Operator Ordering

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Time-Ordering T

Definition

> The time ordering operator T takes any product of operators and changes the order so that each operator has only later operators to the left, and earlier operators to the right.

We get the same operators in the product after time ordering, just in a different order. We must assume that all operators are at different times. If they are not, then give all operators a distinct time and take the limit of equal times at the end of the calculation
e.g. $(\hat{\phi}(t))^4 = \lim_{t_i \to t} \hat{\phi}(t_1)\hat{\phi}(t_2)\hat{\phi}(t_3)\hat{\phi}(t_4)$.

For two operators $\widehat{A}$ and $\widehat{B}$ we have

$$\mathrm{T}(\widehat{A}(t_1)\widehat{B}(t_2)) = \theta(t_1 - t_2)\widehat{A}(t_1)\widehat{B}(t_2) + \theta(t_2 - t_1)\widehat{B}(t_2)\widehat{A}(t_1) \tag{1}$$

where the "Heaviside step function" is defined as[1] $\theta(t) = +1$ if $t > 0$ and $\theta(t) = 0$ if $t < 0$.

More generally if we write a product of n operators, $\widehat{A}(t_i)$ $(i = 1, 2, \ldots, n)$, each defined at a specific and distinct time t_i then there is a permutation of $(1, 2, \ldots, n)$, say $(a_1, a_2, \ldots, a_n)$ such that $t_{a_i} > t_{a_{i+1}}$ $\forall\ i = 1, 2, \ldots n$. Time-ordering of a product of operators returns those operators in the order of this a permutation i.e.

$$\mathrm{T}\left(\prod_{i=1}^{n} \widehat{A}(t_i)\right) = \widehat{A}(t_{a_1})\widehat{A}(t_{a_2})\ldots\widehat{A}(t_{a_n}) \quad \text{where} \quad t_{a_i} > t_{a_j}\ \forall\ i > j\,. \tag{2}$$

An important property which is implicit in all our work is that "time ordering is distributive over addition". That is if we apply time ordering to a sum of products this is equal to the sum of the result of time ordering each product. So

$$\mathrm{T}\left(\widehat{A}\widehat{B}\widehat{C} + \widehat{X}\widehat{Y}\widehat{Z}\right) = \mathrm{T}\left(\widehat{A}\widehat{B}\widehat{C}\right) + \mathrm{T}\left(\widehat{X}\widehat{Y}\widehat{Z}\right)\,. \tag{3}$$

Normal Ordering N

Normal ordering works in terms of a split of fields into two parts, $\phi_i(t_i) \equiv \phi_i = \phi_i^+ + \phi_i^-$.
Notation: the plus superscript + used here is *not* the same as the dagger † used for hermitian conjugation and they mean different things, $\phi^+ \not\equiv \phi^\dagger$.

In the most general case, the **normal ordered product** of fields, N(fields), takes a product of 'split' fields, e.g. ϕ^+ or ϕ^-, and gives you back a product of exactly the same 'split' fields but in a different order. It does not change the $\pm$ labels on any field. Normal ordering is defined such that all ϕ_i^+ are moved to the right of all ϕ_i^-, switching terms as few times as possible. In the most general definition there is no change of the order within the subset of ϕ_i^+, nor are there changes in order within the subset of ϕ_i^-. So the normal order of a product of such split fields is that

[1] The "Heaviside step function", often referred to simply as a "theta function", is not well defined at $t = 0$ — it is not a function in the strict sense. See the note on equal times in the handout entitled "The Feynman Propagator and Cauchy's Theorem".

1. the plus-fields ϕ_i^+ are always to the right of the minus-fields ϕ_i^-
2. the order within the set of plus-fields ϕ_i^+ is unchanged from the original product
3. the order within the set of minus-fields $\phi_i^-$is unchanged from the original product.

For $\mathrm{N}[\phi_i\phi_j]$ we have that

$$\mathrm{N}[\phi_i\phi_j] = \mathrm{N}[(\phi_i^+ + \phi_i^-)(\phi_j^+ + \phi_j^-)] = \phi_i^+\phi_j^+ + \underbrace{\phi_j^-\phi_i^+}_{\text{(order changed)}} + \phi_i^-\phi_j^+ + \phi_i^-\phi_j^- \,. \tag{4}$$

Again we have that normal ordering is distributive over addition, that is we expand out expressions into a sum over products of split fields and the result of normal ordering such an expression is the sum of the normal ordered results for each of the products. So

$$\mathrm{N}\left(\phi_1\phi_2\ldots\phi_n\right) = \mathrm{N}\left((\phi_1^+ + \phi_1^-)(\phi_2^+ + \phi_2^-)\phi_2\ldots(\phi_n^+ + \phi_n^-)\right) = \mathrm{N}\Big(\sum_{\pm}\phi_1^\pm\phi_2^\pm\ldots\phi_n^\pm\Big) \tag{5}$$

$$= \sum_{\pm}\Big(\mathrm{N}\left(\phi_1^\pm\phi_2^\pm\ldots\phi_n^\pm\right)\Big) \tag{6}$$

Typical Simplification

Note that in almost all cases you will have additional properties such as that all plus-fields commute between themselves and all minus fields commute within themselves. That is you will often find that

$$\left[\phi_i^+, \phi_j^+\right] = \left[\phi_i^-, \phi_j^-\right] = 0\,. \tag{7}$$

That means the order within the ϕ^+ is irrelevant in the definition of the Normal ordered product and ensures normal ordered products are symmetric under interchange of fields. This property needs to be stated if such simplifications are to exploited in your work with normal ordering.

Choosing the Best Field Split

You will always choose the split so that $\langle N(\text{fields})\rangle = 0$ for whatever expectation values your problem requires. This in turn will mean the appropriate contraction, the propagator, changes with the expectation value required. Here $\langle\ldots\rangle$ denotes some generic expectation value.

Vacuum Expectation Values. For this course, and for most QFT applications, we are interested in vacuum expectation values so we require $\langle 0|N(\text{fields})|\,0\rangle = 0$. In this case the most useful definition of the split of fields is then $\phi_i^+ \sim \hat{a}_i$ (pure annihilation operator term) and $\phi_i^- \sim \hat{a}_i^\dagger$ (pure creation operator term). Happily this obeys (7) making algebraic manipulations much simpler.

Thermal Expectation Values. (Not needed for course) To provide an example of an expectation value where the usual normal ordering is *not* useful, consider a system of relativistic particles in thermal equilibrium., e.g. gas of cold atoms or the early universe (if expansion is slow). In this case our expectation values for any operator $\widehat{\mathcal{O}}$ in terms of some basis states $\{|\,\psi\rangle\}$ are of the form

$$\langle\widehat{\mathcal{O}}\rangle = \frac{1}{Z}\sum_{\psi}\langle\psi|e^{-\beta\widehat{H}}\widehat{\mathcal{O}}|\,\psi\rangle\,, \qquad Z = \langle 1\rangle = \sum_{\psi}\langle\psi|e^{-\beta\widehat{H}}|\,\psi\rangle\,. \tag{8}$$

For any one QHO mode, $\hat{a}$, using the basis of normalised n-quanta eigenstates $|\,n\rangle$, we find that

$$\langle\hat{a}^\dagger\hat{a}\rangle = \frac{1}{Z}\sum_{n}\langle n|\exp\{-n\beta\omega\}n|\,n\rangle = \frac{1}{e^{\beta\omega}-1}\,. \tag{9}$$

This is clearly not zero yet $(\hat{a}^\dagger\hat{a})$ is normal ordered if we use the usual vacuum normal ordering. See optional questions on Problem Sheet 5 for the best split and normal order for such thermal problems.