

4 QFT for Real Scalar Fields ϕ

We have seen the link between ^{classical} field theory ϕ and SHO through normal modes.
 $\Rightarrow$ Focus on QHO quantised SHO

4.1 QHO = Quantum Harmonic Oscillator

Q.M. may be defined via
 "CCR" = "canonical commutation relations"
 = standard

for position & momentum operators $\hat{q}_i, \hat{p}_i$
 for $i=1, \dots, N$ particles

$$[\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}$$

($u_i = q_i$
in 1D case)

AND $[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$ one definition of $\hbar$
 Classically

We work in normal modes, linear combinations of q & p , repeat for $\hat{q}$ & $\hat{p}$

$$\left. \begin{aligned} \hat{Q}_i &= \sum_j U_{ij} \hat{q}_j \\ \hat{P}_i &= \sum_j U_{ij}^* \hat{p}_j \end{aligned} \right\} \text{N.B. } \begin{aligned} Q_i^+ &= Q_i \Leftrightarrow U_{ij} = U_{ij}^* \\ P_i^+ &= P_i \end{aligned}$$

$$\begin{aligned} \Rightarrow [\hat{Q}_i, \hat{P}_j] &= \sum_{k,l} U_{ik} U_{jl}^* [\hat{q}_k, \hat{p}_l] \\ &= \sum_{k,l} U_{ik} U_{jl}^* i\hbar \delta_{kl} \\ &= i\hbar [U U^T]_{ij} \end{aligned}$$

QFT
 In
 B1
 Q3
 in general

Find
 P3, Q3

$$\Rightarrow [\hat{Q}_i, \hat{P}_j] = i\hbar \delta_{ij} \quad \text{IFF} \quad U U^T = 1$$

since $\hat{q}_i, \hat{p}_i$ hermitian.

$\Rightarrow$ Unitary ($\Rightarrow$ linear) transformations preserve c.c.r.

- Unitary transformations are "canonical transformations" they preserve c.c.r.
- ⇒ Can use $\{\hat{q}_i, \hat{p}_i\}$ OR $\{\hat{Q}_i, \hat{P}_i\}$ as both equally good QM variables (standard)
- Best to use $\{\hat{Q}_i, \hat{P}_i\}$ in QFT as 1D example showed
- In QFT each mode i corresponds to different k value (as 1D shows)

Focus on one ^{QHO} mode with characteristic

$P = P^+$ (normalised)
e.g.
 $P_n + P_{-k} = 0$ in 1D case

$$\hat{H} = \frac{1}{2m} \hat{P}^2 + \frac{1}{2} m \omega^2 \hat{Q}^2$$

NOTE
NOT unique
generalisation
of $H = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 q^2$
due to OPERATOR
ORDER

Best solved with ^{dimensionless} creation $\hat{a}^+$ & annihilation $\hat{a}$ operators

$$\hat{a} := \frac{1}{\sqrt{2}} \left(\frac{\hat{Q}}{l} + i \frac{l}{\hbar} \hat{P} \right),$$

$$\Rightarrow \hat{a}^+ := \frac{1}{\sqrt{2}} \left(\frac{\hat{Q}}{l} - i \frac{l}{\hbar} \hat{P} \right), \quad (\hat{a}^+ \neq \hat{a} \Rightarrow \text{not hermitian})$$

where $l = \sqrt{\frac{\hbar}{m\omega}}$ is characteristic length (EFS check units)

⇒ c.c.r become $[\hat{a}, \hat{a}^+] = 1$

AND $[\hat{a}, \hat{a}] = [\hat{a}^+, \hat{a}^+] = 0$

EFS
PS2
Q7

$$\Rightarrow \hat{q} = \frac{l}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger), \quad \hat{p} = \frac{i\hbar}{\sqrt{2}l} (\hat{a}^\dagger - \hat{a})$$

$$\Rightarrow \hat{H} = \frac{1}{2m} \frac{\hbar^2}{2l^2} (\hat{a} - \hat{a}^\dagger)(\hat{a}^\dagger - \hat{a}) + \frac{1}{2} m \omega^2 \frac{l}{2} (\hat{a}^\dagger + \hat{a})(\hat{a} + \hat{a}^\dagger)$$

$$\Rightarrow \hat{H} = \frac{1}{2} \hbar \omega (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) = \hbar \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

Value depends on
operator order in
original $\hat{H}$ } Zero Point
Energy

If have many normal modes $\hat{Q}_k, \hat{P}_k \leftrightarrow \hat{a}_k, \hat{a}_k^\dagger$
ONE OSCILLATOR PER NORMAL MODE

$$\Rightarrow \hat{H} = \sum_k \hbar \omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right)$$

as each mode independent = commutes with others
 k = mode label, usually the wavenumber will do
rather than an integer

SOLUTION for Energy Eigenvalues

PS1

Energy eigenstates are $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle$ ($n=0,1,2,\dots$)

They contain n "quanta" $\hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle$

Energy value is $E_n = \frac{1}{2} + n \hbar \omega$

(FREE SCALAR)

QFT

QM $[q_i(t), p_j(t)] = i\hbar \delta_{ij}$, $[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$

Position $\rightarrow$ Conjugate Momentum
particle i in space $\hat{q}_i$

$$\hat{p}_j = \frac{\partial L}{\partial \dot{\hat{q}}_j}$$

QFT $\approx$ "Position" of field i in field space $\hat{\phi}_i$ $\rightarrow$ Conjugate Momentum Density $\hat{\pi}_j = \frac{\partial L}{\partial \dot{\hat{\phi}}_j}$

Defⁿ Axiom

$$[\hat{\phi}_i(x, t), \hat{\pi}_j(x', t)] = i\hbar \delta_{ij} \delta^3(x - x'),$$

AND $[\hat{\phi}_i(x, t), \hat{\phi}_j(x', t)] = [\hat{\pi}_i(x, t), \hat{\pi}_j(x', t)] = 0$

NOTES

① these are EQUAL TIME COMMUTATION RELATIONS
= ETCR for real scalar fields

They DEFINE QFT.

 $\Rightarrow$ ANY OPERATOR on ANY STATES s.t. ETCR true is VALID

② QM $\hat{q}_i(t) \Rightarrow$ QFT $\hat{\phi}_i(x, t)$

Geographical: QUANTISED Space

COORDINATE

i.e. QM is a QFT in 1 time zero space dimensional space-time

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DynamicsWe want $k^2 = m^2$ so classical solⁿ of KB.

$$H = \frac{1}{2} \Pi^2 + \frac{1}{2} m \phi^2$$

$$\phi(x, t) = \int \frac{d^3k}{2\omega_k} \left(e^{-ikx} A_k + e^{+ikx} B_k \right)$$

$$kx = \omega_k t - \underline{k} \cdot \underline{x}$$

 A_k^* as $\phi \in \mathbb{R}$ $\Downarrow$ QuantiseAXIOM

$$\begin{aligned} \hat{\phi}(x, t) &= \int \frac{d^3k}{2\omega_k} \left(e^{-ikx} \hat{A}_k + e^{+ikx} \hat{A}_k^+ \right) \\ &= \int \frac{d^3k}{\sqrt{2\omega_k}} \left(e^{-ikx} \hat{a}_k + e^{+ikx} \hat{a}_k^+ \right) \end{aligned}$$

where $\hat{A}_k = \sqrt{2\omega_k} \hat{a}_k$; EFS $\hat{\phi} = \hat{\phi}^+$ hermitian
c.i. $\hat{\phi} = \hat{\phi}^+$

$$\Rightarrow \hat{\Pi}(x, t) = \partial_t \hat{\phi} = \int \frac{d^3k}{\sqrt{2}} -i\omega_k \left(e^{-ikx} \hat{a}_k - e^{+ikx} \hat{a}_k^+ \right)$$

AXIOM $\hat{H} = \frac{1}{2} \hat{\Pi}^2 + \frac{1}{2} m \hat{\phi}^2$

Annihilation & Creation Qs.

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EFS
PS4

$$\text{ETCR} \Rightarrow [\hat{a}_k, \hat{a}_p^\dagger] = \delta^3(\underline{k} - \underline{p})$$

$$[\hat{a}_k, \hat{a}_p] = [\hat{a}_k^\dagger, \hat{a}_p^\dagger] = 0$$

and

$$\hat{H} = \int d^3p \, \omega_p \left(\hat{a}_p^\dagger \hat{a}_p + \frac{1}{2} \right)$$

i.e. • Our free quantum field is a sum of an infinite number of QHO, one for each momentum.

- Each degree of freedom at momentum p
 $\Leftrightarrow$ One QHO

N.B. without interactions, number of particles/quantum in each mode is conserved $[\hat{H}, \hat{a}_k^\dagger \hat{a}_k] = 0$

5 Hilbert Space -

- space of possible states

Use energy/number eigenstate basis

$$\begin{aligned}
 \hat{a}_p^\dagger |0\rangle &= |p\rangle && \text{Empty vacuum} \\
 \hat{a}_p^\dagger \hat{a}_q^\dagger |0\rangle &= |p, q\rangle && \text{1-particle state} \\
 &\vdots && \text{2-particle state} \\
 &\text{etc.}
 \end{aligned}$$

For instance

$$\hat{a}_p^\dagger |p\rangle = \omega_p |p\rangle \quad \text{where} \quad \omega_p = \sqrt{p^2 + m^2} \geq 0$$

Norm?

More formally, the state space is a tensor product of QM state spaces

$$|n\rangle_{k_1} \otimes |n\rangle_{k_2} \otimes \dots$$

$$\begin{aligned}
 \text{e.g.} \\
 \hat{a}_p^\dagger |0\rangle = |p\rangle &= |0\rangle_{k_1} \otimes |0\rangle_{k_2} \otimes \dots \otimes \underbrace{\hat{a}_p^\dagger |0\rangle_p}_{|1\rangle_p} \otimes \dots
 \end{aligned}$$

Normalisation

$$\langle 0 | 0 \rangle = 1$$

$$\begin{aligned}
 \Rightarrow \langle p | k \rangle &= \langle 0 | \hat{a}_p \hat{a}_k^\dagger | 0 \rangle \quad \downarrow \text{c.r.} \\
 &= \delta^3(p - k) \langle 0 | 0 \rangle
 \end{aligned}$$