

5.4FEYNMAN DIAGRAMS

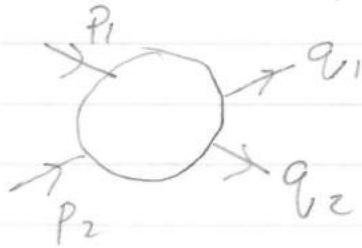
To illustrate

- We will use SYTh (Scalar Yukawa Theory)

Tong
§3.3.3
(3.25)

$$H_{\text{int}} = g \int d^3x \psi^\dagger(x) \psi(x) \phi(x)$$

- Consider $\psi\psi \rightarrow \psi\psi$ scattering



$$\mathcal{M} = \langle f | S | i \rangle$$

Initial state: 2 ψ 's momenta p_1 & p_2

Final state: 2 ψ 's momenta q_1 & q_2

(cut done in 12)

F2

Done

States

Here

$$|i\rangle = \sqrt{2\Omega_{p_1}} \sqrt{2\Omega_{p_2}} \hat{b}_{p_1}^{\dagger} \hat{b}_{p_2}^{\dagger} |0\rangle$$

Note that

$$\psi^{\dagger}(y) |0\rangle = \int \frac{d^3k}{\sqrt{2\Omega_k}} (\hat{b}_k^{\dagger} e^{+ik \cdot y} + c e^{-ik \cdot y}) |0\rangle$$

$$= \int \frac{d^3k}{\sqrt{2\Omega_k}} e^{+ik \cdot y} |k\rangle = \int \frac{d^3k}{2\Omega_k} e^{i\Omega_k t - i\vec{k} \cdot \vec{y}} |k\rangle$$

$$\Rightarrow \int d^3y e^{-ip \cdot y} 2\Omega_p (\psi^{\dagger}(y) |0\rangle) \quad \begin{cases} y_0 = t \\ p_0 = +\Omega_p \end{cases}$$

$$= \int d^3k \frac{2\Omega_p}{2\Omega_k} e^{+i(\Omega_p - \Omega_k)t} \delta^3(\vec{p} - \vec{k}) |k\rangle$$

$$= |k\rangle$$

$$\Rightarrow M = \langle f | S | i \rangle$$

$$= \prod_{i=1,2} \left(\int d^3y_i e^{-ip_i \cdot y_i} 2\Omega_{p_i} \right) (p_i \text{ in})$$
$$\prod_{f=1,2} \left(\int d^3z_f e^{+iq_f \cdot z_f} 2\Omega_{q_f} \right) (q_f \text{ out})$$

correspond to cut

$$\times \langle 0 | T \psi(z_1) \psi(z_2) S \psi^{\dagger}(y_1) \psi^{\dagger}(y_2) | 0 \rangle$$

4-point Green Function

$G(z_1, z_2, y_1, y_2)$ = Vacuum expectation value
of $\langle 0 | \psi \psi \psi^{\dagger} \psi^{\dagger} | 0 \rangle$
written in I picture

where $z_f^{\mu=0} = t_f \rightarrow +\infty$, $y_i^{\mu=0} = t_i \rightarrow -\infty$

Example $\psi\psi \rightarrow \psi\psi$ in SYTh (PS6, Q4)

We need $G = \langle 0 | T \psi(z_1) \psi(z_2) \psi^\dagger(y_1) \psi^\dagger(y_2) | 0 \rangle$

$$= \sum_n G_n(y_1, y_2, z_1, z_2)$$

where $G_n = \langle 0 | T \psi(z_1) \psi(z_2) \psi^\dagger(y_1) \psi^\dagger(y_2) \cdot \frac{(-ig)^n}{n!} \prod_{i=1}^n \left(\int d^4x_i \psi^\dagger(x_i) \psi(x_i) \right) | 0 \rangle$

$| 0 \rangle$

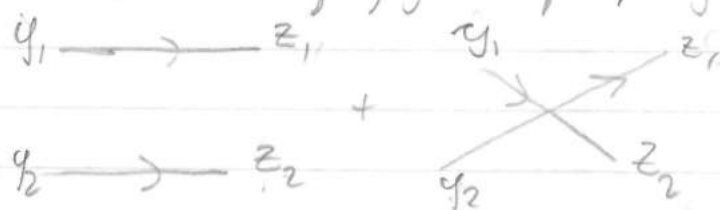
(a) $G_0 \propto g^0$

EFS EFS find two contributions to G_0 corresponding to terms in $\mathcal{L}_0$ in PS6, Q4ii

$$\Delta_\psi(y_1 - z_1) \Delta_\psi(y_2 - z_2) + \Delta_\psi(y_1 - z_2) \Delta_\psi(y_2 - z_1)$$

where $\Delta_\psi(x - y) = \overline{\psi(x)} \psi^\dagger(y)$

i.e. no **scattering**, just propagation



$\Rightarrow$ In momentum space find $\mathcal{M} \propto \delta^4(p_1 - q_1) \delta^4(p_2 - q_2) + \delta^4(p_1 - q_2) \delta^4(p_2 - q_1)$

PS6 Q4iii

EFS All odd powers of g are zero, follows from odd number of ϕ 's in expression + Wick's theorem $\Leftrightarrow \phi \psi / \psi^\dagger = 0$

(c) $\underline{G_2} \quad \underline{O(q^2)}$

$$G_2(y_1, y_2, z_1, z_2) = \frac{(-iq)^2}{2} \int d^4x_1 \int d^4x_2 \Gamma(x_1, x_2, y_1, y_2, z_1, z_2)$$

where

$$\Gamma = \langle 0 | T \psi(z_1) \psi(z_2) \psi^\dagger(x_1) \psi(x_1) \phi(x_1)$$

$$\psi^\dagger(x_2) \psi(x_2) \phi(x_2) \psi^\dagger(y_1) \psi(y_2)$$

$|0\rangle$

$\uparrow$
 $\mathbb{1}0$ field expectation
 value!

Apply Wick's theorem & use $\langle 0 | : \text{fields} : | 0 \rangle = 0$

$$\Rightarrow \Gamma = \langle 0 | T \psi(z_1) \psi(z_2) \psi^\dagger(x_1) \psi(x_1) \phi(x_1) \psi^\dagger(x_2) \psi(x_2) \phi(x_2) \psi^\dagger(y_1) \psi^\dagger(y_2) | 0 \rangle$$

$$= \textcircled{A} \overbrace{\psi(z_1) \psi(z_2) \psi^\dagger(x_1) \psi(x_1) \phi(x_1)}^{\text{contraction 1}} \overbrace{\psi^\dagger(x_2) \psi(x_2) \phi(x_2)}^{\text{contraction 2}} \underbrace{\psi^\dagger(y_1) \psi^\dagger(y_2)}_{\text{contraction 3}}$$

+ Many other terms with 5 contractions [945

$$\Gamma = \Gamma_A + (\text{rest})$$

$$\Gamma_A = \Delta_\psi(z_1 - x_1) \Delta_\psi(z_2 - x_2) \Delta_\phi(x_1 - x_2)$$

$$\Delta_\psi(x_1 - y_1) \Delta_\psi(x_2 - y_2)$$

NOTES

(i) Trivial Zero Terms

From $[a, b^\dagger] = 0$, $[a, a] = 0$ etc find

EFB
PS5

$$\begin{aligned} \text{a) } \overbrace{\psi \phi} &= 0 & \text{contractions of different} \\ \overbrace{\psi^\dagger \phi} &= 0 & \text{fields zero} \end{aligned}$$

b) $\overbrace{\psi \psi} = \overbrace{\psi^\dagger \psi^\dagger} = 0$ - Complex field with itself are zero

$\Rightarrow$ c) contractions non-zero only for field & (field) † h.c.

$$\overbrace{\psi(x) \psi^\dagger(y)} = \Delta_\psi(x-y)$$

$$\overbrace{\phi(x) \phi(y)} = \Delta_\phi(x-y)$$

same form as $\overbrace{\phi \phi}$ except ω_k, M not ω_k, m

$\Rightarrow$ so many terms zero.

[*** Only contractions between a field & its h.c. are non-zero in standard cases.]

e.o.L20 (ii) Some contribution to G all from other terms in Γ
30/11/15

e.g. x_1 & x_2 from H_{int} are identical
so can swap roles

$$\Gamma_B = \underbrace{\psi(z_1) \psi(z_2) \psi^+(x_1) \psi(x_1) \phi(x_1)}_{\psi^+(x_2) \psi(x_2) \phi(x_2) \psi^+(y_1) \psi^+(y_2)}$$

will give identical contribution to G as Γ_A

*** In fact permutation of internal vertex coord counts $\frac{1}{n!} H_{int}$

Help!!! 9.7.5.3.1 = 945 distinct pairs of

ALL ONE DIAGRAM ONE NUMBER 5 contractions here
Combinatorics FACTOR

(i) $\frac{1}{n!}$ from $S = T \sum \frac{(H_{int})^n}{n!}$ is largely cancelled

by permutation of x in H_{int}
e.g. $x_1 \leftrightarrow x_2$ in Γ_2 above

(ii) Each interaction term chosen with normalisation
to match permutations of fields

e.g. $\frac{\lambda}{4!} \phi^4$ as

$$\underbrace{\phi(x) \phi(x) \phi(x) \phi(x)}$$

$$\frac{g}{1!} \psi^+ \psi \phi$$

↑

No identical fields

4! ways of matching these
contractions

see
later

Handout

All possible Diagrams = 1, 2, 3, ... ?

* F. rules for 6×7 space

PSp94

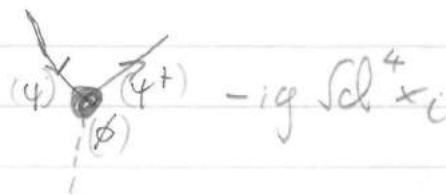
Coordinate Space Feynman Rules for 6×7 fields $10 \times$

AIM ONE DIAGRAM CAPTURES ALL CONTRIBUTIONS OF SAME VALUE

! Each term one H_{int} factor represented by a vertex with

- Unique coordinate x_i
- Integration $\int d^4x$
- $(-i) \times (\text{coupling constant})$ e.g. $-ig$ in SYTh (provided symm accounted for)
- legs/edges for each field in term

e.g. SYTh = Scalar Yukawa Theory, $H_{int} = g \phi^3 \psi \psi^\dagger$



e.o.L20
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3 Connect pairs of vertices with lines representing Δ_F

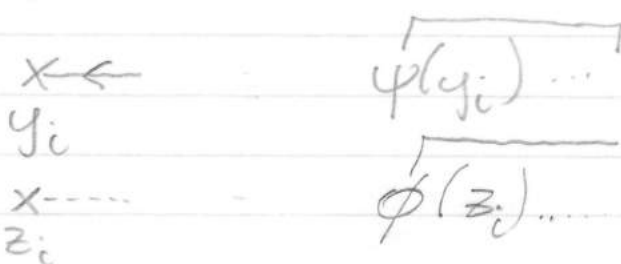
e.g. $\begin{matrix} (\phi) & (\phi) \\ \bullet & \bullet \\ x_i & x_j \end{matrix} \text{ --- } \Delta_F(x_i - x_j) = \overline{\phi(x_i) \phi(x_j)} = \langle 0 | T \phi(x_i) \phi(x_j) | 0 \rangle$

a) If charged field, $\psi \neq \psi^\dagger$, add arrow pointing FROM $\psi^\dagger$ TO ψ - Needed for

$\begin{matrix} x_i & x_j \\ \bullet & \bullet \\ (\psi) & (\psi^\dagger) \end{matrix} \text{ --- } \Delta_F(x_i - x_j) = \overline{\psi(x_i) \psi^\dagger(x_j)}$

$= \langle 0 | T \psi(x_i) \psi^\dagger(x_j) | 0 \rangle$
 $= \int d^4p e^{-ip(x_i - x_j)} \frac{i}{(p^2 - m^2 + i\epsilon)}$

3 For initial/final states have external vertex with one leg & one coordinate y_i / z_i but no integration



[There is a $\int d^4p e^{-ipx}$ factor when converting to $\mathcal{M}(\{A\})$ with p^h INTO diagram.

(Coord. Space F. Rules continued)

4 Divide by the "Symmetry Factor S "

(PSP93) Symmetry Factor & Combinatorics

Many terms found expanding $\langle \text{OIT}(\text{fields}) \rangle_0$ using Wick give same expression but we can avoid over counting

(i) $\frac{1}{n!}$ from $S = Te^{-iH_{int}t} = \sum \frac{(iH_{int}t)^n}{n!}$

This cancels with $n!$ ways of assigning labels x_i to each vertex

$\Rightarrow$ DO NOT consider these permutations
 $\sim \int dx_1 H(x_1) \int dx_2 H(x_2)$ contains

e.g. $\frac{1}{1 \times} \text{diagram} = \frac{1}{2!} \cdot \left(\text{diagram}_1 + \text{diagram}_2 \right)$

UNLABELLED $\uparrow$ $n!$ from S

Same as internal coordinates are integrated over, dummy indices.

C.O.L.21

4/12/15

(ii) Permutations at each vertex

Do not distinguish legs of same type at a vertex

= Include this factor in definition of coupling constant

Example? ϕ^4

i.e. $L_{int} = \pm (\text{coeff}) / (\# \text{Permutations})$

(ii) Permutations of Fields at Vertex

Example $\mathcal{L}_{int} = -\frac{\lambda}{4!} \phi^4$

$$H_{int} = -\mathcal{L}_{int}$$

One interaction factor produces terms like:

$$\dots \left(-i \frac{\lambda}{4!} \int d^4x \phi^1(x) \phi^2(x) \phi^3(x) \phi^4(x) \right) \dots$$
$$\dots \phi(x_A) \dots \phi(x_B) \dots \phi(x_C) \dots \phi(x_D)$$

$$= \frac{4!}{4!} \cdot -i\lambda \cdot \Delta(x-x_A) \Delta(x-x_B) \Delta(x-x_C) \Delta(x-x_D)$$

ALL $4!$ terms in WICK give ONE contribution = ONE Feynman Diagram.

SO DON'T DISTINGUISH IDENTICAL LEGS
AT A VERTEX

=
NO NEED TO INCLUDE PERMUTATION
CONSTANT IN F. RULES.

Here

$$\text{Diagram of a vertex with 4 legs} \equiv -i\lambda \int d^4x$$

Generally

$$\mathcal{L}_{int} = - \frac{(\text{c.c.}) (\text{fields})}{(\# \text{ permutations of fields})}$$

$$= \text{Diagram of a vertex with } n \text{ legs} \cdot -i(\text{c.c.}) \int d^4x$$

NO other constants

(ii) Symmetry Factor S Definition

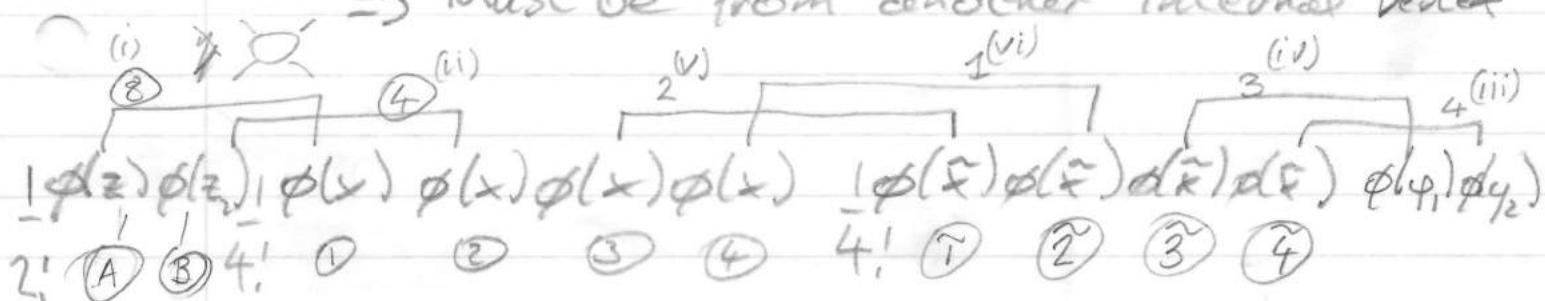
S = Number of permutations of internal lines which give same diagram

e.g. suppose we consider

$$x_A = x_B = \tilde{x}$$

in above example 1

$\Rightarrow$ Must be from another internal vertex



$$(3C = \Delta(x_3 - x_4))$$

Normally we have 4 distinct terms

$$3C, 4D, \tilde{1}\tilde{A}, \tilde{2}\tilde{B}$$

$$3C, 4D, \tilde{2}\tilde{A}, \tilde{1}\tilde{B}$$

$$4C, 3D, \tilde{1}\tilde{A}, \tilde{2}\tilde{B}$$

$$4C, 3D, \tilde{2}\tilde{A}, \tilde{1}\tilde{B}$$

to cancel

$$\frac{1}{4!}$$

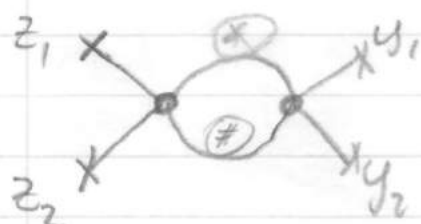
Here only two distinct terms in Wick

$$3\tilde{1}, 4\tilde{2}$$

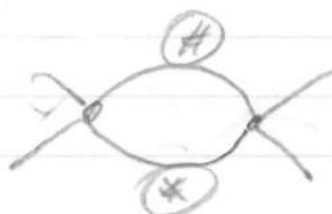
$$3\tilde{2}, 4\tilde{1}$$

$\Rightarrow \frac{1}{2}$ left over

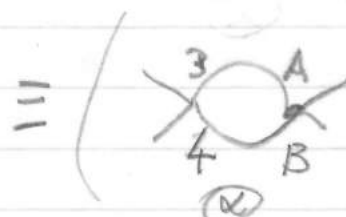
SHOW WICK



One Diagram



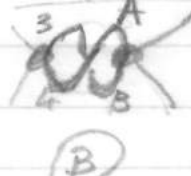
Symmetry $\Rightarrow S=2$



$\neq 3A, 4B$

Term

in Wick



$3B, 4A$

Term

in Wick

$4B, 3A$

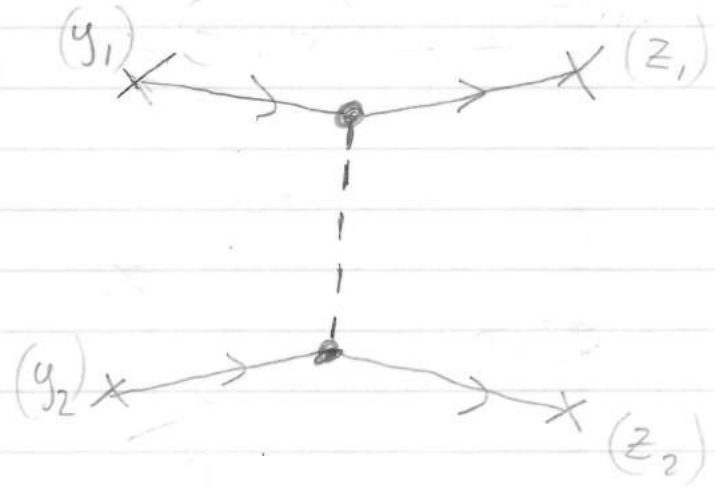
$= \alpha$

SAME AGAIN

NOT new term in Wick
 $\Rightarrow S=2$

Theorem

Each ^{distinct} contribution to $\mathcal{M}_G$ corresponds to a topologically distinct Feynman diagram

e.g. $\left(\frac{\Gamma_A + \Gamma_B}{2!} \right) =$  $(\mathcal{L}=1)$

(i) $\longrightarrow$ (f)

See PS for more examples
PS6

$$= (-iq)^2 \int d^4x_1 d^4x_2$$

$$\times \Delta(x_1 - y_1) \Delta_\psi(z_1 - x_1) \Delta_\phi(x_1 - x_2) \Delta_\psi(x_2 - y_2) \Delta_\psi(z_2 - x_2)$$

COL 22/11/15

Do this
for
next
year

RECIPE

- 1 Write down external legs for initial/final state fields
- 2 Write down n vertices if working at order n .
Do this in all possible ways if more than one type of vertex
- 3 Connect up legs of external lines/vertices in all possible distinct ways.