Operator Ordering

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**Time-Ordering T**

The time ordering operator $T$ takes any product of operators, where each operator is defined at one time, and changes the order so that every operator has only later operators to the left, earlier operators to the right.

So time ordering always takes a product of operators and reorders the product so that for any operator, say $\hat{A}(t)$, this has later time operators to its left and operators with earlier times are to the right of $\hat{A}(t)$. We get the same operators in the product after time ordering, just in a different order.

Typically these operators are fields in the interaction representation.

For example for two operators we can write

$$T(\hat{A}(t_1)\hat{B}(t_2)) = \theta(t_1 - t_2)\hat{A}(t_1)\hat{B}(t_2) + \theta(t_2 - t_1)\hat{B}(t_2)\hat{A}(t_1)$$

(1)

where the Heaviside function is defined as

$$\theta(t) = +1 \text{ if } t > 0 \text{ and } \theta(t) = 0 \text{ if } t < 0.$$ 

More generally if we write a product of $n$ operators, $\hat{A}(t_i)$ ($i = 1, 2, \ldots, n$), each defined at a specific time $t_i$ then there is a permutation of $(1, 2, \ldots, n)$, say $(a_1, a_2, \ldots, a_n)$ such that $t_{a_i} > t_{a_{i+1}} \forall i = 1, 2, \ldots n$. Time-ordering of a product of operators returns those operators in the order of this $a$ permutation i.e.

$$T \left( \prod_{i=1}^{n} \hat{A}(t_i) \right) = \hat{A}(t_{a_1})\hat{A}(t_{a_2})\cdots\hat{A}(t_{a_n}) \text{ where } t_{a_i} > t_{a_j} \forall i > j.$$ 

(2)

An important property which is implicit in all our work is that “time ordering is distributive over addition”. That is if we apply time ordering to a sum of products this is equal to the sum of the result of time ordering each product. So

$$T \left( \hat{A}\hat{B}\hat{C} + \hat{X}\hat{Y}\hat{Z} \right) = T \left( \hat{A}\hat{B}\hat{C} \right) + T \left( \hat{X}\hat{Y}\hat{Z} \right).$$ 

(3)

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1It is not well defined at $t = 0$ in general or at least you need to be much more careful about its definition — it is not a function in the strict sense. Sometimes it helps to think of it representing a value of $1/2$ at $t = 0$ if needed.
Normal Ordering N

Normal ordering works in terms of a split of fields into two parts, \( \phi_i(t_i) \equiv \phi_i^+ + \phi_i^- \). In the most general case the normal ordered product of fields, \( N(\text{fields}) \), takes a product of ‘split’ fields, e.g. \( \phi^+ \) or \( \phi^- \), and gives you back a product of exactly the same ‘split’ fields but in a different order. It does not change the \( \pm \) labels on any field. Normal ordering is defined such that all \( \phi_i^+ \) are moved to the right of all \( \phi_i^- \), switching terms as few times as possible. In the most general definition there is no change of the order within the subset of \( \phi^+_i \), nor are there changes in order within the subset of \( \phi^-_i \). So the normal order of a product of such split fields is that

1. the plus-fields \( \phi_i^+ \) are always to the right of the minus-fields \( \phi_i^- \)
2. the order within the set of plus-fields \( \phi_i^+ \) is unchanged from the original product
3. the order within the set of minus-fields \( \phi_i^- \) is unchanged from the original product.

For \( N[\phi_i \phi_j] \) we have that

\[
N[\phi_i \phi_j] = N[(\phi_i^+ + \phi_i^-)(\phi_j^+ + \phi_j^-)] = \phi_i^+ \phi_j^+ + \phi_i^- \phi_j^- + \phi_i^+ \phi_j^- + \phi_i^- \phi_j^+. \tag{4}
\]

Again we have that normal ordering is distributive over addition, that is we expand out expressions into a sum over products of split fields and the result of normal ordering such an expression is the sum of the normal ordered results for each of the products. So

\[
N(\phi_1 \phi_2 \ldots \phi_n) = N\left(\sum_{\pm} (\phi_1^+ \phi_2 \ldots \phi_n^\pm) \phi_2 \ldots (\phi_n^+ + \phi_n^-)\right) \tag{5}
\]

\[
= N\left(\sum_{\pm} (\phi_1^+ \phi_2 \ldots \phi_n^\pm)\right) \tag{6}
\]

\[
= \sum_{\pm} \left( N(\phi_1^+ \phi_2^+ \ldots \phi_n^\pm) \right) \tag{7}
\]

Typical Simplification

Note that in almost all cases you will have additional properties such as that all plus-fields commute between themselves and all minus fields commute within themselves. That is you will often find that

\[
[\phi_i^+, \phi_j^+] = [\phi_i^-, \phi_j^-] = 0. \tag{8}
\]

That means the order within the \( \phi^+ \) is irrelevant in the definition of the Normal ordered product and ensures normal ordered products are symmetric under interchange of fields. This property needs to be stated if such simplifications are to exploited in your work with normal ordering.

Best choice for Vacuum Expectation Values

You will always choose the split so that \( \langle N(\text{fields}) \rangle = 0 \) for whatever expectation values your problem requires. For this course, and for most QFT applications, we are interested in vacuum expectation values so we require \( \langle 0 | N(\text{fields}) | 0 \rangle = 0 \). In this case the most useful definition of the split of fields is then \( \phi_i^+ \sim \hat{a}_i \) (pure annihilation operator term) and \( \phi_i^- \sim \hat{a}_i^\dagger \) (pure creation operator term). Happily this obeys (8) making algebraic manipulations much simpler.