

5.3

WICK'S THEOREM 1950

Key tool (but related to general properties of gaussian noise)

Notation

$$\phi_i \equiv \hat{\phi}_i(x_i)$$

(so some $\phi_i = \phi_j$)

where $\hat{\phi}_i$ is any bosonic field
(possibly some/all the same field)
 $x_i \equiv x_i^m$ are distinct coordinates.
May take some $x_i^m = x_j^m$ at end

Normal Ordering

- Split every field into two parts.

$$\hat{\phi}_i = \hat{\phi}_i^+ + \hat{\phi}_i^-$$

plus sign NOT dagger † of h.c.

Arbitrary choice, so choose^{it} to simplify calculation

- A NORMAL ORDERED expression has all $\hat{\phi}^-$ to the left of all $\hat{\phi}^+$

e.g. $Ex = \hat{\phi}_3^- \hat{\phi}_6^- \hat{\phi}_1^- \hat{\phi}_2^+ \hat{\phi}_5^+ \hat{\phi}_4^+$

is normal ordered

Define NORMAL ORDER operator $N(\text{fields})$

$$N(\phi_1 \phi_2 \dots \phi_m)$$

(Acts on all fields
to right if no brackets)

$$= N((\phi_1^+ + \phi_1^-)(\phi_2^+ + \phi_2^-) \dots (\phi_m^+ + \phi_m^-))$$

1 Replace all ϕ_i by $(\phi_i^+ + \phi_i^-)$ and expand

2 Consider each of the 2^m terms,
each a product of $\phi_i^\pm$

3 Replace a ^{neighboring} pair $\phi_i^+ \phi_j^- \xrightarrow{\text{by}} \phi_j^- \phi_i^+$

4 Repeat 3 until no more $\phi^+ \phi^-$
i.e. ϕ^- are to left of all ϕ^+

BEST CHOICE

For us choose $\phi^\pm$ split s.t.

$$(i) \langle N(\text{fields}) \rangle = 0 \quad (ii) [\phi_i^+, \phi_j^+] = [\phi_i^-, \phi_j^-] = 0$$

Usual case

$$(iii) [\phi_i^+, \phi_j^-] \propto \mathbb{I} \text{ c-number}$$

If we want $\langle \mathcal{O} \rangle = \langle 0 | \mathcal{O} | 0 \rangle$
vacuum expectation values
then choose

$$\phi^+ = \int \frac{d^3k}{\sqrt{2\omega_k}} e^{-ikx} \hat{a}_k \sim \text{annihilation}$$

$$\phi^- = \int \frac{d^3k}{\sqrt{2\omega_k}} e^{+ikx} \hat{a}_k^\dagger \sim \text{creation}$$

as then

$$\Rightarrow [\phi_i^+, \phi_j^+] = [\phi_i^-, \phi_j^-] = 0$$

$$\text{AND } \langle N(\text{fields}) \rangle = \langle 0 | \phi^- \phi^- \dots \phi^+ \phi^+ | 0 \rangle = 0$$

In this course I will use std notation

$$: (\text{fields}) : \equiv N(\text{fields}) \quad \text{when } \phi^+, \phi^- \text{ chosen as here.}$$

Other cases.

Different choice of ϕ^+, ϕ^- is better if

(i) Symmetry Breaking

$$\langle 0 | \phi | 0 \rangle \neq 0$$

(ii) Many Body Problems

$$\sum_n \langle n | e^{-\beta E_n} \hat{a}^\dagger \hat{a} | n \rangle = \frac{1}{e^{\beta \omega} - 1} \neq 0$$

(iii) Curved Space Time

Two Point Functions - One field $\phi_i = \phi$ $i=1,2$ Defⁿ of Δ
 Define CONTRACTION of two fields

$$\text{Define } \Delta \text{ s.t. } T \phi(x) \phi(y) = N \phi(x) \phi(y) + \Delta(x, y)$$

$$\Rightarrow N \phi(x) \phi(y) = \underbrace{\phi^+(x) \phi^+(y)}_{\text{order irrelevant}} + \underbrace{\phi^-(x) \phi^+(y)}_{\text{only term different from } \phi(y) \phi(x)}$$

$$+ \underbrace{\phi^-(y) \phi^+(x)}_{\text{order irrelevant}} + \underbrace{\phi^-(x) \phi^-(y)}_{\text{order irrelevant}}$$

Only term different from $\phi(x) \phi(y)$

$x_0 > y_0$ $T \phi(x) \phi(y) = \phi(x) \phi(y)$

$$\Rightarrow \Delta(x, y) = \phi^+(x) \phi^-(y) - \phi^-(y) \phi^+(x)$$

$$= [\phi^+(x), \phi^-(y)]$$

$x_0 < y_0$ $T \phi(x) \phi(y) = \phi(y) \phi(x)$

$$\text{Exploit } [\phi^+, \phi^+] = [\phi^-, \phi^-] = 0$$

$$\Rightarrow \Delta(x, y) = [\phi^+(y), \phi^-(x)]$$

$$\Rightarrow \Delta(x, y) = \theta(x_0 - y_0) [\phi^+(x), \phi^-(y)]$$

$$+ \theta(y_0 - x_0) [\phi^+(y), \phi^-(x)]$$

Now if $[\phi^+, \phi^-] \propto \hat{1}$ a c-number

e.g. when $\phi^+ \sim \int \hat{a}$ $\phi^- \sim \int \hat{a}^\dagger$

$\Rightarrow$ CONTRACTION is a c-number

Invariably choose this to be so.

If it is a c-number AND if $\langle N(\text{fields}) \rangle = 0$
then

$$\langle T \phi(x) \phi(y) \rangle = \langle N \phi(x) \phi(y) \rangle + \overset{=0}{\Delta(x,y)}$$

$$\Rightarrow \Delta_F(x-y) = \Delta(x-y)$$

FEYNMAN PROPAGATOR = CONTRACTION

General Fields

Same result for best choice of split

$$\Delta_{F;ij}(x-y) \hat{1} = \Delta_{ij} \hat{1} = \underbrace{T \hat{\phi}_i \hat{\phi}_j - N \hat{\phi}_i \hat{\phi}_j}_{\text{MANY contractions zero if fields unrelated}}$$

MANY contractions
zero if fields
unrelated

See PS 5

Wick's Theorem

For ANY collection of fields, we have

$$T \phi_1 \phi_2 \dots \phi_n = \sum_{\text{ALL POSSIBLE CONTRACTIONS, 0 to } n/2} N(\overbrace{\phi_1 \dots \phi_n}^{\text{diagram of contractions}})$$

(Here $\phi_i = \phi_i(x_i)$ different locations x but $\phi_i = \phi$ allowed)

Defn A contraction of a pair of fields in an expression means replacing those operators with $\Delta_{12} = T \phi_1 \phi_2 - N \phi_1 \phi_2$

and is denoted

$$\left. \begin{aligned} \hat{A} \overbrace{\phi_1(x_1) \phi_2(x_2)}^{\text{contraction}} \hat{C} \\ = \hat{A} \hat{B} \hat{C} \underbrace{\Delta_{12}(x_1, x_2)} \end{aligned} \right\} \text{where } \Delta_{12} = \Delta_{12}(x_1, x_2) = T \phi_1 \phi_2 - N \phi_1 \phi_2$$

Simplification

must be c-number

Provided $[\phi_1^+(x_1), \phi_2^+(x_2)] = [\phi^-(x_1), \phi^-(x_2)] = 0$ (invariant sol)

$$\Delta_{12}(x_1, x_2) \equiv \theta(t_1 - t_2) [\phi_1^+(x_1), \phi_2^-(x_2)] + \theta(t_2 - t_1) [\phi_2^+(x_2), \phi_1^-(x_1)]$$

$$\Rightarrow \Delta_{12}(x_1, x_2) = \langle T \phi_1 \phi_2 \rangle - \langle N \phi_1 \phi_2 \rangle$$

Why use Wick?

If $\langle N \phi_1 \dots \phi_m \rangle = 0 \quad \forall m, \text{ any fields}$

$$\Rightarrow \mathcal{U} \sim \sum_m \langle T \phi_1 \dots \phi_m \rangle = \sum_m (\overbrace{\phi_1 \phi_2 \phi_3 \phi_3}^{\text{all contractions}} \dots) = \sum \text{products of c-number } \Delta$$

SEE PSS
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So Wick's Theorem is:-

$$T \phi_1 \phi_2 \dots \phi_n = \sum_{\text{ALL possible contractions}} N \phi_1 \phi_2 \dots \phi_n$$

$$= N(\phi_1 \phi_2 \dots \phi_n) \quad 0 \text{ contractions}$$

$$\{ N(\overbrace{\phi_1 \phi_2}^+ \phi_3 \dots \phi_n) \} \quad 1 \text{ contraction}$$

$$\{ N(\overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^+ \dots \phi_n) \} \quad \vdots \quad \left\{ \binom{n}{2} \text{ terms in total} \right\}$$

Note always coeff. of 1!

$$\{ N(\overbrace{\phi_1 \phi_2}^+ \overbrace{\phi_3 \phi_4}^+ \phi_5 \dots \phi_n) \} \quad 2 \text{ contractions}$$

$$\vdots \quad \left\{ \frac{1}{2} \frac{n!}{2!2!(n-4)!} \text{ terms} \right\}$$

+

 $\vdots$

$$\{ \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^+ \dots \overbrace{\phi_{n-1} \phi_n}^+ \} \quad \left(\frac{n}{2} \right) \text{ contractions}$$

$$\vdots \quad \left\{ \begin{array}{l} \text{Total} \\ [n-1(n-3)\dots 1] \\ \text{terms} \end{array} \right\} \quad n!!$$

Note $N(\phi_1 \dots \overbrace{\phi_i \dots \phi_j}^+ \dots \phi_n)$

$$= \overbrace{\phi_i \phi_j}^+ N(\phi_1 \dots \cancel{\phi_i} \dots \cancel{\phi_j} \dots \phi_n)$$

as c-number so commutes

Proof of Wick's THEOREM

Assume (i) $[\phi_i^+, \phi_j^+] = [\phi_i^-, \phi_j^-] = 0$ for simplicity

(ii) $[\phi_i^+, \phi_j^-] \propto \delta_{ij}$ c-number necessary

① T ordering is symmetric

$$T(\phi_1 \phi_2 \dots \phi_n) = T(\phi_{a_1} \phi_{a_2} \dots \phi_{a_n})$$

where $\begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$ is permutation

Why? T ordering fixes same order in both cases.

② N ordering is symmetric

$$N(\phi_1 \phi_2 \dots \phi_n) = N(\phi_{a_1} \phi_{a_2} \dots \phi_{a_n})$$

Why? N fixes order between $\phi^+ \triangle \phi^-$ but leaves order within ϕ^+ fixed

$$\phi_1^+ \phi_2^- \phi_3^+ = \phi_2^- \phi_1^+ \phi_3^+$$

1 then 3

③ HENCE

W.L.O.G. choose $t_1 > t_2 > \dots > t_n$

$$T \phi_1 \phi_2 \dots \phi_n = \phi_1 \phi_2 \dots \phi_n$$

(4) Assume Wick is true for $(n-1)$ fields or less

$$\begin{aligned}
 T\phi_1\phi_2\cdots\phi_n &= \phi_1 T\phi_2\phi_3\cdots\phi_n \\
 &= \phi_1 \sum_{\substack{\text{ALL} \\ \text{CONTRACTIONS} \\ \text{OF } 2 \dots n \text{ fields}}} N(\overbrace{\phi_1 \dots \phi_n}^{\text{rest of}}) \\
 &\quad (\Delta \dots \Delta) N(\text{fields})
 \end{aligned}$$

For each term $\phi_1 (\Delta \dots \Delta) N(\text{fields})$
 we need to add term with no contractions
 & all terms with one ϕ_1 contraction

$$\phi_1 \Delta \dots \Delta N(\text{r.o.f.}) = \Delta \Delta \dots \Delta N(\phi_1, \text{r.o.f.}) \quad \textcircled{A}$$

$$+ \Delta \Delta \dots \Delta \sum_{(i,j)} N(\overbrace{\phi_1 \dots \phi_n}^{\text{r.o.f.}}) \quad \textcircled{B}$$

So consider any one of these $\phi_1 N(\text{r.o.f.})$ terms.

$$\begin{aligned}
 \phi_1 N(\phi_2 \dots \phi_m) &= (\phi_1^+ + \phi_1^-) N(\phi_2 \dots \phi_m) \\
 &= \phi_1^+ N(\phi_2 \dots \phi_m) \\
 &\quad + N(\phi_1^- \phi_2 \dots \phi_m)
 \end{aligned}$$

↑
 Since ϕ_1^- will always be 1st
 under N operation

⇒ This term is part of A

We still want $N(\phi_1^+ \phi_2 \dots \phi_m)$ for A
 and all Δ_{ij} contractions for B.

$$\phi_1^+ N(\phi_2 \dots \phi_m) = N(\phi_2 \dots \phi_m) \phi_1^+ \quad \textcircled{C}$$

$$+ [\phi_1^+, N(\phi_2 \dots \phi_m)] \quad \textcircled{D}$$

Now

$$\textcircled{C} = N(\phi_2 \phi_3 \dots \phi_m) \phi_1^+ = N(\phi_2 \dots \phi_m \phi_1^+)$$

$\uparrow$ leaves ϕ_1^+ on far right

$$= N(\phi_1^+ \phi_2 \dots \phi_m) = \text{other half of term } \textcircled{A}$$

since N will push ϕ_1^+ to front of ϕ_j^+ 's but order within ϕ_j^+ is irrelevant as they all commute, so can push it to back = far left.

Other terms $\textcircled{D}$ & $\textcircled{B}$

$$[A, BC] = [A, B]C + B[A, C]$$

$$\Rightarrow [A, B_1 B_2 \dots B_m] = \sum_j B_1 B_2 \dots B_{j-1} [A, B_j] B_{j+1} \dots B_m$$

Apply to any term in $N\phi_2 \dots \phi_m \sim \sum \phi^- \dots \phi^- \phi^+ \dots \phi^+$

to see we get a sum over

$$\textcircled{D} = \sum_j N(\phi_2 \dots \underbrace{[\phi_1^+, \phi_j^-]} \dots \phi_m)$$

$$= [\phi_1^+, \phi_j^-]$$

$$= \Delta_{ij} \quad \text{as } t_i > t_j$$

so we have

$$\phi_1 N \phi_2 \dots \phi_m = N \phi_1 \phi_2 \dots \phi_m + \sum_{\substack{(1,j) \\ \text{contractions}}} N(\overbrace{\phi_1 \phi_2 \dots \phi_j \dots \phi_m})$$

Do this for each N term in $T\phi_2 \dots \phi_n$ which have $(n-1-2m)$ fields under N and m contractions.

Each normal ordered term gets

- (A) ϕ_1 absorbed to front of normal ordering
- ⊥ (B) ϕ_1 contracted to one field in normal ordering.

This gives us WICK for n fields if true for $(n-1)$ fields.

True for $n=2$ - definition of Δ_{ij}

$\Rightarrow$ True $\forall n$

QED.