

5

INTERACTING QFT ϕ^4 I.1Consider $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

$$\Rightarrow \text{Classical e.o.m. } \partial_\mu \partial^\mu \phi + m^2 \phi + \frac{\lambda}{3!} \phi^3 = 0$$

~~~~~

Non-linear

$\Rightarrow$  solutions of Free  
equation different  $\phi^4$   
are MIXED

See this in QFT as  $\hat{\phi}^4$  term contains  
(using free fields)

$$\hat{\phi}^4 \sim \underbrace{\hat{a}_{p_1}^\dagger \hat{a}_{p_2}^\dagger \hat{a}_{p_3}^\dagger \hat{a}_{p_4}}_{\text{creation/annihilation}} \delta(p_1 + p_2 + p_3 - p_4) \text{ etc}$$

#s of  $\phi$  particles & their momenta  
changing

These NON-LINEARITIES  
capture INTERACTIONS between  
particles.

# Tong §3.1 Interaction Picture

We can solve free QFT EXACTLY so makes sense to build interacting theory around this.

Suppose we split (in ANY picture)

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$

|      |                                                       |                                                                                           |
|------|-------------------------------------------------------|-------------------------------------------------------------------------------------------|
| ↑    | ↑                                                     | ↑                                                                                         |
| Full | Solvable<br>Free<br>Quadratic<br>$O(\phi^n) n \leq 2$ | Difficult<br>Interactions / Non Linearities<br>Cubic or higher<br>$\phi^n \quad n \geq 3$ |

In the INTERACTION PICTURE (Subscript I.)  
- operators carry just  $H_0$ 's time dependence

Tong (3.11)  
p81

$$\Rightarrow \mathcal{O}_I(t) := e^{i\hat{H}_0 t} \mathcal{O}_S e^{-i\hat{H}_0 t} \quad (H_0 = H_{0,S})$$

$$\text{cf } \mathcal{O}_H(t) = e^{i\hat{H} t} \mathcal{O}_S e^{-i\hat{H} t} \quad \hat{H} = \hat{H}_S = \hat{H}_H$$

- states carry rest of time dependence

$$|\psi, t\rangle_I = e^{+iH_0 t} |\psi, t\rangle_S$$

as then

± Expectation values unchanged

$$\begin{aligned} \langle \psi, t | \mathcal{O}_S | \psi, t \rangle_S &= \langle \psi, t | (e^{-i\hat{H}_0 t} \mathcal{O}_I(t) e^{+i\hat{H}_0 t}) | \psi, t \rangle_S \\ &= \langle \psi, t | \mathcal{O}_I(t) | \psi, t \rangle_I \end{aligned}$$

Note convention here I, H, S all agree at  $t=0$

## Important Cases

(i)  $H_0$   $H_{0I}(t) = e^{iH_0 t} \hat{H}_{0S} e^{-iH_0 t} = \hat{H}_{0S}$

$H_{0S}$  is time independent  
 $\Rightarrow$  BUT  $H_{int I}, H_I$  are NOT as  $[H_S, H_{0S}], [H_{int S}, H_{0S}] \neq 0$

(ii)  $\hat{\phi}_I(t)$   $\hat{\phi}_I(t) = e^{i\hat{H}_0 t} \hat{\phi}_S e^{-i\hat{H}_0 t}$   
 $= (\hat{\phi}_H(t))|_{H_{int}=0}$

i.e.  $\hat{\phi}_I(t)$  is exactly as we had for free field case for our quadratic  $H_0$

Usual  $H_0 \Rightarrow \hat{\phi}_I(t) = \int \frac{d^3k}{\sqrt{2\omega_k}} (\hat{a}_k e^{-ikx} + \hat{a}_k^\dagger e^{+ikx})$   
 $k_0 = +\omega_k \geq 0$

## (iii) Products

Let  $\mathcal{O}_{12S} = \mathcal{O}_{1S} \mathcal{O}_{2S}$

$\Rightarrow \mathcal{O}_{12I}(t) = e^{iH_0 t} \mathcal{O}_{1S} \mathcal{O}_{2S} e^{-iH_0 t}$   
 $\quad \quad \quad e^{-iH_0 t} \cdot e^{+iH_0 t}$

$= \mathcal{O}_{1I}(t) \cdot \mathcal{O}_{2I}(t)$

(iv)  $H_{int}$  Suppose  $H_{int S} = \int d^3x \left( \frac{\lambda}{4!} \right) \hat{\phi}_S^4$

$\Rightarrow H_{int, I} = \int d^3x \left( \frac{\lambda}{4!} \right) \cdot \hat{\phi}_I(t)$

Typical Calculation

We want to know  $\mathcal{M} = \langle f, t_f | i, t_i \rangle$   
 the amplitude of the overlap of the  
 initial state evolved in time to  $t_f$   
 to compare with possible final state measured  
 at time  $t_f$ .

e.g.  $i$  = protons at LHC

$f$  = leptons measured as output

Our pet theory predicts  $|\mathcal{M}|^2$  as probability  
 for this process

$$\Rightarrow \mathcal{M} = {}_S \langle f, t_f | e^{i\hat{H}_S(t_f - t_i)} | i, t_i \rangle_S = \langle f, t_f | i, t_f \rangle$$

$$\text{as S.Equ}^n \quad \frac{d}{dt} | \psi, t \rangle_S = \hat{H}_S | \psi, t \rangle_S$$

but picture irrelevant so

$$\mathcal{M} = {}_I \langle f, t_f | i, t_i \rangle_I$$

$$= {}_I \langle \underbrace{f, t_f}_{\text{Measured output}} | \underbrace{U(t_f, t_i)}_{\text{Final}} | \underbrace{i, t_i}_{\text{Input state}} \rangle_I$$

$$\text{where } | \psi, t \rangle_I = U(t, t_0) | \psi, t_0 \rangle_I$$

Properties of  $U$ 

$$|\psi, t\rangle_I = U(t, t_0) |\psi, t_0\rangle_I$$

$$(i) \quad U(t, t) = 1$$

$$(ii) \quad U(t_2, t_1) U(t_1, t_0) = U(t_2, t_0)$$

(iii) Unitary

Normalisation constant

$$\Rightarrow \int_I \langle \psi, t | \psi, t \rangle_I = \int_I \langle \psi, t_0 | (U(t, t_0))^{\dagger} U(t, t_0) | \psi, t_0 \rangle_I$$

 $= \langle \psi, t_0 | \psi, t_0 \rangle$  needed

$$\Rightarrow [U(t, t_0)]^{\dagger} U(t, t_0) = 1 \quad \underline{\text{UNITARY}}$$

$$(iv) \quad \Rightarrow U(t, t_0)^{\dagger} = U(t_0, t) \quad \text{from (i)}$$

N.B. (ii)  $\int_I \langle \psi, t | \psi, t \rangle_I = \int_I \langle \psi, t_0 | (U(t, t_0))^{\dagger} U(t, t_0) | \psi, t_0 \rangle_I$



Time Evolution Equations

To find  $U(t, t_0)$  need time evolution of  $|4, t\rangle_I = e^{iH_0 t} |4, t_0\rangle_S$

S. Equ<sup>n</sup>  $i \frac{d}{dt} |4, t\rangle_S = H_S |4, t\rangle_S$

LHS  $i \frac{d}{dt} (e^{-iH_0 t} |4, t\rangle_I) = H_S e^{-iH_0 t} |4, t\rangle_I + e^{-iH_0 t} i \frac{d}{dt} |4, t\rangle_I$

RHS  $H_S |4, t\rangle_S = H_S e^{-iH_0 t} |4, t\rangle_I$

$\Rightarrow i \frac{d}{dt} |4, t\rangle_I = e^{iH_0 t} (H_S - H_{0S}) e^{-iH_0 t} |4, t\rangle_I$

$i \frac{d}{dt} |4, t\rangle_I = H_{int, I}(t) |4, t\rangle_I$

$\Rightarrow i \frac{d}{dt} U(t, t_0) = H_{int, I}(t) U(t, t_0)$

So I.p.c state evolution controlled by  $H_{int, I}$  made from I.p.c fields  $\hat{\phi}_I$  of known (free  $H_0$ ) time evolution

Time Evolution Sol<sup>n</sup>

$$\frac{d}{dt} |\psi, t\rangle_I = \hat{H}_{int, I}(t) |\psi, t\rangle_I$$

Looks like " $|\psi, t_2\rangle = e^{-i \int_{t_1}^{t_2} dt \hat{H}_{int, I}(t)} |\psi, t_1\rangle_I$ "

WRONG!

$\hat{H}_{int, I}$  is an operator  $\Rightarrow$  operator ordering issues

Correct answer

Let  $\varepsilon \rightarrow 0^+$  ( $1 \gg \varepsilon > 0$ )

$$\Rightarrow \frac{1}{\varepsilon} (i |\psi, t+\varepsilon\rangle_I - i |\psi, t\rangle_I) \simeq \hat{H}_{int, I}(t) |\psi, t\rangle_I$$

$$\Rightarrow |\psi, t+\varepsilon\rangle_I \simeq (1 - i\varepsilon \hat{H}_{int, I}(t)) |\psi, t\rangle_I$$

$$\simeq \exp \{ -i\varepsilon \hat{H}_{int, I}(t) \} |\psi, t\rangle_I$$

$$\Rightarrow |\psi, t+2\varepsilon\rangle_I \simeq e^{-i\varepsilon \hat{H}_{int, I}(t+\varepsilon)} e^{-i\varepsilon \hat{H}_{int, I}(t)} |\psi, t\rangle_I$$

$$\Rightarrow |\psi, t+N\varepsilon\rangle_I \simeq e^{-i\varepsilon \hat{H}_{int, I}(t+(N-1)\varepsilon)} \cdot e^{-i\varepsilon \hat{H}_{int, I}(t+(N-2)\varepsilon)}$$

$$\dots e^{-i\varepsilon \hat{H}_{int, I}(t+\varepsilon)} e^{-i\varepsilon \hat{H}_{int, I}(t)}$$

$$\cdot |\psi, t\rangle_I$$

$$\Rightarrow |\psi, t+N\varepsilon\rangle_I = T \exp \left\{ -i\varepsilon \sum_{n=0}^{N-1} \hat{H}_{int, I}(t+n\varepsilon) \right\} |\psi, t\rangle_I$$

$$= T \exp \left\{ -i \int dt \hat{H}_{int, I}(t) \right\} |\psi, t\rangle_I$$

Why no Campbell-Baker-Hausdorff formula

since  $e^A e^B = e^{A+B + \frac{1}{2}[A,B] + \dots}$  True

AND  $= T e^{A+B}$  True  
if  $A, B$  have times

EPS Prove