A Recipe for Perturbation Theory

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1. Select Initial i and Final f States.

We are looking for the probability $p_{i\to f}$ that $i \to f$ so we can compare the number of f events seen against the number of times we started our experiment with state i.

2. Write Probability $p_{i\to f}$ as a Matrix Element $\mathcal{M}_{i\to f}$.

There may be various kinematic factors as the initial and final state are not always expressed in terms of single energies and momenta, e.g. a total cross-section is an integral over final particles seen at all angles relative to the incoming beam of particles. However in all cases $p_{i\to f}$ is proportional to the modulus squared of a Matrix element $\mathcal{M}_{i\to f}$ of the form

$$p_{i\to f} \propto \left|\mathcal{M}_{i\to f}\right|^2 , \qquad \mathcal{M}_{i\to f} \;=\; {}_{\mathrm{I}}\langle f, t_f|S|i, t_i\rangle_{\mathrm{I}} \tag{1}$$

where the S-matrix is

$$S = U(+\infty, -\infty) = \mathrm{T}\exp(-i\int_{-\infty}^{+\infty} d^4x\, \mathcal{H}_{\mathrm{int,I}})\,, \qquad H_{\mathrm{int,I}} = \int d^3x\, \mathcal{H}_{\mathrm{int,I}}\,. \tag{2}$$

3. Write the Matrix element $\mathcal{M}$ as a Green Function G

$$\begin{aligned}\mathcal{M}(\{q_f\},\{p_i\}) \;=\; & \lim_{z_f^0\to+\infty}\left(\prod_f \int d^3z_f\, e^{+iq_fz_f} 2\omega_f\right) \\ & .\lim_{y_i^0\to-\infty}\left(\prod_i \int d^3y_i\, e^{-ip_iy_i} 2\omega_i\right) .G(\{z_f\},\{y_i\})\,. \end{aligned}\tag{3}$$

where $\omega_f = \sqrt{(\boldsymbol{q}_f)^2 + m_f^2}$ and $\omega_i = \sqrt{(\boldsymbol{p}_i)^2 + m_i^2}$ are the dispersion relations for each field (i's and f's) in the initial and final state. A Green function is a vacuum expectation value of the time ordered products of fields and the S-Matrix so

$$G(\{x_i\}) \;=\; \langle 0|\mathrm{T}\left(\big(\prod_i \phi_i(x_i)\big)S\right)|\,0\rangle\,. \tag{4}$$

In these notes, each ϕ_i can represent any scalar field in the theory[1].

4. Expand S-matrix

Expand the S-matrix of (2) in the Green functions (4) to produce a sum over terms, each of which is a vacuum expectation value of the time-ordered products of fields.

$$G(\{x_i\}) \;=\; \sum_{N=0}^{\infty}\frac{1}{N!}\left(\prod_{j=1}^{N}\int d^4x_j\right)\,\langle 0|\mathrm{T}\left(\big(\prod_i \phi_i(x_i)\big)\big(\mathcal{H}_{\mathrm{int,I}}(x_j)\big)^N\right)|\,0\rangle\,. \tag{5}$$

[1]Perhaps $\phi_3 \equiv \phi_3(x_3) = \psi^\dagger(x_3)$, or ϕ_1 and ϕ_5 can be the same scalar field but at different coordinates x_1^μ and x_5^μ. This notation is used to keep expressions generic.

The $\mathcal{H}_{\mathrm{int,I}}(x_j)$ is a sum of products of fields, each with a coupling constant coefficient, e.g. for SYTh we have that $\mathcal{H}_{\mathrm{int,I}}(x) = g\phi(x)\psi(x)\psi^\dagger(x)$.

5. Apply Wick's Theorem

Use Wick's Theorem to write each time-ordered product of fields in our expression (5) as a sum of several terms, each of which is a product of contractions of pairs of fields and Normal ordered products of remaining fields. Wick's theorem states that the time ordered product of such fields is equal to a sum over normal ordered products of the field where *distinct* pairs of fields are contracted in all possible ways. That is

$$\begin{aligned}
\mathrm{T}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\right) &= \mathrm{N}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\right) \\
&\quad + \sum_{(i,j)} \mathrm{N}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_i\ldots\hat{\phi}_j\ldots\hat{\phi}_n\right) \\
&\quad + \sum_{(i,j),(k,l)} \mathrm{N}\left(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_i\ldots\hat{\phi}_k\ldots\hat{\phi}_j\ldots\hat{\phi}_l\ldots\hat{\phi}_n\right) \\
&\quad + \qquad \vdots \\
&\quad + \sum_{m \text{ distinct pairs}} \mathrm{N}\left(n \text{ fields containing } m \text{ contracted pairs}\right) \\
&\quad + \qquad \vdots .
\end{aligned} \tag{6}$$

Note that in these sums i, j, k, l etc. must all be distinct values and we never repeat the same set of field pairs in any one sum (treating (i,j) and (j,i) as identical pairs). Also note that this form is true *whatever* the times of the fields are. Changing the values of the times can make an explicit change to the order of fields on the left hand side but the operators on the right hand side will still be written in the same order, for example with ϕ_1 on the left and ϕ_n on the right of a normal ordered product containing these fields if neither of these fields is in some contraction.

6. Choose best field splits $\phi_i^\pm$

Choose the split your fields $\phi_i = \phi_i^+ + \phi_i^-$ to ensure that the expectation value of any normal ordered product is zero and that the contraction of any pair of fields is a c-number.

$$\text{Choose } \phi_i^\pm \text{ s.t.} \quad \langle \mathrm{N}\big(\hat{\phi}_1\hat{\phi}_2\ldots\hat{\phi}_n\big)\rangle = 0, \quad \Delta_{ij} \propto \widehat{\mathbf{1}}. \tag{7}$$

You will find that the contraction is equal to the expectation value of the time ordered product of a couple of fields and these are the different propagators in the theory.

For our case of vacuum expectation values, the split we need is where we put annihilation operator parts of field in the plus parts of fields and the creation operator pars of fields in the minus parts of the field. The propagator in this case is the Feynman propagator for scalar fields discussed we were considering free field theory. You will also find in our case (and most situations) that the only contractions which are not zero are those between a field and its hermitian conjugate field.

After we apply the vacuum expectation value to the Wick theorem expansion of the time-ordered products in (5), the expression for the Green function (5) is now a sum of terms, each term representing a products of contractions (i.e. propagators) only, with various factors of $-ig \int d^4x$ of similar coming from the S matrix expansion. Many of these terms are identical numerically.

7. Encode the expression in terms of Feynman diagrams.

Many different terms from the last step which are numerically identical are now encoded in one diagram. Mostly this repetition is obvious and there are no problems so the symmetry factor is 1. However, in some cases the expansion has now caused as much repetition as you might imagine, e.g. because of symmetries (automorphisms of the graph), leading to a non-trivial symmetry factor. This is hard to evaluate.

8. Coordinate to Momentum Space

The Feynman rules produced so far are in coordinate space. If desired change from coordinate to momentum space diagrams for the Green function in momentum space, $G(\{p_i\})$ where

$$G(\{p_i\}) = \left(\prod_{i=1}^{n} \int d^4x_i \, \exp\{-ip_i y_i\} \right) G(\{y_i\}) \, . \tag{8}$$

- The diagrams remain unchanged so the symmetry factors remain unchanged.
- The Feynman rules change and diagrams now represent a contribution to the Fourier transform of the original coordinate space Green function.
- Energy-momentum conservation means we know that $G(\{p_i\}) \propto \tilde{\delta}^4(\sum_i p_i)$.

Ultimately it is more natural to work in momentum space as particles are produced in energy momentum eigenstates so the original initial and final states are expressed in terms of these parameters.