

NOW ON
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Classically, we study propagation by solving the e.o.m., here the K-G equⁿ.

As a PDE, we solve KG using

GREEN FUNCTIONS $G(x,y) = G(x-y)$ by symmetry
 $G(x,y)$ is eff'd at x if hit system at y

$$(\partial_t^2 - \nabla^2 + m^2) G(x,y) = -i\delta^4(x-y)$$

$$\text{Let } G(x-y) = \int d^4p G(p) e^{-ip(x-y)}$$

$$\Rightarrow (*) = (\partial_t^2 - \nabla^2 + m^2) G(x-y)$$

$$= \int d^4p (p_0^2 - \mathbf{p}^2 + m^2) G(p) e^{-ip(x-y)}$$

$$\text{so if } G(p) = \frac{i}{p^2 - m^2} \text{ we get}$$

$$(*) \stackrel{?}{=} \int d^4p e^{-ip(x-y)} \stackrel{?}{=} -i\delta^4(x-y) \text{ as required}$$

$$\text{so } G(x-y) \stackrel{?}{=} \int d^4p \frac{i}{p^2 - m^2} e^{-ip(x-y)}$$

Too Quick! (a) PDE solⁿ/Green functions depend on boundary conditions

(b) what do we do about poles at $p_0 = \pm \omega_p \Leftrightarrow p^2 = m^2$

$$G(x,y) = \int_{-\infty}^{+\infty} dp_0 \int d^3p \frac{1}{(p_0 - \omega_p)(p_0 + \omega_p)} i e^{-ip_0(x-y)} e^{+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$$

$$p^2 - m^2 = p_0^2 - \mathbf{p}^2 - m^2 = (p_0 - \omega_p)(p_0 + \omega_p)$$

Answers Push poles off Re axis $\Leftrightarrow$ Different B.C

Retarded Propagator Δ_R G_R

B.C. G_R is zero if $x_0 < y_0$

Write this as $(\epsilon \rightarrow 0^+)(1 \gg \epsilon > 0)$

$t > 0$ $t < 0$

ω_p $-\omega_p$ $+\omega_p$

Im Re

$2\pi i \cdot 4$ P_0

$$i \Delta_R(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{i}{(p_0 + i\epsilon)^2 - \omega_p^2} e^{-ip(x-y)}$$

$$(p_0 - \omega_p + i\epsilon)(p_0 + \omega_p + i\epsilon)$$

$\frac{1}{p_0}$

$-\omega_p - i\epsilon$ $+\omega_p - i\epsilon$

- To do the p_0 integral we note that if
- a) $(x_0 - y_0) > 0$ we must complete p_0 contour at $\text{Im}(p_0) < 0$
- $e^{-ip_0(x_0 - y_0)} \sim e^{-i(-i\omega_p)(+ve)} \rightarrow 0$
- $\Rightarrow$ Pick up both poles
- b) $(x_0 - y_0) < 0$ complete at $\text{Im}(p_0) > 0 \Rightarrow$ No poles
- $e^{-ip_0(x_0 - y_0)} \sim e^{-i(+i\omega_p)(+ve)} \rightarrow 0$

EFS

residues

$$\Rightarrow \Delta_R(x-y) = \theta(x_0 - y_0) \int \frac{d^3 p}{2\omega_p} \left\{ e^{-i\omega_p(x_0 - y_0) + ip \cdot (x-y)} - e^{+i\omega_p(x_0 - y_0) + ip \cdot (x-y)} \right\}$$

Residue at pole

$$i \Delta_R(x-y) = \theta(x_0 - y_0) (D(x-y) - D(y-x))$$

$D(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ip \cdot x}$ $D(y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ip \cdot y}$

$\Rightarrow$ clearly solves K.G pde with initial conditions

also then $\Delta_R(x-y) = 0$ if $x_0 < y_0$ (nothing before (y))

EFS

Also from defⁿ of D we see

$$i \Delta_R(x-y) = \langle 0 | \theta(x_0 - y_0) [\hat{\phi}(x), \hat{\phi}(y)] | 0 \rangle$$

See PS4 for similar example Δ_A

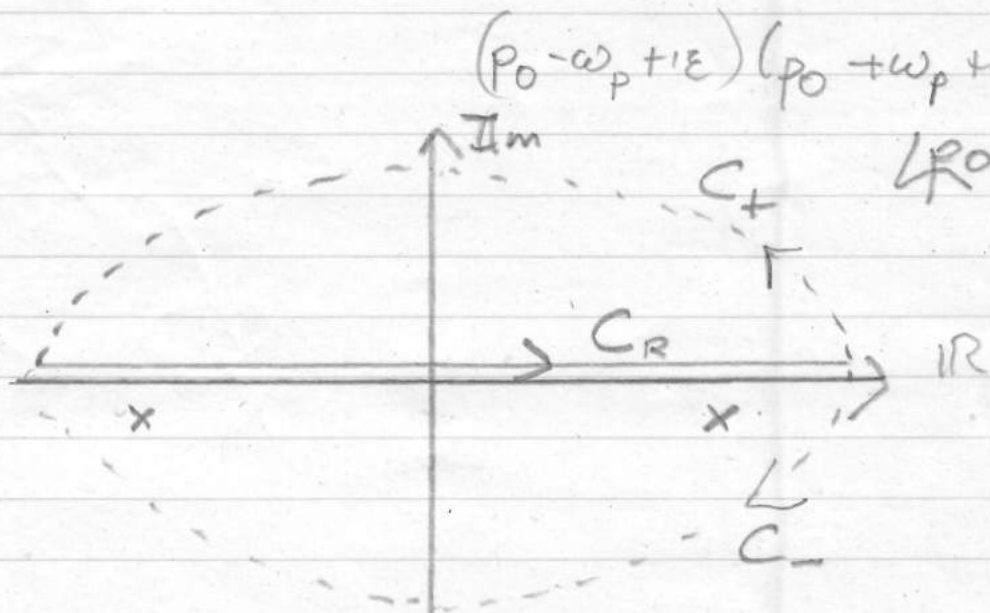
Retarded Green Function G_R

Boundary condition: $G_R(x, y) = 0$ if $x_0 < y_0$

Only reacts to delta function impulse at y^u

Solution

$$G_R(x, y) = G_R(x - y) = \int d^4 p \underbrace{\frac{i}{(p_0 + i\varepsilon)^2 - \omega_p^2}}_{(p_0 - \omega_p + i\varepsilon)(p_0 + \omega_p + i\varepsilon)} e^{-ip(x-y)}$$



coll
2/11/15

To do $\int dp_0$ we note

$$a) (x_0 - y_0) > 0 \quad \int_{C_-} dp_0 e^{-ip_0(x_0 - y_0)} = 0$$

$$e^{-i(-i\omega)(+ve)} \sim e^{-|\omega|}$$

$$b) (x_0 - y_0) < 0 \quad \int_{C_+} dp_0 e^{-ip_0(x_0 - y_0)} = 0$$

$$\Rightarrow \int_{C_R} dp_0 = \theta(x_0 - y_0) \int_{C_R + C_-} dp_0 + \theta(x_0 + y_0) \int_{C_R + C_+} dp_0$$

$$\Rightarrow G_R(x-y) = \Theta(x_0-y_0) \int d^3p e^{+ip_0(x-y)}$$

$$\left(\frac{e^{-i\omega_p(x_0-y_0)}}{2\omega_p} - \frac{1}{2\omega_p} e^{+i\omega_p(x_0-y_0)} \right)$$

change $p_i \rightarrow -p_i$
in this term

EPS $G_R(x-y) = \Theta(x_0-y_0) (D(x-y) - D(y-x))$

Clearly solves K-G as $G(p) \sim \frac{1}{p^2 m^2}$
& B.C. clearly retarded

Free QFT?

$$D(x-y) = \langle 0 | \hat{\phi}(x) \hat{\phi}(y) | 0 \rangle$$

$$\Rightarrow G_R(x-y) = \langle 0 | \Theta(x_0-y_0) [\hat{\phi}(x), \hat{\phi}(y)] | 0 \rangle$$

EPS $\underbrace{\Delta_R(x-y)}_{G_R(x-y)} \hat{1} = \Theta(x_0-y_0) [\hat{\phi}(x), \hat{\phi}(y)] = \hat{\Delta}_R$

Tong
62.7.1Feynman Propagator Δ_F

The most important for QFT, only see this later

Tong
(2.93) -
(2.95)

$$\Delta_F(x-y) := \langle 0 | T \hat{\phi}(x) \hat{\phi}(y) | 0 \rangle$$

$$\text{where } T(\hat{A}(t) \hat{B}(t')) = \begin{cases} \hat{A} \hat{B} & \text{if } t > t' \\ \hat{B} \hat{A} & \text{if } t < t' \end{cases}$$

$\uparrow \quad \uparrow$
 BIGGEST SMALLEST
 time

$T = \underline{\text{Time Ordering}}$, puts operators in order of their time largest (smallest) on left (right)

$$\Rightarrow \Delta_F(x-y) = \Theta(x_0 - y_0) D(x-y) \quad \textcircled{1} \\ + \Theta(y_0 - x_0) D(y-x) \quad \textcircled{2}$$

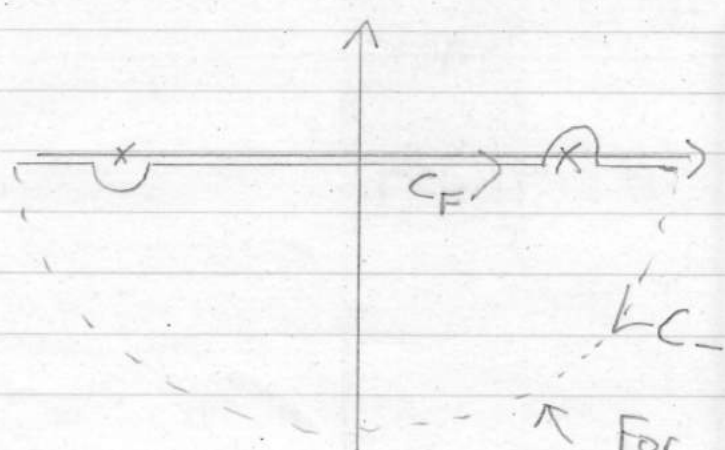
1st Term

$$\textcircled{1} = \Theta(x_0 - y_0) D(x-y)$$

$$= \Theta(x_0 - y_0) \int \frac{d^3 p}{(2\pi)^3} e^{-i\omega_p(x_0 - y_0) + i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})}$$

$$= \Theta(x_0 - y_0) \int \frac{d^3 p}{(2\pi)^3} \int \frac{d p_0}{2\pi i} \frac{1}{p_0 - \omega_p} \cdot \frac{1}{(p_0 + \omega_p)} e^{-i p_0(x_0 - y_0)} x e^{+i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})}$$

Reverse and
Distort contour to



Only pole
at $p_0 = +i\omega$
inside

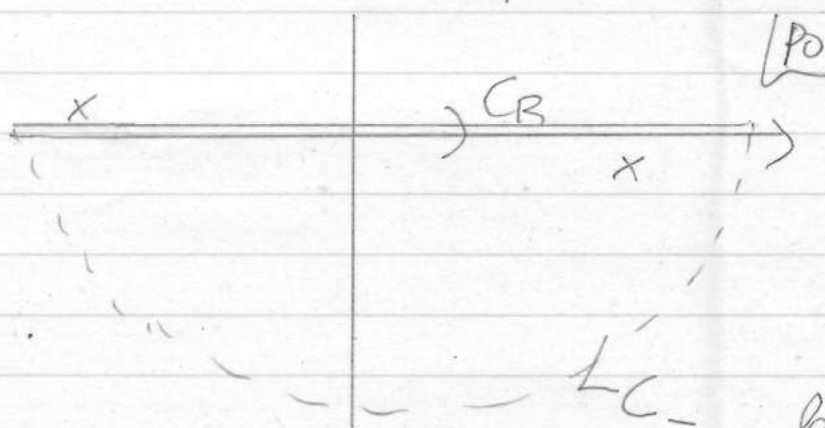
$$|e^{-ip_0(x_0 - y_0)}| \rightarrow 0$$

$$\Rightarrow \oint_{C_-} f(p_0) = 0$$

so can drop this

* $\downarrow \frac{1}{\epsilon} = -i$ matches contour
reverse sign.

$$\Rightarrow \textcircled{1} = \Theta(x_0 - y_0) \oint_{C_F} f(p_0) \frac{i}{(p_0 - \omega)(p_0 + \omega)} e^{-ip(x-y)}$$



OR

push poles
above/below

by $i\epsilon$

but leave
 $\oint_{C_F}$ along
real axis

$$\textcircled{1} = \Theta(x_0 - y_0) \oint_{C_F} f(p_0) \frac{i}{(p_0 - \omega - i\epsilon)(p_0 + \omega + i\epsilon)} e^{-ip(x-y)}$$

$$\Rightarrow \textcircled{1} = \theta(x_0 - y_0) \int d^4 p \frac{i}{p_0^2 - (\omega_p - i\epsilon)^2} e^{-ip(x-y)}$$

$\uparrow$
 Integrate for real p_0

$p^2 - \omega^2 + 2i\omega\epsilon$
 $\uparrow$
 +ive so $\equiv$ to $i\epsilon$

EVS Likewise 2nd term comes when we complete to upper half $\odot$ valid when $x_0 < y_0$

$$\Rightarrow \Delta_F(x-y) = \int d^4 p \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

EFS $(\partial_t^2 - \nabla^2 + m^2) \Delta_F(x-y) = -i \delta(x-y)$

with B.C. $e^{-i\omega t}$ only as $t \rightarrow +\infty$
 $e^{+i\omega t}$ " " $t \rightarrow -\infty$

EFS

e.o.L. 12

6/5/11