UNIVERSITY OF LONDON

MSci EXAMINATION May 2007

**for Internal Students of Imperial College of Science, Technology and Medicine**

*This paper is also taken for the relevant Examination for the Associateship*

# QUANTUM FIELD THEORY

**For Fourth-Year Physics Students**

Thursday 17th May 2007: 14.00 to 16.00

*Answer THREE questions.*

*All questions carry equal marks.*

*Marks shown on this paper are indicative of those the Examiners anticipate assigning.*

## General Instructions

Write your CANDIDATE NUMBER clearly on each of the THREE answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

**You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.**

**1.** Consider the classical real scalar field $\phi(x)$ with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 \ .$$

(i) What is the momentum density $\pi(x)$ conjugate to $\phi(x)$? Show that the Hamiltonian $H$ is given by

$$H = \int \mathrm{d}^3x \left(\tfrac{1}{2}\pi^2 + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \tfrac{1}{2}m^2\phi^2\right) \ .$$

[4 marks]

(ii) Using the Klein–Gordon equation $\left(\partial_\mu\partial^\mu + m^2\right)\phi = 0$, show that the tensor

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \tfrac{1}{2}\eta^{\mu\nu}\left(\partial_\lambda\phi\partial^\lambda\phi - m^2\phi^2\right)$$

is conserved, that is $\partial_\mu T^{\mu\nu} = 0$.

[3 marks]

(iii) What do the corresponding conserved charges

$$Q^\mu = \int \mathrm{d}^3x T^{0\mu}$$

represent physically? Show explicitly that one component of $Q^\mu$ can be related to $H$. Noether's theorem relates these charges to a symmetry of the action. What is the symmetry?

[5 marks]

(iv) Now consider

$$M^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu.$$

Using $\partial_\mu T^{\mu\nu} = 0$ (or otherwise) show that $\partial_\mu M^{\mu\nu\rho} = 0$.

[2 marks]

(v) Show that, for the plane wave solution $\phi(x) = A\mathrm{e}^{-\mathrm{i}p\cdot x} + A^*\mathrm{e}^{\mathrm{i}p\cdot x}$, the conserved charges arising from $M^{\mu\nu\rho}$ are given by

$$Q^{\mu\nu} = -2\int \mathrm{d}^3x\, p^0|A|^2\left(x^\mu p^\nu - x^\nu p^\mu\right),$$

and that

$$Q^\mu = 2\int \mathrm{d}^3x\, p^0|A|^2 p^\mu.$$

(You may assume that oscillating terms integrate to zero.)

Given the first expression, what do the charges $Q^{ij}$ with $i, j = 1, 2, 3$ represent? What symmetry of the action leads to the conserved currents $M^{\mu\nu\rho}$?

[6 marks]

[TOTAL 20 marks]

**2.** Consider a free real scalar field with, in the Heisenberg picture,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left( a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x} \right)$$

$$\pi(x) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_{\mathbf{p}}}{2}} \left( a_{\mathbf{p}} e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x} \right) ,$$

where $p^0 = E_{\mathbf{p}}$ and $\pi(x)$ is the momentum density conjugate to $\phi(x)$.

(i) The equal time commutation relations (ETCRs) state that

$$\left[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})\right] = i\delta^3(\mathbf{x} - \mathbf{y}).$$

What are the ETCRs for $\left[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})\right]$ and $\left[\pi(t, \mathbf{x}), \pi(t, \mathbf{y})\right]$ ? [2 marks]

(ii) Show that

$$a_{\mathbf{k}} = \int d^3x \frac{e^{ip\cdot x}}{\sqrt{2E_{\mathbf{k}}}} \left[E_{\mathbf{k}}\phi(t, \mathbf{x}) + i\pi(t, \mathbf{x})\right] .$$

[4 marks]

(iii) Hence, using the ETCRs, show that

$$\left[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger\right] = (2\pi)^2\delta^3(\mathbf{p} - \mathbf{q}),$$

$$\left[a_{\mathbf{p}}, a_{\mathbf{q}}\right] = 0, \qquad \left[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger\right] = 0 .$$

[6 marks]

(iv) Using the results from part 2(iii), show that at unequal times

$$\left[\phi(x), \phi(y)\right] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left( e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right).$$

[4 marks]

(v) Show that when $x - y$ is spacelike, $\left[\phi(x), \phi(y)\right] = 0$ . What is the physical implication of this result?

[4 marks]

[TOTAL 20 marks]

**3.** Consider the classical Dirac field $\psi(x)$ satisfying the Dirac equation

$$\left(\mathrm{i}\gamma^\mu \partial_\mu - m\right) \psi(x) = 0 \, ,$$

where the gamma matrices $\gamma^\mu$ satisfy $\left\{\gamma^\mu, \gamma^\nu\right\} = 2\eta^{\mu\nu}\mathbb{I}$. Under a Lorentz transformation $x^\mu \mapsto x'^\mu = \left(\Lambda^{-1}\right)^\mu{}_\nu x^\nu$ the Dirac field maps as

$$\psi(x) \mapsto \psi'(x) = \Lambda_{\frac{1}{2}}\, \psi(\Lambda^{-1}x) \, ,$$

where $\Lambda_{\frac{1}{2}}$ is a matrix acting on the spinor $\psi$.

(i) Show that

$$\partial_\mu = \left(\Lambda^{-1}\right)^\nu{}_\mu \partial'_\nu \, ,$$

where $\partial'_\nu = \partial/\partial x'^\nu$ . [2 marks]

(ii) You are given that $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$. Hence show that if $\psi(x)$ satisfies the Dirac equation then

$$\left(\mathrm{i}\gamma^\mu \partial_\mu - m\right) \psi'(x) = \Lambda_{\frac{1}{2}} \left(\mathrm{i}\gamma^\mu \partial'_\mu - m\right) \psi(x') = 0 \, .$$

Comment on the significance of this result. [6 marks]

(iii) Using the definition $\bar{\psi}(x) = \psi^\dagger\gamma^0$ and $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ show that if $\psi(x)$ satisfies the Dirac equation then

$$\mathrm{i}\partial_\mu\bar{\psi}(x)\gamma^\mu + m\bar{\psi}(x) = 0 \, .$$

[4 marks]

(iv) Hence show that the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved, that is $\partial_\mu j^\mu = 0$. [3 marks]

(v) The matrix $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ has the property that $\left\{\gamma_5, \gamma_\mu\right\} = 0$. Show that the 'axial vector' current $j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ satisfies

$$\partial_\mu j_5^\mu = 2\mathrm{i}m\bar{\psi}\gamma_5\psi \, .$$

Comment on what this result implies about the symmetries of the Dirac equation when $m = 0$. [5 marks]

[TOTAL 20 marks]

**4.** Consider a free Dirac field in the Heisenberg picture, with

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{s=1}^{2} \left[ a_\mathbf{p}^s u^s(p) e^{-ip\cdot x} + b_\mathbf{p}^{s\dagger} v^s(p) e^{ip\cdot x} \right]$$

where $p^0 = E_\mathbf{p}$. The non-vanishing anti-commutation relations are

$$\left\{ a_\mathbf{p}^r, a_\mathbf{q}^{s\dagger} \right\} = \left\{ b_\mathbf{p}^r, b_\mathbf{q}^{s\dagger} \right\} = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) \delta^{rs} \; .$$

(i) Define the vacuum state $|0\rangle$, the single fermion state $|\mathbf{p}, s, -\rangle$, and the single anti-fermion state $|\mathbf{p}, s, +\rangle$.
Define the two-fermion state, and show that the particles have Fermi statistics.
[4 marks]

(ii) The Hamiltonian can be written as

$$\begin{aligned} H &= \; : \int d^3x \bar{\psi} \left( -i\gamma^i \nabla_i + m \right) \psi \; : \\ &= \int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^{2} E_\mathbf{p} \left( a_\mathbf{p}^{s\dagger} a_\mathbf{p}^s + b_\mathbf{p}^{s\dagger} b_\mathbf{p}^s \right) . \end{aligned}$$

To what are the normal ordered expressions $: a_{\mathbf{p}_1}^{s_1} a_{\mathbf{p}_2}^{s_2\dagger} :$, $: a_{\mathbf{p}_1}^{s_1} b_{\mathbf{p}_2}^{s_2\dagger} :$ and $: a_{\mathbf{p}_1}^{s_1} b_{\mathbf{p}_2}^{s_2\dagger} b_{\mathbf{p}_3}^{s_3\dagger} :$ equal?
[2 marks]

(iii) Show that

$$\left[ AB, C \right] = A \left\{ B, C \right\} - \left\{ A, C \right\} B \; .$$

[2 marks]

(iv) Hence show that $|0\rangle$, $|\mathbf{p}, s, -\rangle$ and $|\mathbf{p}, s, +\rangle$ are eigenstates of $H$ and give the eigenvalues. Comment very briefly on the physical meaning of this result. [6 marks]

(v) Furthermore, show that $\psi(x)$ satisfies the Heisenberg equation of motion

$$\partial_t \psi = i \left[ H, \psi \right] .$$

[6 marks]

[TOTAL 20 marks]

**5.** (i) Consider the interacting Yukawa theory with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}M^2\phi^2 + \bar{\psi}\left(\mathrm{i}\gamma^\mu\partial_\mu - m\right)\psi - g\phi\bar{\psi}\psi.$$

Define the free Lagrangian density $\mathcal{L}_0$ and the interaction Lagrangian density $\mathcal{L}_{\text{int}}$. Give the corresponding interaction Hamiltonian $H_{\text{int}}$. [3 marks]

(ii) In the Heisenberg picture operators $\mathcal{O}_H(t)$ evolve with time, while states $|\psi\rangle_H$ are fixed, that is

$$\partial_t\mathcal{O}_H = \mathrm{i}\big[H_H, \mathcal{O}_H\big], \qquad \partial_t\,|\psi\rangle_H = 0\,,$$

where $H_H$ is the total Hamiltonian in the Heisenberg picture.

What are the corresponding expressions for the evolution of operators $\mathcal{O}_I(t)$ and states $|\psi, t\rangle_I$ in the interaction picture, in terms of $H_{0,I}$ and $H_{\text{int},I}$ (the free and interaction Hamiltonians in the interaction picture)? [3 marks]

(iii) Let operators in the two pictures be related by

$$\mathcal{O}_I(t) = U(t,t_0)\mathcal{O}_H(t)U(t,t_0)^{-1}\,,$$

where $U(t_0,t_0) = 1$. Show that, for any operator $\mathcal{O}_I(t)$ ,

$$\partial_t\mathcal{O}_I = \big[(\partial_t U)\,U^{-1} + \mathrm{i}H_I, \mathcal{O}_I\big]\,,$$

where $H_I$ is the total Hamiltonian in interaction picture. [6 marks]

(iv) Hence show that

$$\partial_t U = -\mathrm{i}H_{\text{int},I}U\ .$$

Show that this equation is consistent with $U$ being a unitary operator. [4 marks]

(v) As an expansion in $H_{\text{int},I}$, show explicitly that

$$U(t,t_0) = 1 - \mathrm{i}\int_{t_0}^{t}\mathrm{d}t_1\,H_{\text{int},I}(t_1) + \ldots$$

is a solution for $U(t,t_0)$ up to first order in $H_{\text{int},I}$. [4 marks]

[TOTAL 20 marks]

**6.** Consider the scattering of two $\phi$ particles with momenta $p_1$ and $p_2$ to two $\phi$ particles with momenta $p_1'$ and $p_2'$ in scalar $\lambda\phi^4$ theory. The $S$-matrix is given by

$$S = T \exp\left(-\mathrm{i}\int \mathrm{d}^4x : \tfrac{1}{4!}\lambda\phi^4 :\right) ,$$

where $\phi(x)$ in the interaction picture has the expansion

$$\phi(x) = \phi^+(x) + \phi^-(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}p\cdot x} + a_{\mathbf{p}}^\dagger \mathrm{e}^{\mathrm{i}p\cdot x}\right) ,$$

with $p^0 = E_{\mathbf{p}}$ and the only non-zero commutator is $\left[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger\right] = (2\pi)^2\delta^3(\mathbf{p}-\mathbf{q})$.

(i) Define the states $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$ for the two incoming particles with momenta $p_1$ and $p_2$ and the two outgoing particles with momenta $p_1'$ and $p_2'$ respectively. [2 marks]

(ii) Given that

$$\left[\phi^+(x), a_{\mathbf{p}}^\dagger\right] = \frac{1}{\sqrt{2E_{\mathbf{p}}}}\mathrm{e}^{-\mathrm{i}p\cdot x} ,$$

show that

$$\phi^+(x)^2\,|\mathrm{in}\rangle = 2\mathrm{e}^{-\mathrm{i}p_1\cdot x}\mathrm{e}^{-\mathrm{i}p_2\cdot x}\,|0\rangle .$$

[5 marks]

(iii) Using $\phi^-(x) = \phi^+(x)^\dagger$ hence show that

$$\langle\mathrm{out}|\,\phi^-(x)^2 = 2\mathrm{e}^{\mathrm{i}p_1'\cdot x}\mathrm{e}^{\mathrm{i}p_2'\cdot x}\,\langle 0| .$$

[2 marks]

(iv) Show that

$$:\phi(x)^4: \; = \phi^+(x)^4 + 4\phi^-(x)\phi^+(x)^3 + 6\phi^-(x)^2\phi^+(x)^2 + 4\phi^-(x)^3\phi^+(x) + \phi^-(x)^4 .$$

[3 marks]

(v) Using the results from parts 6(ii), 6(iii) and 6(iv), show that $S^{(1)}_{\phi\phi\to\phi\phi}$, the contribution to $\langle\mathrm{out}|\,S\,|\mathrm{in}\rangle$ at order $\lambda$, is given by

$$S^{(1)}_{\phi\phi\to\phi\phi} = -\mathrm{i}\lambda\,(2\pi)^4\delta^4(p_1' + p_2' - p_1 - p_2) .$$

[6 marks]

(vi) Draw the Feynman diagram corresponding to $S^{(1)}_{\phi\phi\to\phi\phi}$. [2 marks]

[TOTAL 20 marks]

**End**