vector to vary, say $S_1 = |S|, |S| - 1, \ldots, -|S|$. Finally we could let the spins take continuous values $S_1 \in \mathbb{R}$ — a continuous spin model. In this case the theory becomes equivalent to a theory of a scalar field theory $\phi_i(a) \equiv S_i(a)$ on a Euclidean space-time lattice, a common approximation used in practical QFT calculations.

However, one should be *every* careful with this spin/field analogy. In this case it is not obvious what the particle side of the picture should be — the scalar field represents a spin zero particle in QFT yet we started with objects in our spin models of higher spins than that. The length of the scalar field vector $\boldsymbol{\phi}$ is not to be thought of as a physical spin as in the original model with $\boldsymbol{S}$. Following on from this we see that the spins $\boldsymbol{S}$ are limited to live in at most the three-dimensional space of our physical experience. However, in the 'continuous spin' version, the field $\boldsymbol{\phi}$ need not be pointing in any direction of real space. For instance we could have many scalar fields, many scalar particles, each field component with a value at each space-time point for each scalar particle, just as the density of particles varies from point to point. In this case the index of the fields, d, runs over the number of scalar particles *not* the space-time dimension. These fields live in a different space, called the **internal space**. At each space-time point we have a new label or coordinate but this has nothing to do with the space (or space-time) of the model. The axes of the internal space are labelled by different types of particle, e.g. we would need a seven dimensional space to describe a system of pions, protons, neutrons and their anti-particles. As we will see, we tend to split these internal spaces up into subspaces, each for the particles of closely related properties. For instance we might focus on the three-dimensional subspace needed for the three pions. These three scalar fields would be better represented as three components of a scalar field vector $\boldsymbol{\phi}$ living in a three dimensional internal space.

2.2 From Classical to Quantum Theory

In a quantum system, the classical solution for a field is only one of the most probable values for a field but as such it can still play a central role in QFT . Further the quantum expectation value of the classical eom is also satisfied in QFT even though many field configurations other than the classical field configurations contribute. It is therefore worthwhile studying the classical eom and their solutions as many results will remain true in the full QFT and as such they often form a starting point for a study in the full QFT .

The Lagrangian density $\mathcal{L}$ by itself gives us a complete description of a classical theory of the dynamics of fields $\boldsymbol{\phi}$, e.g. we can derive classical equations of motion from it. From this view point it is not obvious why we should associate the d fields $\phi_j(x)$, which are just some real functions of space-time variables x, with particles — point like objects characterised by certain definite numbers such as energy, momentum, charge etc. In fact this link is *only* possible if we look at the full QFT . Only QFT can show us why m is related to the mass of some particle, λ a measure of the strength of interactions between particles.

However there is a good rationale for studying in this book the classical field theories rather than the full QFT . What we are implicitly assuming in this course is that the full quantum solution, $\bar{f}_{\mathrm{q}}(x)$, is of the form

$$\bar{f}_{\mathrm{q}}(x) = \bar{f}_{\mathsf{Cl}}(x) + \delta \bar{f}(x) \tag{2.2.1}$$

where the classical field, $\bar{f}_{\mathsf{Cl}}(x)$, is the solution to the classical equations of motion, and $\delta\bar{f}(x)$ is a small quantum correction, i.e $\sim O(\hbar)$. If this is true then when we study classical field theory and its symmetries we will get a good qualitative picture of the physics of the full quantum theory as the quantum effects generate only 'small' changes. Of course this is merely hope at this stage. The full quantum theory must be studied to see if its valid, or at least classical predictions compared against experimental data. In both cases it is known that there are many problems where this classical approximation does at least give use a good overview and its a genuinely useful tool.

At the same time there are several situations where the classical picture is misleading and the quantum solution may not always be obtained by a small-order ($O(\hbar)$) correction to the classical solution. These include

- Bound states — such as the mesons and baryons in QCD (quantum chromodynamics), the theory of strong nuclear forces. Their existence can not be checked directly from the classical analysis, e.g. we can not calculate their masses. However, the symmetry in classical theory is preserved in the quantum theory and from this we can predict some of the properties of the bound states, such as their conserved charges.
- Phase transitions, where thermal fluctuations balance quantum fluctuations.
- Anomalies, where quantum effects destroy classical symmetry.

The last point is particularly important. Symmetry is at the core of what we are doing in this book, and there is no way in the classical theory to see if the quantum theory has all the symmetry of the classical theory. Luckily, it is relatively simple in QFT to see if anomalies are present or not.

We will need to draw on quantum theory for a few aspects. The most important is to make a connection between particles and fields; e.g.

$$\text{pions } \pi \longleftrightarrow \pi(x), \qquad \text{photons } \gamma \longleftrightarrow A^\mu(x) \tag{2.2.2}$$

One way of thinking of particle/field duality is to imagine that peaks in a field can be thought of as corresponding to the positions of a particles. This follows, assuming that the zero field value is the lowest energy state, because a peak in a field represents at least a localised lump of energy, one property at least of what we call a particle.

The propagation of quanta (packets of conserved quantities) is described in QFT by *propagators*. A **free propagator**, i.e. no interactions, is a solution to the equations of motion of the quadratic part of the Lagrangian. The quadratic requirement implies no terms of three or more fields. The full solution to QFT is usually built on top of the free non-interacting theory and its solutions, so we shall focus on the quadratic part of the Lagrangian.

Degrees of Freedom

Another better way of saying that a field has a N independent ways of moving energy in any given direction is to say it represents N particles. To make the link with particles, one really needs to turn to QFT and in particular talk about how many different types of annihilation and creation operators are needed to build the quantum version of the field. A field made up of N independent real fields require N distinct types of annihilation/creation operator pairs $\hat{a}_i(\boldsymbol{k}), \hat{a}_i^\dagger(\boldsymbol{k})$, $i = 1, 2, \ldots, N$ where $\boldsymbol{k}$ is the momentum label. Distinct here means that for a given energy and momentum, there are different operators, and we distinguish them with the label i. We do not count here the fact that there are different operators for each momentum k, afterall the same particle can move at different momenta. More precisely, distinct annihilation and creation operators in QFT means they commute with each other, even for the same momentum k,

$$[\hat{a}_i(\boldsymbol{k}), \hat{a}_j^\dagger(\boldsymbol{k}')] \propto \delta_{ij}\delta^3(\boldsymbol{k}-\boldsymbol{k}'). \tag{2.2.3}$$

Thus this field represents d distinct particles which can run around at any momentum k.

There are many different ways mathematically of rewriting the field $\boldsymbol{\phi}(x)$ but we will always need d-independent real functions to describe it fully. It can be quite hard to check that an given redefinition does indeed allow you to describe *all* possible values of $\boldsymbol{\phi}(x)$. For instance trying to use the d fields $\eta_i(x) := \exp(\phi_i(x)\boldsymbol{\phi}^2)$ will be confusing and complicated. Any appropriate redefinition can not change the physics, but it can help reveal (usually it will just obscure) the true physics. However, we will almost always work with appropriate and simple field definitions and such field redefinitions will not be a concern. An exception is the use of the unitary gauge in chapter 7.

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However, there is one last criterion, namely that $\boldsymbol{\phi} = 0$ be a lowest energy solution for the theory, so that the field $\boldsymbol{\phi}$ represents fluctuations around the lowest energy or **vacuum** energy solution. This is the case of **unbroken symmetry** and in the case of (2.3.2) occurs if $m^2 > 0, \lambda > 0$. It corresponds to having a vacuum expectation value in the full QFT for all components of the $\boldsymbol{\phi}$ equal to zero $\langle 0|\boldsymbol{\phi}_i(x)|0\rangle = 0$ (and therefore constant in space and time). We call this the **vacuum** solution for $\boldsymbol{\phi}$. If we rewrote the Lagrangian density of a Hamiltonian density (the latter is roughly kinetic plus potential energy, the former is just the difference) it is then clear that zero field is also the lowest energy value classically too *provided* $m^2 > 0, \lambda > 0$. It therefore makes sense to think of $\boldsymbol{\phi}$ being small and representing small quantum fluctuations about this vacuum solution. Then, provided λ is small, terms $O(\phi^3)$ or $O(\phi^4)$ etc are going to be very small and won't alter the qualitative picture given by the larger $O(\boldsymbol{\phi}^2)$ terms, *provided* the **coupling constant** λ is small too.

2.3 Typical Lagrangians in Particle Physics

The form of the Lagrangian needed to describe relativistic particles are very specific, the form specified by symmetries such as space-time symmetries, and by quantum ideas such as renormalisability. Different parts are related to different types of physical properties of the particles being described. As a result the same terms appear again and again and are referred to using standard terminology. This we will outline below as a full understanding requires a proper QFT treatment. We will also focus only on scalar fields in 3+1 dimensions. Similar ideas apply to other fields in four-dimensions, and we will note these as these are introduced in later sections, gauge fields in section 6 and fermions in section 8. Generalisations to other dimensions or to non-relativistic theories are also straight forward.

Fields of Particles of similar properties

When particles have the same spin and mass and they interact at relatives strengths which are simple multiples of each other, then there is some deeper relationship between the particles. For instance the three pions are all scalar particles of essentially the same mass, their EM charges are $+1, 0$ and -1 so their EM interactions are closely related, and one finds that their strong and weak interactions are closely related. When particles have such similar properties, there is a **symmetry** relating these particles and there is a mathematical **group** associated with this theory. A major part of this book is to see how to express these similarities between particles in terms of Lagrangians and fields. At this stage, all we need to note is that when particles share common features in this way, it will turn out to be extremely convenient to put the fields associated with all the particles into a single *vector* of fields. For instance, for the pions we might define a three component field

$$\mathbf{\Phi}(x) = \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \\ \Phi_3(x) \end{pmatrix} \tag{2.3.1}$$

where $\Phi^1(x)$ could be the field describing the π^+ particle, $\Phi^2(x)$ might be for the π^0 and the π^- is linked to the third component $\Phi^3(x)$.

Free and Interacting parts

The first division and most important division of any Lagrangian is into free and interacting terms. The **free field** part of any Lagrangian is that part made up of terms which are at most quadratic in the fields, $O(F^2)$. The remaining cubic and higher terms are called **interaction terms**. We will see one reason for these names in a moment. In this case then we have

$$\mathcal{L} \;=\; \underbrace{\frac{1}{2}(\partial_\mu \phi).(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2}_{\text{Free field terms}} \;-\; \underbrace{\lambda\phi^4}_{\text{Interaction term}} \tag{2.3.2}$$

The **e.o.m.** (equation of motion) is[4]

$$\partial^\mu \partial_\mu \phi + m^2 \phi + 4\lambda\phi^3 = 0 \tag{2.3.3}$$

The terms of two or less fields in the Lagrangian appear as linear terms in the eom. These represent propagating wave solutions. Since one can add two solutions to a linear equation to get another solution, these represent waves that pass through each other without noticing other solutions. They are non-interacting terms and represent what we would identify as an isolated partilce or wave in a detector. Terms which have more than two fields in the Lagrangian are interactions as these lead to non-linear terms in the eom. Such non-linear terms mean that the sum of two solutions to the linear parts, i.e. non-interacting proagating waves, are no longer guaranteed to be a solution. That is the non-linear terms distrupt the solutions to the linear part, i.e. the non-linearity is an interaction.

The naming is clearer in Fourier space where the equation of motion becomes

$$(k^2 - m^2)\phi(k) \;=\; 4\lambda \sum_{k_1,k_2} \phi(k_1)\phi(k_2)\phi(k - k_1 - k_2) \tag{2.3.4}$$

$$\phi(k) \;:=\; \int d^4x \, e^{ikx}\phi(x) \tag{2.3.5}$$

We see that it is only the term proportional to lambda which mixes the different energies and three-momenta. In a particle language such mixing is caused by the scalar particles interacting. The term with coefficient λ controls the mixing or *interactions* of the different components of the fields, hence the name.

[4] **EFS 2.3.1**: Derive the eom of (2.3.3) for the single real self-interacting scalar field ϕ of (2.3.2).

There are other divisions of the Lagrangian which will be encountered, namely

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi).(\partial^\mu \phi)}_{\text{Kinetic Terms}} - \underbrace{V(\phi)}_{\text{Potential}} , \tag{2.3.6}$$

$$V(\phi) := \underbrace{\frac{1}{2} m^2 \phi^2}_{\text{Mass term}} + \underbrace{\lambda \phi^4}_{\text{Quartic interaction term}} \tag{2.3.7}$$

These are discussed below.

2.3.1 Free Fields

The terms which are no more than quadratic in the fields are called the **free field terms**. These describe the propagation of particles (i.e. lumps of conserved measurable quantities such as energy, momentum, spin, charges) without any interactions. This can be seen in the classical theory from the eom in fourier space (2.3.4). For each k, the free field eom ($\lambda = 0$) is satisfied only is $k^2 = m^2$. Thus each $\phi(k)$ describes some sort of amplitude of a wave of momentum $\boldsymbol{k}$ and energy $\sqrt{(\boldsymbol{k}^2 + m^2)}$.

More importantly for QFT , with zero interactions one can build on these classical solutions **QFT** (Quantum Field Theory) exactly in any dimensions if these are the only terms present. With interactions present, there are virtually no exactly solvable QFT theories[5]. As a result, free theories play a central role in all QFT approximate solutions to the physically interacting theories.

First let us split the free part of the Lagrangian into two more pieces

$$\mathcal{L}_0 = \underbrace{\frac{1}{2}(\partial_\mu \phi).(\partial^\mu \phi)}_{\text{Kinetic Terms}} - \underbrace{\frac{1}{2} m^2 \phi^2}_{\text{Mass terms}} \tag{2.3.8}$$

Note that a subscript 0 is often (but not always) used to denote the free part of the Lagrangian[6].

Kinetic vs. Potential Terms

The $\frac{1}{2}\partial_\mu \boldsymbol{\phi}.\partial^\mu \boldsymbol{\phi}$ derivative terms are the **kinetic terms** as no propagation of energy, momentum, charge etc. occurs without these terms. They have two derivatives and are quadratic in the fields. The remaining terms, with no derivatives only in simple cases, are called the potential terms.

The naming is best understood by looking at the Hamiltonian density $\mathcal{H}$ defined to be

$$\mathcal{H} := \Pi_i \frac{\partial \phi_i}{\partial t} - \mathcal{L} \tag{2.3.9}$$

$$\Pi_i := \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \tag{2.3.10}$$

in terms of the canonical momenta Π_i. For our O(N) example we have

$$\mathcal{H} := \frac{1}{2}\Pi_i \Pi_i + \frac{1}{2}\nabla_x \boldsymbol{\phi}_i \nabla_x \boldsymbol{\phi}_i + \underbrace{V(\boldsymbol{\phi}.\boldsymbol{\phi})}_{\text{Potential}} , \tag{2.3.11}$$

The Hamiltonian is the total energy, kinetic plus potential energy. The derivative terms are the kinetic energy terms, e.g. the $(\nabla \boldsymbol{\phi})^2$ is the momentum squared term which in the non-relativistic limit would be the usual kinetic term $p^2/2m = -(i\hbar\nabla)^2/2m$. The rest, here V, must be the potential energy terms.

[5] Some special examples exist in 1+1 dimensions where conformal symmetry plays a vital role in allowing one to solve these theories.

[6] When discussing renormalisation in QFT , it is used to denote 'bare' (usually infinite) quantities.

Mass Terms

The quadratic term with no derivatives $\frac{1}{2}m^2\phi^2$ is the **mass term**. If there are only kinetic and mass terms for the fields ϕ in the Lagrangian density, then *and only then* may one interpret m as the mass of the particle described by the real field $\phi(x)$. This m is a classical mass and in the full quantum theory, the value of the physical mass will be different as the mass gets **renormalised** in the quantum theory.

Free Field Solutions

The free field case, when all interaction terms are zero ($\lambda = 0$ in (2.3.2)), is exactly solvable in any number of space-time dimensions. In the case of our scalar field example here, the equation of motion we obtain for the free field is called the **Klein-Gordon** equation

$$\partial^\mu \partial_\mu \phi(x) + m^2 \phi(x) = 0 \qquad \textbf{Klein-Gordon equation} \tag{2.3.12}$$

In fact we can always solve the free field case exactly, both in quantum and classical theories and for any fields (in flat space-time). For this mathematical reason, *almost all* analytic solutions for field theories, classical or quantum, are built around free field solutions. The solutions are best seen by working with the Fourier transform of the fields $\phi(k)$, which gives for the free part

$$(k^\mu k_\mu - m^2)\phi_j(k) = 0 \Rightarrow \phi_j(k) \sim \exp\{\pm i\omega t\} \exp\{\pm i\boldsymbol{k}.\boldsymbol{x}\} \tag{2.3.13}$$

where

$$k_0 = \pm\omega, \quad \omega = |\sqrt{\boldsymbol{k}^2 + m^2}|. \tag{2.3.14}$$

Boundary conditions decide precisely what combinations of these exponentials form the solution. This tells us several things. First the free field solution $\phi(x)$ is a combination of packets of definite energy ω and three-momentum $\boldsymbol{k}$. Therefore it makes sense to represent (approximate) particles seen in detectors by free fields carrying the same energy and momentum as the particles. From this analogy it is also clear from (2.3.14) that m is then to be linked with the mass of the particle represented by these classical fields, since (2.3.14) is the standard energy momentum relationship for a relativistic particle..

2.3.2 Interacting theories

When we have $\lambda \neq 0$ we have an interacting theory, not just a free theory. The interactions spoil the simple free field solution. One must resort to other techniques to find a solution. For small λ, **perturbation theory** can be used to produce *approximate* solutions giving ϕ as a power series in λ. For large λ **non-perturbative** methods[7] must be used, such as straight numerical approximations.

To make sense, we want the **coupling constant** λ to be positive so that the potential energy density is **bounded below**, it has an absolute minimum. Thinking of $\phi_j(x)$ as small (quantum fluctuations) then, provided λ is small, cubic, quartic and higher[8] terms - $O(\phi^3)$ or $O(\phi^4)$, are going to be very small and won't alter the qualitative picture given by the larger quadratic $O(\phi^2)$ terms, *provided* the **coupling constant** λ is small too. The terms $O(\phi^3)$ or higher are usually called the **interaction terms**[9].

[7] In field theory, "perturbation theory" invariably means expansions in these coupling constants coefficients λ. Most analytic "non-perturbative solutions" are also perturbation series but in some other parameter.

[8] In four space-time dimensions, a renormalisable (finite) *quantum* theory will not have anything higher than quadratic field terms. Even though we are studying classical theories here, we will tend to stick to this limitation.

[9] More advanced considerations - renormalisation, or going beyond weak coupling perturbation theory ($\lambda \gtrsim 1$) - leads to a weakening of this simple division of what is an 'interaction' and what is a 'free' term. It is usually only involves adding quadratic or linear field terms to the interaction part, the cubic and quartic terms stay in the interaction part almost always.

2.3.3 Other terms no higher than Quadratic

Initially, the Lagrangians will be of the form given in (2.3.2). However, other terms will be encountered, both in symmetry breaking and when sources are introduced. It is important to know how to deal with terms which are linear or quadratic as they spoil the interpretation of the m coefficient as a mass. Terms which are cubic or higher will invariably be treated as interactions.

Linear Terms

There must be *no linear term* in the fields if one is to interpret the quadratic terms in the way suggested above, namely propagation and mass terms for particles. If you have them, get rid of them.

The QFT remains completely solvable with such linear terms. For instance an appropriate redefinition of the fields using a constant shift, $\tilde{\phi}(x) = \phi(x) + v$ for some appropriate constant v, can always be found which removes linear terms, but it will change the coefficients of the other terms both quadratic (mass) and any the interactions too. This will be used in **symmetry breaking** where v will be linked to the **vacuum** state (lowest energy state) changing from that which is empty of real particles (though full of virtual particles in the quantum theory) to one full of real particles.

Another area where such linear terms are encountered will be in the full quantum theory. There one will soon encounter a term like $\boldsymbol{j}(x).\phi(x)$ term, where $\boldsymbol{j}(x)$ are N external **sources**. I will mention it in this section as it is strictly less than quadratic and it does not alter the solubility of the quadratic QFT as mentioned above. However this linear $j\phi$ term is best thought of as representing an interaction between particles described by the ϕ field and external 'classical' sources $\boldsymbol{j}$. Something *like* the three different components of a magnetic field applied by experimentalists, say to bend charged particles, but this specific example requires a more sophisticated term to be added.

Note though that a $\boldsymbol{j}.\phi$ term will usually be encountered where it is to be thought of as *unphysical*, added only for mathematical convenience and has no physical meaning. Clearly in this case one only gets sensible physics if $\boldsymbol{j} = 0$. The mathematical trick is to take derivatives with respect to the components of the current $\boldsymbol{j}_i(x)$ (functional derivatives technically) and then to set $\boldsymbol{j} = 0$. It allows one to generate useful quantities in a mathematically concise way, i.e. its a trick to summarise all QFT as a generating functional $Z[j]$[10]. This is just the same idea as generating functions in mathematics, simple functions of a 'dummy' variable whose multiple derivatives with respect to the dummy variable, followed by setting the dummy variable to zero, generate series of well known functions e.g. Bessel's functions.

Quadratic mixing terms

There must also be *no mixing* of different field components in one quadratic term if one is to interpret the mass and kinetic terms as those describing the propagation of real physical particles of mass m. So no $\phi_i(x)\phi_j(x)$ $(i \neq j)$ terms if the propagation terms are to be interpreted as I have said above.

Should you encounter such mixing terms, then a redefinition of fields through a linear unitary transformation between the fields will always remove such terms but again this will change any cubic and quartic interactions, usually making them more complicated.

[10]Usually its called Z but not always!

2.3.4 Other restrictions on Lagrangians

When constructing a classical Lagrangian to be used for QFT , it is common to restrict the choice of terms further, reflecting deeper ideas from the quantum physics.

Real Lagrangian

Actions and Lagrangians ought to be real. If they are not the simple probabilistic interpretation of the action S though $\exp\{-(i/\hbar)S\}$, e.g. in the path integral, fails in QFT . It indicates an instability in the QFT .

Renormalisability

In QFT the famous ultra-violet infinities, which plague all these theories, can be removed systematically in perturbation theory to leave finite answers for physical quantities provided that all coupling constants (constant coefficients of terms in the Lagrangian) have positive or no dimensions, $[M]^n, n \geq 0$,[11] in natural units. The scalar field has natural units of $M^{d/2-1}$ in d space-time dimensions[12]

Thus in $d = 4$ dimensions, one is allowed a $g\phi^3$ type term in scalar field theories by this argument, but a $g\phi^6$ term is not allowed.[13]

By default, you should always try to construct a Lagrangian which is renormalisable, or at least highlight the fact that your Lagrangian is not renormalisable. Non-renormalisable theories may can give good descriptions of low energy processes and, while not fundamental, they have many practical applications.

Locality

A key requirement in most QFT , is that the physics is local. That is physics at one space-time point is only effected by information in the neighbourhood that point. This is equivalent to demanding that each term is a polynomial of finite order in derivatives. Thus the kinetic term is a second order polynomial in derivatives ∂^μ, but a term like $\ln[M^4 + (\partial^\mu\phi)(\partial_\mu\phi)]$ is not (think of the expansion of the logarithm, it has terms of an infinite number of derivatives requiring information from an infinite distance from one point if we are to calculate them. A term like $[(\partial^\mu\phi)(\partial_\mu\phi)]^2$ is allowed by this rule[14].

Example 1 Single real scalar field
$\phi(x) \in \mathbb{R}$ describes a single scalar (spin zero) particle which is its own anti-particle such as the π^0. In 3+1 dimensions the most general renormalisable Lagrangian density for $\phi \in \mathbb{R}$ is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi(x))(\partial^\mu\phi(x)) - \frac{1}{2}m^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x) \equiv \mathcal{L}_0 - \frac{\lambda}{4!}\phi^4(x) \qquad (2.3.15)$$

We can make the following points:

(i) $\mathcal{L}$ is *real* if $\phi(x), m^2, \lambda \in \mathbb{R}$. If we do not limit our variables to be real then we would have a complex action and it would represent an unstable theory;

[11] M indicates some appropriate mass scale.

[12] **EFS 2.3.2**: By assuming that the kinetic term is always of the same form, prove that the scalar field has natural units of $M^{d/2-1}$ in d space-time dimensions. Hence deduce that form of the mass term is always $m^2\phi^2$ for a scalar field in any dimension. Why is $1+1$ dimensional space-time special?

[13] **EFS 2.3.3**: Which is the largest power of scalar fields allowed in any term in (a) $d = 3$, (b) $d = 6$ dimensions if the theory is to be renormalisable?

[14] **EFS 2.3.4**: Is this term ever renormalisable?

(ii) The first two terms are the terms quadratic in the fields. They form free part of the Lagrangian, $\mathcal{L}_0$. By themselves they would give the Klein-Gordon equation of motion.

(iii) The first component is called the kinetic term, even though it contains both $\dot{\phi}$ and $\nabla\phi$ expressions. Compare to the classical case where kinetic energy is only linked to time derivatives, e.g. $\frac{1}{2}m\dot{x}^2$. Without this term, the quanta do not propagate.

(iv) The m coefficient is the mass of the scalar particle, *provided* the quadratic terms have the standard form given in (2.3.15)

(v) Three or more fields per term are covered in the interaction terms. Here λ is a coupling constant, a measure of interaction strength, (cross-section $\propto \lambda^2$).

(vi) For the interaction term λ you will see in the texts, variously: λ, $\frac{\lambda}{4}$ or even $\frac{\lambda}{4!}$, as in our real case, (2.3.15). The precise term used is not important as long as one is consistent within the problem.

(vii) **renormalisability** can remove all the infinities of quantum field theory, provided all coefficients have units $[M]^n$, $n \geq 0$. This implies there are no $g\phi^6$ terms in equation (2.3.15).

(viii) $g\phi^3$ *is* allowed by these rules; however, we choose to leave it out for reasons of

- symmetry: $\phi \longleftrightarrow -\phi$;
- simplicity

(ix) $\mathcal{L}$ is a **Lorentz scalar**;

(x) this describes *one degree of freedom*: one distinct particle mode. What this means in quantum field theory is that $\hat{\phi}$ is described with *one* pair of annihilation/creation operators

$$\hat{a}_{\boldsymbol{k}} \qquad \hat{a}^{\dagger}_{\boldsymbol{k}} \tag{2.3.16}$$

That is, this particle is its own anti-particle;

(xi) if we wish to interpret m in equation (2.3.15) as mass, the first two terms must be quadratic, i.e. *free*. Thus *no linear terms*, such as $v(\phi(x))^1$, are allowed. Removal of linear terms is quite easy using a change of variables such as

$$\phi(x) = \eta^{(x)} + c \tag{2.3.17}$$

where c is chosen so that there are no $\left(\eta^{(x)}\right)^1$ terms;

(xii) this Lagrangian is *local* (in space-time), i.e. the behaviour of the field at x^μ depends upon its value throughout a small neighbourhood of x^μ. Thus $\phi(x)$ is only affected by field values an infinitesimal distance away from x^μ, and therefore only finite powers of ∂_μ are allowed.
We note that without this assumption, one can run into severe problems with, for example, special relativity and quantum gravity.

□

In practice, we will always want to reduce our problem to the quadratic form in (2.3.15). In fact, every formula can be written in this form. However, it is quite common to use complex fields to describe a problem.

Example 2 Complex Scalar Field
We could write

$$\Phi(x) = \frac{1}{\sqrt{2}}\big(\phi_1(x) + i\phi_2(x)\big) \in \mathbb{C} \tag{2.3.18}$$

and analyse $\phi_1, \phi_2 \in \mathbb{R}$ as above. However, it is common to work with Φ (for example, Φ can be an eigenstate of charge where the $\phi_i \in \mathbb{R}$ can not). We therefore need to obtain the canonical form

$$\mathcal{L} = (\partial_\mu \Phi(x))^\dagger (\partial^\mu \Phi(x)) - m^2 \Phi^\dagger(x)\Phi(x) - \frac{\lambda}{4}\left(\Phi^\dagger(x)\Phi(x)\right)^2 \tag{2.3.19}$$

where the third term is chosen to give $U(1)$ symmetry (that is, it gives a phase invariance). Once again, we note that the first two terms must be quadratic for m to be interpretable as a mass. This prohibits mathematically viable things like $\left(\Phi^\dagger\right)^2 + (\Phi)^2$, which *are* real scalars, but not particularly useful fields.

(1) The particle content of one $\Phi(x)$ field is *two particles* of equal mass m, thus giving rise to two degrees of freedom. These are a particle/anti-particle pair, e.g. π^+ and π^-.[15]
In quantum field theory, two pairs of annihilation/creation operators, $\hat{a}^\dagger$, $\hat{a}$ and $\hat{b}^\dagger$, $\hat{b}$ are needed to describe the quantum field $\hat{\Phi}$: a pair for each *distinct* particle, e.g.:

$$\left[\hat{a}, \hat{a}^\dagger\right] = i\hbar \qquad \left[\hat{b}, \hat{b}^\dagger\right] = i\hbar \qquad \left[\hat{a}, \hat{b}^\dagger\right] = 0 \tag{2.3.20}$$

(2) The factors of $\frac{1}{2}$ or $\frac{1}{\sqrt{2}}$, *et cetera*, come from standard required normalisation of commutation relations in quantum field theory.

The relationship between real and complex scalar fields will be investigated in more detail and looking at the symmetry aspects in section **??**. □

Example 3 Several components
Consider $\boldsymbol{\phi}(x)$, equivalently $\phi_i(x)$, $i = 1, \ldots, d$; c.f.

$\psi^\alpha(x)$	Dirac spinor	$\alpha = 0,1,2,3$
$A^\mu(x)$	EM gauge field[16]	$\mu = 0,1,2,3$

So to find the mass of such fields we must reduce $\mathcal{L}$ to the form[17]

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \boldsymbol{\phi}(x)\right)\left(\partial^\mu \boldsymbol{\phi}(x)\right) - \frac{1}{2}\sum_{j=1}^{d} m_j^2 \phi_j(x)\phi_j(x) + O\left(\phi^3\right) \tag{2.3.22}$$

where the $O\left(\phi^3\right)$ terms represent interactions. We usually work with

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \boldsymbol{\phi}\right)\left(\partial^\mu \boldsymbol{\phi}\right) - \frac{1}{2}\boldsymbol{\phi}(x)(\mathsf{M}^2)\boldsymbol{\phi}(x) + \text{interactions} \tag{2.3.23}$$

[15] Contrast this to the case before, where the particle is its own anti-particle.
[16] This has four components but two redundancies due to the polarisations of the two underlying fields. This is not important here, but may arise in quantum field theory.
[17] Note that since the vector space of all $\boldsymbol{\phi}(x)$ is, in essence, an artifice, the use of up and down indices on $\phi_i(x)$ is not necessary; although, it *does* become necessary when working with the group representation. The metric on the d-dimensional vector space of $\boldsymbol{\phi}$ is simply

$$g^{ij} = \operatorname{diag}(+1, \ldots, +1) \tag{2.3.21}$$

Here the constant matrix M^2 is real and can be chosen to be symmetric, so that $(\mathsf{M}^2)^\mathsf{T} = (\mathsf{M}^2)$. From some basic results of **quadratic forms** we know there exists an orthogonal matrix O which diagonalises M, i.e.

$$\mathsf{O} \cdot \mathsf{O}^\mathsf{T} = \mathbb{1} \qquad \mathsf{O}\left(\mathsf{M}^2\right)\mathsf{O}^\mathsf{T} = \mathrm{diag}\left(\lambda_1, \ldots\right) \tag{2.3.24}$$

where the λ_i are eigenvalues of M^2.
Thus, if we define $\boldsymbol{\eta}(x)$ such that

$$\boldsymbol{\eta}(x) = \mathsf{O}\boldsymbol{\phi}(x) \tag{2.3.25}$$

i.e. a linear redefinition of $\boldsymbol{\phi}$, then the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \boldsymbol{\eta}\right)\left(\partial^\mu \boldsymbol{\eta}\right) - \frac{1}{2}\mu_j \lambda_j \eta_j \eta_j + \text{interactions} \tag{2.3.26}$$

where the eigenvalues of M^2 are $(\text{mass})^2$ parameters for the η_j fields. Now, all quadratic mixing terms, e.g. $\phi_1\phi_2$, are gone in the $\eta(x)$ variables.
Special case $\mathsf{M}^2 = m^2\mathbb{1}$, so that $\frac{1}{2}m^2\boldsymbol{\phi}\cdot\boldsymbol{\phi}$ gives rise to $O(d)$ symmetry.

□