Different Pictures for QFT

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Schrödinger picture definitions

Most people first encounter QM in terms of the Schrödinger picture where the states carry all the time dependence of the problem. This becomes our assumption in QFT that in the Schrödinger picture (here indicated with S subscripts) operators, $\widehat{\mathcal{O}}_{\rm S}$, are time-independent while the states carry all the time dependence as specified by the Schrödinger equation. That is (with $\hbar = 1$)

$$i\frac{d}{dt}|\psi, t\rangle_{\rm S} = \widehat{H}_{\rm S}\,|\psi, t\rangle_{\rm S}\,, \tag{1}$$

$$\Rightarrow \quad |\psi, t\rangle_{\rm S} = \exp\{-i\widehat{H}_{\rm S}(t-s)\}|\psi, s\rangle_{\rm S}\,. \tag{2}$$

Note that we are assuming that the full Hamiltonian $\widehat{H}_{\rm S}$ has no explicit time dependence[1].

Heisenberg picture definitions

The operators now carry all the time and states are constant. The Schrödinger and Heisenberg pictures identical at some reference time which here we will choose to be at $t = 0$. We define the evolution of operators in the Heisenberg picture as

$$\widehat{\mathcal{O}}_{\rm H}(t) = \exp\{+i\widehat{H}_{\rm S}t\}\,\widehat{\mathcal{O}}_{\rm S}\,\exp\{-i\widehat{H}_{\rm S}t\} \tag{3}$$

which is equivalent to

$$\frac{d}{dt}\widehat{\mathcal{O}}_{\rm H}(t) = +i\left[\widehat{H}_{\rm S}, \widehat{\mathcal{O}}_{\rm H}(t)\right]\,. \tag{4}$$

It is quick to see from this definition that the full Hamiltonian operator is the same in both Schrödinger and Heisenberg pictures for all times[2]

$$\widehat{H}_{\rm S} = \widehat{H}_{\rm H}(t=0) = \widehat{H}_{\rm H}(t)\,. \tag{5}$$

So we will now drop the picture index on the full Hamiltonian operator and just use $\widehat{H}$. Thus we have for the Heisenberg picture that (Tong equations (2.77) and (2.78))

$$\widehat{\mathcal{O}}_{\rm H}(t) = \exp\{+i\widehat{H}t\}\widehat{\mathcal{O}}_{\rm H}(t=0)\exp\{-i\widehat{H}t\}\,, \qquad \widehat{\mathcal{O}}_{\rm H}(t=0) = \widehat{\mathcal{O}}_{\rm S}\,. \tag{6}$$

$$\frac{d}{dt}\widehat{\mathcal{O}}_{\rm H}(t) = +i\left[\widehat{H}, \widehat{\mathcal{O}}_{\rm H}(t)\right]\,. \tag{7}$$

[1] An example of where we have explicit time dependence would be where we have electrically charged particles but there is some time-dependent 'external' magnetic field applied by some experimentalist.

[2] Indeed this is true for any operator linked to a conserved quantity, i.e. for any operator which commutes with the Hamiltonian.

Interaction picture definitions

In QFT most calculations are worked in the Interaction picture. To define the interaction picture we start by splitting the Hamiltonian into two parts, $H = H_0 + H_{\rm int}$. In principle this is an arbitrary split and sometimes this can be exploited. In any practical work[3] the H_0 will be always be quadratic in the fields, with any terms which are higher order than quadratic in the fields, ϕ^n for $n > 2$, always placed in what is called the interaction part of the Hamiltonian $H_{\rm int}$.

The complication is that since $[H_0, H_{\rm int}] \neq 0$ in general this leads, after some consideration, to the fact that these parts are not usually the same in all pictures so we must label add a subscript to indicate which picture we are working in: S for Schrödinger picture objects, H for Heisenberg picture objects or I for the Interaction picture objects. In many places in the course, we may drop the subscript where the context makes it clear what picture we are using. For instance, once we have discussed the different pictures, all remaining discussions of interacting fields will be done in the interaction picture and all objects will be in the interaction picture unless clearly stated.

We defined the Interaction picture by demanding that the operators carry the time dependence defined by the free Hamiltonian. This means the states in the Interaction picture carry the time dependence defined by the interaction Hamiltonian. Again we define states and operators at a reference time $t = 0$ are identical in all three pictures. More precisely we define (Tong equation (3.11))

$$i\frac{d}{dt}|\psi, t\rangle_{\rm I} = H_{\rm int,I}\,|\psi, t\rangle_{\rm I}\,, \tag{8}$$

$$\widehat{\mathcal{O}}_{\rm I}(t) = \exp\{+i\widehat{H}_{0,\rm S}t\}\widehat{\mathcal{O}}_{\rm S}\exp\{-i\widehat{H}_{0,\rm S}t\}\,, \tag{9}$$

We can rewrite the time evolution in the interaction picture in terms of a new operator[4] $\widehat{U}$ where

$$|\,\psi, t\rangle = \widehat{U}(t,s)|\,\psi, s\rangle\,. \tag{10}$$

The time evolution operator $\widehat{U}$ used in the interaction picture satisfies (Tong section 3.1.1)

$$i\frac{d}{dt}U(t,t_0) = H_{\rm int,I}(t)U(t,t_0)\,. \tag{11}$$

After some work we find that for $t > s$ (the only case we will use) it may be expressed as

$$\widehat{U}(t,s) = \mathrm{T}\left(\exp\{-i\int_s^t dt' H_{\rm int,I}(t')\}\right) \text{ for } t > s\,, \tag{12}$$

where $\mathrm{T}(\ldots)$ indicates the operators are time-ordered.

The S-matrix is often encountered as we evolve states from $t = -\infty$ (where we notionally prepare the input state such as pairs of particles collided in an accelerator) to $t = +\infty$, where we observe particles in a detector. The S-matrix is simply the time evolution operator in the interaction picture between these two extreme times

$$\hat{S} = \widehat{U}(+\infty, -\infty) = \mathrm{T}\left(\exp\{-i\int_{-\infty}^{+\infty} dt' H_{\rm int,I}(t')\}\right)\,. \tag{13}$$

Perturbation theory is the approximation in which we expand the S-matrix to a finite order in the Taylor series of this exponential.

[3] Some formal derivations do not care what how we define this split but when you come to do actual calculations we need the free part to correspond to an exactly solvable theory. The only exactly solvable theory in QFT is that of a non-interacting field theory so H_0 is invariably quadratic in all fields.

[4] We will not add a I subscript as we ought to. This $\widehat{U}$ operator is only needed in the Interaction picture.