The Feynman Propagator and Cauchy’s Theorem

Tim Evans
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Feynman Propagator

We would like to derive an integral expression for the Feynman propagator

$$\Delta_F(x − y) = \langle 0 | T\phi(x)\phi(y) | 0 \rangle,$$

where the time-ordered product is defined by

$$T\phi(x)\phi(y) = \theta(x^0 − y^0)\phi(x)\phi(y) + \theta(y^0 − x^0)\phi(y)\phi(x),$$

with the theta function \(\theta(t)\) given by \(\theta(t) = 1\) if \(t > 0\) and \(\theta(t) = 0\) if \(t < 0\). Using the field expansion

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}),$$

it is straightforward to show that

$$\Delta_F(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[ \theta(x^0)e^{-ip\cdot x} + \theta(-x^0)e^{ip\cdot x} \right],$$

where \(p^\mu = (\omega_p, p)\) and \(p \cdot x = p^\mu x_\mu = \omega_p t - p \cdot x\) in the exponentials.

Rewriting using Cauchy’s theorem

We now want to use a result from complex analysis. Suppose an analytic function \(f(z)\) has simple poles at \(z = z_i\) where \(i = 1, \ldots, n\). This means that near \(z = z_i\) the function diverges as

$$f(z) = \frac{R_i}{z - z_i} + \ldots$$

where the remaining terms are finite as \(z \to z_i\) and \(R_i\) is known as the residue at \(z = z_i\). Cauchy’s theorem states

$$\int_C f(z)dz = 2\pi i \sum_i R_i$$

1Based on notes from Prof. Waldram.
where the sum is over those points $z = z_i$ enclosed by the closed curve $C$.

Now consider the function

$$f(p^0, p) = \frac{e^{-ip \cdot x}}{p^2 - m^2} = \frac{e^{-ip^0 x^0} e^{ip \cdot x}}{(p^0 - \omega_p)(p^0 + \omega_p)}$$  \hspace{1cm} (7)

where, by definition $\omega_p = +\sqrt{p^2 + m^2}$. Clearly, as a function of $p^0$, there are simple poles at $p^0 = \pm \omega_p$. The corresponding residues are given by

$$R_{\pm} = \pm \frac{e^{\mp i \omega_p x^0} e^{ip \cdot x}}{2\omega_p}.$$  \hspace{1cm} (8)

Now consider the modified function

$$f_\epsilon(p^0, p) = \frac{e^{-ip \cdot x}}{p^2 - m^2 + i\epsilon}$$  \hspace{1cm} (9)

where $\epsilon$ is small and positive. The poles now shift slightly off the real axis to

$$p_{\pm}^0 = \pm \sqrt{p^2 + m^2 - i\epsilon} = \pm \omega_p \left(1 - \frac{i\epsilon}{2 \omega_p^2} \right)$$  \hspace{1cm} (10)

where we have expanded in $\epsilon$. To leading order in $\epsilon$ the residues are unchanged.

We now integrate $f_\epsilon(p^0, p)$ along the real axis

$$I = \int_{-\infty}^{\infty} dp^0 f_\epsilon(p^0, p).$$  \hspace{1cm} (11)

If $x^0 > 0$ then $e^{-ip^0 x^0} \to 0$ (and hence $f_\epsilon(p^0, p) \to 0$) as $\text{Im} p^0 \to -\infty$. This means we evaluation $I$ by closing the integration contour below:

$$I = -\int_{C_+} dp^0 f_\epsilon(p^0, p) = -2\pi i R_+$$  \hspace{1cm} (12)

On the other hand if $x^0 < 0$ then $e^{-ip^0 x^0} \to 0$ (and hence $f_\epsilon(p^0, p) \to 0$) as $\text{Im} p^0 \to \infty$. This means we evaluation $I$ by closing the integration contour above:

$$I = \int_{C_-} dp^0 f_\epsilon(p^0, p) = 2\pi i R_-$$  \hspace{1cm} (13)
Thus we can write

$$\int d^4 p f_c(p^0, p) = \int d^3 p \int_{-\infty}^{\infty} dp^0 f_c(p^0, p)$$

$$= \int d^3 p \left[ -\theta(x^0)2\pi i R_+ + \theta(-x^0)2\pi i R_- \right]$$

$$= -2\pi i \int d^3 p \left[ \theta(x^0)\frac{e^{-i\omega p x^0}e^{ip\cdot x}}{2\omega p} + \theta(-x^0)\frac{e^{i\omega p x^0}e^{ip\cdot x}}{2\omega p} \right]$$

$$= -2\pi i \int \frac{d^3 p}{2\omega p} \left[ \theta(x^0)e^{-ip\cdot x} + \theta(-x^0)e^{ip\cdot x} \right]$$

$$= -(2\pi)^4 i \Delta_F(x)$$

(14)

where in going from the third to fourth line we changed integration variables from $p$ to $-p$ in the second term. In other words, we have shown that

$$\Delta_F(x) = i \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\epsilon}.$$