

(Tony 2.6) TIME & QFT
p35

So far all work has been at fixed times.
e.g. $\hat{a}_p^+ |0\rangle = |p\rangle$ has no time
ETCR

Three ways of introducing time, first two are:-

① Schrödinger Picture = S.Pic.

- Work with operators with no explicit time
- States carry all time

$$\Rightarrow \hat{\phi}_S(\underline{x}) = \hat{\phi}(\underline{x}, t=0) \quad (t=0 \text{ arbitrary choice})$$

$\uparrow$ $\uparrow$
 S.Pic. No explicit t dependence NOTN

$$= \int \frac{d^3k}{\sqrt{2\omega_k}} \left(\hat{a}_k e^{+ik \cdot x} + \hat{a}_k^\dagger e^{-ik \cdot x} \right)$$

and for any state ψ

$$i \frac{d}{dt} |\psi, t\rangle_S = \hat{H}_S |\psi, t\rangle_S$$

(same $\forall$ states)

Schrödinger Equation
another fundamental
axiom.

This is valid but unhelpful approach in QFT
especially with relativity as Sch. Equⁿ gives
time a special role.

(B) Heisenberg Picture = H, A_c

- Let operators carry all time
- States have no explicit time dependence

$$\Rightarrow \hat{O}_H(\underline{x}, t) = e^{i\hat{H}_S t} \underbrace{\hat{O}_S(\underline{x})}_{\substack{\text{NOT} \\ \text{NO TIME}}} e^{-i\hat{H}_S t}$$

(Same for ALL operators)

$$\hat{O}_S(\underline{x}) \stackrel{\text{N.A.}}{=} \hat{A} = \sum_n \frac{\hat{A}^n}{n!}$$

and $|\psi\rangle_H = |\psi, t=0\rangle_S$

NOTE (i) $H_H = H_S \Rightarrow$ drop subscript for H

" (ii) that time evolution is equivalent to

$$\frac{d\hat{O}_H}{dt} = i[\hat{H}, \hat{O}_H]$$

Consistency

Can work in either picture,
completely equivalent

e.g. to find $\langle \hat{O} \rangle$ at time t
in state ψ we need

$$\begin{aligned} \left(\langle \psi | \hat{O} | \psi \rangle \right)_t &= {}_S \langle \psi(t, \underline{x}) | \hat{O}_S(\underline{x}) | \psi(t, \underline{x}) \rangle_S \\ &= {}_S \langle \psi(0, \underline{x}) | e^{+i\hat{H}_S t} \hat{O}_S(\underline{x}) e^{-i\hat{H}_S t} | \psi(0, \underline{x}) \rangle_S \\ &= {}_H \langle \psi(\underline{x}) | \hat{O}_H(\underline{x}, t) | \psi(\underline{x}) \rangle_H \end{aligned}$$

FREE Field Operator $\hat{\phi}_H(x, t)$ $\hat{\phi}_S(x)$

$$\hat{\phi}_H = \hat{\phi}(x, t) = e^{i\hat{H}t} \hat{\phi}(x, t=0) e^{-i\hat{H}t}$$

FOR FREE FIELD $H = \sum \omega_k a^\dagger a + \text{const}$ THEN

$$[H, \hat{a}_k^\dagger] = \omega_k \hat{a}_k^\dagger, \quad [H, \hat{a}_k] = -\omega_k \hat{a}_k$$

EPS

[Tong 4.4.1]

PS2.46

PS4

PS

$$\Rightarrow e^{i\hat{H}t} \hat{a}_k^\dagger e^{-i\hat{H}t} = e^{+i\omega_k t} \hat{a}_k^\dagger$$

$$e^{i\hat{H}t} \hat{a}_k e^{-i\hat{H}t} = e^{-i\omega_k t} \hat{a}_k$$

$$\Rightarrow \hat{\phi}_H(x, t) = \int \frac{d^3k}{\sqrt{2\omega_k}} \left\{ e^{-ikx} \hat{a}_k + \hat{a}_k^\dagger e^{+ikx} \right\}$$

where $k_0 = +\omega_k \geq 0$ and time evolution as
given earlier
FOR FREE FIELD.

4.4 TWO-POINT GREEN FUNCTIONS - PROPAGATORS

In free (= non-interacting = quadratic H, L) QFT all information is in expectation values of PAIRS of fields c.f. gaussian random variables

$$\text{e.g. } \langle 0 | \underset{\uparrow}{\hat{\phi}}(x), \underset{\uparrow}{\hat{\phi}}(y) | 0 \rangle = \Delta(x-y)$$

- Two fields, at two points x & y
- Encodes the effect / correlation of disturbance in ϕ field at x on the ϕ field at y - OR VICE-VERSA

Consider the operator $\hat{\Delta}_H(x, y) = [\hat{\phi}_H(x), \hat{\phi}_H(y)]$

$$\Rightarrow \hat{\Delta}_H = \int \frac{d^3p}{\sqrt{2\omega_p}} \int \frac{d^3q}{\sqrt{2\omega_q}} \left[(\hat{a}_p e^{-ipx} + \hat{a}_p^\dagger e^{+ipx}), (\hat{a}_q e^{-iqy} + \hat{a}_q^\dagger e^{+iqy}) \right]$$

(N.B. $p_0 = +\omega_p = \sqrt{p^2 + m^2}$) only non-zero $\propto \delta^3(p-q)$

$$\hat{\Delta}_H = \int \frac{d^3p}{2\omega_p} \left(e^{-ip(x-y)} - e^{+ip(x-y)} \right) \hat{1}_H$$

NOTES /

Unit operator
① equivalent to c-number

$$= \Delta(x-y) \hat{1}_H$$

② Green function/Solⁿ of K-G equation certain boundary conditions (see below) as its sum of $\frac{1}{\sqrt{2\omega_p}} e^{-ip(x-y)}$ where $p_0 = +\omega_p = \sqrt{p^2 + m^2}$

(3) Lorentz Invariant $e^{-ip \cdot (x-y)}$ $\Delta \frac{d^3 p}{2\omega_p}$
 $\Rightarrow \Delta$ is function of $(x-y)^2 = (x'-y')_\mu (x+y)^\mu$

(4) Causal Properties.

If Δ describes physical propagation then it must be ZERO for space-like separations
 $(x-y)^2 < 0$

Consider $x_0 = y_0$

$$\Rightarrow \Delta(x-y) = \int \frac{d^3 p}{2\omega_p} \left(e^{+ip \cdot (x-y)} \right) - \int \frac{d^3 p}{2\omega_p} \left(e^{-ip \cdot (x-y)} \right)$$

switch $\uparrow$ to $-p$
 $p_i' = -p_i$
 to see 2nd term cancels first

$$= 0$$

True $\forall (x-y)$ with $(x_0 - y_0) = 0$

$\Rightarrow$ True $\forall$ values of $(x-y)^2 < 0$ with $(x_0 - y_0) = 0$

But Δ function of $(x-y)^2$ only

$\Rightarrow \Delta(x-y) = 0 \quad \forall (x-y)^2 < 0$ QED

N.B. $(x-y)^2 > 0$ find $\Delta(x-y)$ can be non-zero

e.g. $x=y \Rightarrow \Delta(x-y) \sim \int \frac{d^3 p}{2\omega_p} \sin(\omega_p t)$ EP3

Wightman Function

Example of unphysical 2-pt function
but a useful building block.

Consider $D(x-y) = \langle 0 | \hat{\phi}(x) \hat{\phi}(y) | 0 \rangle$
 (Wightman Function) $= \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} e^{-ipx + iqy} \langle 0 | \hat{a}_p \hat{a}_q^\dagger | 0 \rangle$
 (other terms zero)

ie. this describes creation of a particle at y
and removal " " " " x

$\Rightarrow$ propagation of influence between x & y in
empty vacuum?

$$D(x-y) = \int \frac{d^3 p}{(2\pi)^3} e^{-ip(x-y)} = (D(y-x))^*$$

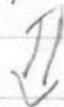
However this D is NOT physically relevant as it
is NON-ZERO for $(x-y)^2 < 0$

EPS e.g. $x_0 = y_0 \Rightarrow D(x-y) \sim e^{-m|x-y|}$ (Bessel's functions)

BACK TO Δ Commutator Again.

The commutator $\Rightarrow \Delta(x-y) = A(x-y) - D(y-x)$
is physically relevant

create quanta
at x which
influences y



For real scalar field particles
is its own anti-particle

we have two modes here, particle propagation with
corresponding anti-particle in opposite direction

? More 2p Greens Functions & Propagators

to
K.G. solⁿ
in Classical
M.T.

For linear p.d.e.s we can solve inhomogeneous p.d.e.s (cut) where $\phi(x)$ not a solution if $\phi(x)$ is using Green functions
ie. here

$$\left(\frac{\partial^2}{\partial x_0^2} - \nabla_x^2 + m^2 \right) G(x, y) = -i \delta^4(x - y)$$

Solution for ϕ at $(x_0, \underline{x})$ given Kick at y // External Kick at $(y_0, \underline{y})$ driving system

This describes how disturbance from interactions PROPAGATES through system BETWEEN Kicks ie. freely // These will come from non-linear interactions

Depends on B.C. (= Boundary Conditions)

eg. nothing before kick $\Rightarrow$ Retarded Propagator
waves absorbed by kick $\Rightarrow$ Advanced Propagator