Imperial College London

MSci EXAMINATION May 2009

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY

For 4th-Year Physics Students

Thursday, 21st May 2009: 10:00 to 12:00

Answer THREE questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. Consider the classical real scalar field $\phi(x)$ with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi).$$

You are given that the classical Poisson bracket satisfies

$$\big\{\phi(t,\boldsymbol{x}),\pi(t,\boldsymbol{y})\big\}_{\mathrm{PB}} = -\big\{\pi(t,\boldsymbol{y}),\phi(t,\boldsymbol{x})\big\}_{\mathrm{PB}} = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}),$$

while $\big\{\phi(t,\boldsymbol{x}),\phi(t,\boldsymbol{y})\big\}_{\mathrm{PB}} = 0$ and $\big\{AB,C\big\}_{\mathrm{PB}} = A\big\{B,C\big\}_{\mathrm{PB}} + B\big\{A,C\big\}_{\mathrm{PB}}$.

(i) What is the definition of the momentum density $\pi(x)$ conjugate to $\phi(x)$? What is it equal to in this case?
Rewrite $\mathcal{L}$ in terms of $\dot\phi$ and $\nabla\phi$ and hence show that the Hamiltonian H is given by

$$H = \int \mathrm{d}^3x\,\left(\tfrac{1}{2}\pi^2 + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi + V(\phi)\right).$$

What is $V(\phi)$ for a free scalar field of mass m? What is the minimum value of H in this case? [6 marks]

ANSWER: *By definition,* $\pi = \partial\mathcal{L}/\partial\dot\phi = \dot\phi$. *Here* $\mathcal{L} = \frac{1}{2}\dot\phi^2 - \frac{1}{2}\nabla\phi\cdot\nabla\phi - V(\phi)$. *Hence*

$$\begin{aligned} H = \int \mathrm{d}^3x\,\left(\pi\dot\phi - \mathcal{L}\right) &= \int \mathrm{d}^3x\,\left(\dot\phi^2 - \tfrac{1}{2}\dot\phi^2 + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi + V(\phi)\right) \\ &= \int \mathrm{d}^3x\,\left(\tfrac{1}{2}\pi^2 + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi + V(\phi)\right), \end{aligned}$$

as required. For a free scalar field of mass m, $V(\phi) = \frac{1}{2}m^2\phi^2$. *Since* H *is then a sum of squares,* $H \geq 0$, *with the minimum realised by* $\phi = 0$.

(ii) Define

$$\begin{aligned} T^{\mu\nu} &= \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\left(\tfrac{1}{2}\partial_\lambda\phi\partial^\lambda\phi - V(\phi)\right), \\ M^{\mu\nu\rho} &= T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu. \end{aligned}$$

Show that $\partial_\mu T^{\mu\nu} = 0$ if ϕ satisfies the equation of motion $\partial_\mu\partial^\mu\phi + V'(\phi) = 0$ (where $V'(\phi) = \mathrm{d}V/\mathrm{d}\phi$) and hence that $\partial_\mu M^{\mu\nu\rho} = 0$. [5 marks]

ANSWER: *By the chain rule,* $\partial_\mu V(\phi) = (\partial_\mu\phi)V'(\phi)$ *where* $V'(\phi) = \mathrm{d}V(\phi)/\mathrm{d}\phi$, *hence*

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= \partial_\mu(\partial^\mu\phi\partial^\nu\phi) - \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu(\partial_\lambda\phi\partial^\lambda\phi) + \eta^{\mu\nu}\partial_\mu V(\phi) \\ &= (\partial_\mu\partial^\mu\phi)\partial^\nu\phi + (\partial^\mu\phi)(\partial_\mu\partial^\nu\phi) - (\partial^\lambda\phi)(\partial^\nu\partial_\lambda\phi) + (\partial^\nu\phi)V'(\phi) \\ &= (\partial^\nu\phi)\left(\partial_\mu\partial^\mu\phi + V'(\phi)\right) \\ &= 0. \end{aligned}$$

We have $\partial_\lambda x^\sigma = \delta_\lambda{}^\sigma$ hence,

$$\begin{aligned}\partial_\mu M^{\mu\nu\rho} &= \partial_\mu \left(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu\right)\\ &= (\partial_\mu T^{\mu\nu})x^\rho + T^{\mu\nu}(\partial_\mu x^\rho) - (\partial_\mu T^{\mu\rho})x^\nu - T^{\mu\rho}(\partial_\mu x^\nu)\\ &= T^{\mu\nu}\delta_\mu{}^\rho - T^{\mu\rho}\delta_\mu{}^\nu\\ &= T^{\rho\nu} - T^{\nu\rho}\\ &= 0,\end{aligned}$$

since by definition $T^{\mu\nu} = T^{\nu\mu}$.

(iii) Consider the integrals

$$Q^\mu = \int \mathrm{d}^3x T^{0\mu}(t,\boldsymbol{x}),$$
$$Q^{\mu\nu} = \int \mathrm{d}^3x M^{0\mu\nu}(t,\boldsymbol{x}).$$

How do they depend on t? What does Q^μ represent physically? What about Q^{ij}, where $i,j=1,2,3$?

Write down the components of Q^μ and show explicitly that one is related to H. [4 marks]

ANSWER: *They are Noether charges, independent of t since the corresponding currents $T^{\mu\nu}$ and $M^{\mu\nu\rho}$ are conserved. Physically Q^μ correspond to the total conserved four-momentum P^μ. The three charges $J^i = \frac{1}{2}\epsilon^{ijk}Q^{jk}$ are the total angular momentum of the field. Explicitly we have*

$$T^{00} = \partial^0\phi\partial^0\phi - \eta^{00}\left(\tfrac{1}{2}\partial_\lambda\phi\partial^\lambda\phi - V(\phi)\right) = \dot{\phi}^2 - \mathcal{L},$$
$$T^{0i} = \partial^0\phi\partial^i\phi - \eta^{0i}\left(\tfrac{1}{2}\partial_\lambda\phi\partial^\lambda\phi - V(\phi)\right) = -\pi\boldsymbol{\nabla}^i\phi.$$

Hence $Q^\mu = (P^0, \boldsymbol{P})$ where

$$\boldsymbol{P} = -\int \mathrm{d}^3x\pi\boldsymbol{\nabla}\phi,$$

and $P^0 = H$.

(iv) Show that $\left\{Q^\mu, \phi(x)\right\}_{\mathrm{PB}} = -\partial^\mu\phi(x)$ and comment on the result.

Comment briefly on what you expect for the Poisson brackets between different components of Q^μ and $Q^{\mu\nu}$. [5 marks]

ANSWER: *The only non-zero contributions to $\left\{T^{0\mu}(t,\boldsymbol{y}),\phi(t,\boldsymbol{x})\right\}_{PB}$ are*

$$\begin{aligned}\left\{T^{00}(t,\boldsymbol{y}),\phi(t,\boldsymbol{x})\right\}_{PB} &= \tfrac{1}{2}\left\{\pi^2(t,\boldsymbol{y}),\phi(t,\boldsymbol{x})\right\}_{PB} = \pi(t,\boldsymbol{y})\left\{\pi(t,\boldsymbol{y}),\phi(t,\boldsymbol{x})\right\}_{PB}\\ &= -\pi(t,\boldsymbol{y})\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}) = -\partial^0\phi(t,\boldsymbol{y})\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}),\\ \left\{T^{0i}(t,\boldsymbol{y}),\phi(t,\boldsymbol{x})\right\}_{PB} &= -\left\{\pi(t,\boldsymbol{y})\boldsymbol{\nabla}^i\phi(t,\boldsymbol{y}),\phi(t,\boldsymbol{x})\right\}_{PB} = -\nabla^i\phi(t,\boldsymbol{y})\left\{\pi(t,\boldsymbol{y}),\phi(t,\boldsymbol{x})\right\}_{PB}\\ &= \nabla^i\phi(t,\boldsymbol{y})\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}) = -\partial^i\phi(t,\boldsymbol{y})\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}).\end{aligned}$$

Since Q^μ are independent of time we can choose to evaluate them at $t = x^0$ so that

$$\begin{aligned}\left\{Q^\mu, \phi(x)\right\}_{PB} &= \int d^3y \left\{T^{0\mu}(t, \boldsymbol{y}), \phi(t, \boldsymbol{x})\right\}_{PB} \\ &= -\int d^3y \partial^\mu \phi(t, \boldsymbol{y}) \delta^{(3)}(\boldsymbol{y} - \boldsymbol{x}) \\ &= -\partial^\mu \phi(x).\end{aligned}$$

This reflects the fact that Q^μ are the generators of the translation symmetry group. Similarly $Q^{\mu\nu}$ are the generators of the Lorentz symmetry group. Collectively $\{Q^\mu, Q^{\mu\nu}\}$ form a representation of the Poincaré group under the Poisson bracket.

[Total 20 marks]

2. Consider a free real scalar field $\phi(x)$ with conjugate momentum density $\pi(x) = \dot{\phi}(x)$. Define the operator

$$a_{\boldsymbol{p}} = \int d^3x \frac{e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}}{\sqrt{2E_{\boldsymbol{p}}}} \left[E_{\boldsymbol{p}}\phi(0,\boldsymbol{x}) + i\pi(0,\boldsymbol{x})\right],$$

where $E_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2}$.

(i) The equal-time commutation relations (ETCRs) state that

$$\left[\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})\right] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}).$$

Are the fields $\phi(x)$ and $\pi(x)$ in the Schrödinger or Heisenberg picture? Why is this picture more natural in a relativistic theory?
What are the ETCRs for $\left[\phi(t,\boldsymbol{x}), \phi(t,\boldsymbol{y})\right]$ and $\left[\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})\right]$? [4 marks]

ANSWER: *The fields operators are in the Heisenberg picture since they explicitly depend on time. This means they are function of all four coordinates x^μ. This is the natural picture in a relativistic theory, since one can then write relativistically invariant expressions. The ETCRs for the remaining commutators are $\left[\phi(t,\boldsymbol{x}), \phi(t,\boldsymbol{y})\right] = \left[\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})\right] = 0$.*

(ii) Derive an expression for $a^\dagger_{\boldsymbol{p}}$ in terms of $\phi(0,\boldsymbol{x})$ and $\pi(0,\boldsymbol{x})$. Using the ETCRs show that

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}),$$
$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}\right] = 0, \qquad \left[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = 0.$$

[6 marks]

ANSWER: *By definition, since ϕ and hence π are Hermitian,*

$$a^\dagger_{\boldsymbol{p}} = \int d^3x \frac{(e^{-i\boldsymbol{p}\cdot\boldsymbol{x}})^*}{\sqrt{2E_{\boldsymbol{p}}}} \left[E_{\boldsymbol{p}}\phi(0,\boldsymbol{x}) + i\pi(0,\boldsymbol{x})\right]^\dagger$$
$$= \int d^3x \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{\sqrt{2E_{\boldsymbol{p}}}} \left[E_{\boldsymbol{p}}\phi(0,\boldsymbol{x}) - i\pi(0,\boldsymbol{x})\right].$$

[This question continues overleaf . . .]

Hence

$$\begin{aligned}
\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] &= \int \mathrm{d}^3x\,\mathrm{d}^3y \frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}}{\sqrt{2E_{\boldsymbol{p}}}} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}}}{\sqrt{2E_{\boldsymbol{q}}}} \left[E_{\boldsymbol{p}}\phi(0,\boldsymbol{x}) + \mathrm{i}\pi(0,\boldsymbol{x}), E_{\boldsymbol{q}}\phi(0,\boldsymbol{y}) - \mathrm{i}\pi(0,\boldsymbol{y})\right] \\
&= \int \mathrm{d}^3x\,\mathrm{d}^3y \frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}+\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}}}{\sqrt{4E_{\boldsymbol{p}}E_{\boldsymbol{q}}}} \Big(\mathrm{i}E_{\boldsymbol{q}}\left[\pi(0,\boldsymbol{x}), \phi(0,\boldsymbol{y})\right] - \mathrm{i}E_{\boldsymbol{p}}\left[\phi(0,\boldsymbol{x}), \pi(0,\boldsymbol{y})\right]\Big) \\
&= \int \mathrm{d}^3x\,\mathrm{d}^3y \frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}+\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}}}{\sqrt{4E_{\boldsymbol{p}}E_{\boldsymbol{q}}}} (-\mathrm{i})(E_{\boldsymbol{p}} + E_{\boldsymbol{q}})\mathrm{i}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \\
&= \int \mathrm{d}^3x \frac{E_{\boldsymbol{p}} + E_{\boldsymbol{q}}}{\sqrt{4E_{\boldsymbol{p}}E_{\boldsymbol{q}}}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}} \\
&= (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}).
\end{aligned}$$

Similarly

$$\begin{aligned}
\left[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}\right] &= \int \mathrm{d}^3x\,\mathrm{d}^3y \frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}}{\sqrt{2E_{\boldsymbol{p}}}} \frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}}}{\sqrt{2E_{\boldsymbol{q}}}} \left[E_{\boldsymbol{p}}\phi(t,\boldsymbol{x}) + \mathrm{i}\pi(t,\boldsymbol{x}), E_{\boldsymbol{q}}\phi(t,\boldsymbol{y}) + \mathrm{i}\pi(t,\boldsymbol{y})\right] \\
&= \int \mathrm{d}^3x\,\mathrm{d}^3y \frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}}}{\sqrt{4E_{\boldsymbol{p}}E_{\boldsymbol{q}}}} \Big(\mathrm{i}E_{\boldsymbol{q}}\left[\pi(t,\boldsymbol{x}), \phi(t,\boldsymbol{y})\right] + \mathrm{i}E_{\boldsymbol{p}}\left[\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})\right]\Big) \\
&= \int \mathrm{d}^3x\,\mathrm{d}^3y \frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}}}{\sqrt{4E_{\boldsymbol{p}}E_{\boldsymbol{q}}}} \mathrm{i}(E_{\boldsymbol{p}} - E_{\boldsymbol{q}})\mathrm{i}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \\
&= \int \mathrm{d}^3x \frac{E_{\boldsymbol{q}} - E_{\boldsymbol{p}}}{\sqrt{4E_{\boldsymbol{p}}E_{\boldsymbol{q}}}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{x}} \\
&= 0,
\end{aligned}$$

since $E_{-\boldsymbol{p}} = E_{\boldsymbol{p}}$. We also have $\left[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = -\left[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}\right]^\dagger = 0$.

(iii) Given

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}p\cdot x} + a^\dagger_{\boldsymbol{p}}\mathrm{e}^{\mathrm{i}p\cdot x}\right),$$

where $p^0 = E_{\boldsymbol{p}}$, use the results from part ii to show that at unequal times

$$\left[\phi(x), \phi(y)\right] = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)}\right).$$

What properties must this commutator have if the field theory is to respect microcausality? [5 marks]

ANSWER: *Substituting we have*

$$
\begin{aligned}
&\big[\phi(x), \phi(y)\big] \\
&\quad = \int \frac{d^3q}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{q}}}\sqrt{2E_{\boldsymbol{p}}}} \big[\big(a_{\boldsymbol{q}} e^{-iq\cdot x} + a^\dagger_{\boldsymbol{q}} e^{iq\cdot x}\big), \big(a_{\boldsymbol{p}} e^{-ip\cdot y} + a^\dagger_{\boldsymbol{p}} e^{ip\cdot y}\big)\big] \\
&\quad = \int \frac{d^3q}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{4E_{\boldsymbol{q}}E_{\boldsymbol{p}}}} \Big(\big[a_{\boldsymbol{q}}, a^\dagger_{\boldsymbol{p}}\big] e^{-iq\cdot x} e^{ip\cdot y} + \big[a^\dagger_{\boldsymbol{q}}, a_{\boldsymbol{p}}\big] e^{iq\cdot x} e^{-ip\cdot y}\Big) \\
&\quad = \int \frac{d^3q}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{4E_{\boldsymbol{q}}E_{\boldsymbol{p}}}} \Big((2\pi)^3 \delta^{(3)}(\boldsymbol{q}-\boldsymbol{p}) e^{-iq\cdot x + ip\cdot y} \\
&\qquad\qquad - (2\pi)^3 \delta^{(3)}(\boldsymbol{q}-\boldsymbol{p}) e^{-iq\cdot x + ip\cdot y}\Big) \\
&\quad = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{q}}} \Big(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}\Big).
\end{aligned}
$$

Microcausality implies that operators commute if they are space-like separated. In particular this requires $\big[\phi(x), \phi(y)\big] = 0$ *if* $(x-y)^2 < 0$.

(iv) You are given that $\delta(x^2 - a^2) = (1/2a)\,[\delta(x-a) - \delta(x+a)]$. Show that the expression for $\big[\phi(x), \phi(y)\big]$ given in part iii can be rewritten as

$$
\big[\phi(x), \phi(y)\big] = \int \frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2) e^{-ip\cdot(x-y)},
$$

where $d^4p = d^3p\, dp^0$. Comment on the Lorentz transformation properties of this expression, and give a brief argument that the ETCRs take the same form in all inertial frames.

What is the significance of this result? [5 marks]

ANSWER: *We have* $p^2 - m^2 = (p^0)^2 - E^2_{\boldsymbol{p}}$. *Using the given relation we then have*

$$
\begin{aligned}
&\int \frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2) e^{-ip\cdot(x-y)} \\
&\quad = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(e^{-ip\cdot(x-y)}\Big|_{p^0=E_{\boldsymbol{p}}} - e^{-ip\cdot(x-y)}\Big|_{p^0=-E_{\boldsymbol{p}}}\right) \\
&\quad = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(e^{-iE_{\boldsymbol{p}}\cdot(x^0-y^0) + i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} - e^{iE_{\boldsymbol{p}}\cdot(x^0-y^0) + i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right) \\
&\quad = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right),
\end{aligned}
$$

where we have changed variables $\boldsymbol{p} \to -\boldsymbol{p}$ *in the second term. Since the new expression is manifestly Lorentz covariant, this shows that* $\big[\phi(x), \phi(y)\big]$ *is Lorentz covariant.*

Thus, evaluating in an arbitrary inertial frame,

$$\begin{aligned}
\left[\phi(t,\boldsymbol{x}),\phi(t,\boldsymbol{y})\right] &= \left[\phi(x),\phi(y)\right]_{x^0=y^0=0} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\left(\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}-\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right) \\
&= 0 \qquad \textit{(odd integrand)}, \\
\left[\phi(t,\boldsymbol{x}),\pi(t,\boldsymbol{y})\right] &= \partial_{y^0}\left[\phi(x),\phi(y)\right]_{x^0=y^0=0} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{i}}{2}\left(\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}+\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right) \\
&= \mathrm{i}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \\
\left[\pi(t,\boldsymbol{x}),\pi(t,\boldsymbol{y})\right] &= \partial_{x^0}\partial_{y^0}\left[\phi(x),\phi(y)\right]_{x^0=y^0=0} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{E_{\boldsymbol{p}}}{2}\left(\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}-\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right) \\
&= 0 \qquad \textit{(odd integrand)}.
\end{aligned}$$

Thus although we chose a particular inertial frame to define the ETCRs, the resulting quantum theory is independent of the original choice of frame, as required for a relativistic theory.

[Total 20 marks]

3. This question is about the free classical Dirac field $\psi(x)$. The Lagrangian is given by

$$\mathcal{L} = \mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi,$$

where the gamma matrices γ^μ satisfy $\left\{\gamma^\mu, \gamma^\nu\right\} = 2\eta^{\mu\nu}\mathbb{1}$ and $\bar{\psi} = \psi^\dagger\gamma^0$.

(i) The field $\psi(x)$ is a spinor. How many components does it have? After quantization what is the spin of the corresponding single-particle states?

Treating $\psi(x)$ and $\bar{\psi}(x)$ as independent fields, show that the Euler–Lagrange equation for $\bar{\psi}(x)$ is the Dirac equation

$$\mathrm{i}\gamma^\mu\partial_\mu\psi - m\psi = 0.$$

[4 marks]

ANSWER: *A spinor field has four components. After quantization, the corresponding single particle states are spin one-half. The Euler–Lagrange equations read* $\partial_\mu(\partial\mathcal{L}/\partial(\partial_\mu\bar{\psi})) = \partial\mathcal{L}/\partial\bar{\psi}$. *Since* $\partial\mathcal{L}/\partial(\partial_\mu\bar{\psi}) = 0$ *we have*

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = \mathrm{i}\gamma^\mu\partial_\mu\psi - m\psi = 0,$$

as required.

(ii) Using $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ show that if $\psi(x)$ satisfies the Dirac equation then

$$\mathrm{i}\partial_\mu\bar{\psi}(x)\gamma^\mu + m\bar{\psi}(x) = 0.$$

Hence show that the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved, that is $\partial_\mu j^\mu = 0$, if ψ satisfies the Dirac equation. [4 marks]

ANSWER: *Taking the conjugate of the Dirac equation*

$$\begin{aligned} 0 &= -\left(\mathrm{i}\gamma^\mu\partial_\mu\psi - m\psi\right)^\dagger\gamma^0 \\ &= \mathrm{i}\partial_\mu\psi^\dagger(\gamma^\mu)^\dagger\gamma^0 + m\psi^\dagger\gamma^0 \\ &= \mathrm{i}\partial_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^0 + m\psi^\dagger\gamma^0 \\ &= \mathrm{i}\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi}. \end{aligned}$$

Hence

$$\begin{aligned} \partial_\mu j^\mu &= \partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right) \\ &= (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi) \\ &= \mathrm{i}m\bar{\psi}\psi - \mathrm{i}m\bar{\psi}\psi \\ &= 0, \end{aligned}$$

as required.

(iii) Show that the complex conjugate of $\mathcal{L}$ is given by

$$\mathcal{L}^* = \mathcal{L} - \mathrm{i}\partial_\mu \left(\bar{\psi}\gamma^\mu\psi\right).$$

Consider the real Lagrangian

$$\mathcal{L}' = \tfrac{1}{2}\mathrm{i}\left[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi\right] - m\bar{\psi}\psi.$$

Treating $\psi(x)$ and $\bar{\psi}(x)$ as independent fields, show that the corresponding Euler–Lagrange equation for $\bar{\psi}(x)$ is again the Dirac equation. [5 marks]

ANSWER: *We have* $(\gamma^0)^\dagger = \gamma^0\gamma^0\gamma^0 = \gamma^0$*, so that*

$$\begin{aligned}
\mathcal{L}^* &= \left(\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi\right)^* \\
&= -\mathrm{i}\left(\psi^\dagger\gamma^0\gamma^\mu\partial_\mu\psi\right)^\dagger - \left(m\psi^\dagger\gamma^0\psi\right)^\dagger \\
&= -\mathrm{i}(\partial_\mu\psi^\dagger)(\gamma^\mu)^\dagger\gamma^0\psi - m\psi^\dagger\gamma^0\psi \\
&= -\mathrm{i}(\partial_\mu\psi^\dagger)\gamma^0\gamma^\mu\gamma^0\gamma^0\psi - m\psi^\dagger\gamma^0\psi \\
&= -\mathrm{i}(\partial_\mu\bar{\psi})\gamma^\mu\psi - m\bar{\psi}\psi \\
&= \mathcal{L} - \mathrm{i}\left[(\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi)\right] \\
&= \mathcal{L} - \mathrm{i}\partial_\mu(\bar{\psi}\gamma^\mu\psi).
\end{aligned}$$

For $\mathcal{L}'$ *we have*

$$\frac{\partial\mathcal{L}'}{\partial(\partial_\mu\bar{\psi})} = -\tfrac{1}{2}\mathrm{i}\gamma^\mu\psi, \qquad \frac{\partial\mathcal{L}'}{\partial\bar{\psi}} = \tfrac{1}{2}\mathrm{i}\gamma^\mu\partial_\mu\psi - m\psi,$$

so that the Euler–Lagrange equation reads

$$\partial_\mu\left(-\tfrac{1}{2}\mathrm{i}\gamma^\mu\psi\right) - \tfrac{1}{2}\mathrm{i}\gamma^\mu\partial_\mu\psi + m\psi = -\mathrm{i}\gamma^\mu\partial_\mu\psi + m\psi = 0,$$

as required.

(iv) The matrix $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ has the property that $\{\gamma_5, \gamma_\mu\} = 0$. Show that the "axial vector" current $j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ satisfies

$$\partial_\mu j_5^\mu = 2\mathrm{i}m\bar{\psi}\gamma_5\psi.$$

What does this result imply about the symmetries of the Dirac equation when $m = 0$?

Identify the corresponding infinitesimal transformation of ψ and demonstrate directly whether or not $\mathcal{L}$ is invariant under this transformation. [7 marks]

ANSWER: *We have*

$$\begin{aligned}
\partial_\mu j_5^\mu &= \partial_\mu\left(\bar{\psi}\gamma^\mu\gamma_5\psi\right) \\
&= (\partial_\mu\bar{\psi})\gamma^\mu\gamma_5\psi + \bar{\psi}\gamma^\mu\gamma_5(\partial_\mu\psi) \\
&= (\partial_\mu\bar{\psi})\gamma^\mu\psi - \bar{\psi}\gamma_5\gamma^\mu(\partial_\mu\psi) \\
&= \mathrm{i}m\bar{\psi}\gamma_5\psi - \mathrm{i}m\bar{\psi}\gamma_5\psi \\
&= 2\mathrm{i}m\bar{\psi}\gamma_5\psi.
\end{aligned}$$

Thus if $m = 0$ the axial current is conserved and we have an additional symmetry of the Dirac equation (by Noether's theorem). The corresponding infinitesimal transformation is

$$\begin{aligned}\delta\psi &= i\alpha\gamma_5\psi,\\ \delta\bar{\psi} &= (i\alpha\gamma_5\psi)^\dagger\gamma^0 = -i\alpha\psi^\dagger\gamma_5^\dagger\gamma^0 = i\alpha\psi^\dagger\gamma^0\gamma_5\gamma^0\gamma^0 = i\alpha\bar{\psi}\gamma_5,\end{aligned}$$

since

$$\gamma_5^\dagger = (i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger = -i\gamma^0(\gamma^3\gamma^2\gamma^1\gamma^0)\gamma^0 = -i\gamma^0(\gamma^0\gamma^1\gamma^2\gamma^3)\gamma^0 = -\gamma^0\gamma_5\gamma_0.$$

Hence

$$\begin{aligned}\delta\mathcal{L} &= i\left[\delta\bar{\psi}\gamma^\mu\partial\psi + \bar{\psi}\gamma^\mu\partial_\mu(\delta\psi)\right] - m\left[\delta\bar{\psi}\psi + \bar{\psi}\delta\psi\right]\\ &= (i\alpha)i\left[\bar{\psi}\gamma_5\gamma^\mu\partial_\mu\psi + \bar{\psi}\gamma^\mu\gamma_5\partial_\mu\psi\right] - (i\alpha)m\left[\bar{\psi}\gamma_5\psi + \bar{\psi}\gamma_5\psi\right]\\ &= -2i\alpha m\bar{\psi}\gamma_5\psi.\end{aligned}$$

Thus we have $\delta\mathcal{L} = 0$ only if $m = 0$ as expected.

[Total 20 marks]

4. Consider a free Dirac field $\psi(x)$ given by

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{2} \left[a^s_{\boldsymbol{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\boldsymbol{p}} v^s(p) e^{ip\cdot x} \right],$$

where $p^0 = E_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2}$. The only non-vanishing anti-commutation relations are

$$\left\{ a^r_{\boldsymbol{p}}, a^{s\dagger}_{\boldsymbol{q}} \right\} = \left\{ b^r_{\boldsymbol{p}}, b^{s\dagger}_{\boldsymbol{q}} \right\} = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \delta^{rs}.$$

The Hamiltonian H and the operator Q are given by

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} E_{\boldsymbol{p}} \left(a^{s\dagger}_{\boldsymbol{p}} a^s_{\boldsymbol{p}} + b^{s\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}} \right),$$

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} \left(a^{s\dagger}_{\boldsymbol{p}} a^s_{\boldsymbol{p}} - b^{s\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}} \right).$$

(i) Define the vacuum state $|0\rangle$ and the single particle states $|\boldsymbol{p}, s, -\rangle$ and $|\boldsymbol{p}, s, +\rangle$. What do the labels $\boldsymbol{p}$, s and $\pm$ denote?
Define a two-particle state, and show that the particles are fermions.
[6 marks]

ANSWER: *By definition*

$$a^s_{\boldsymbol{p}}|0\rangle = b^s_{\boldsymbol{p}}|0\rangle = 0, \qquad |\boldsymbol{p}, s, -\rangle = \sqrt{2E_{\boldsymbol{p}}} a^{s\dagger}_{\boldsymbol{p}}|0\rangle, \qquad |\boldsymbol{p}, s, +\rangle = \sqrt{2E_{\boldsymbol{p}}} b^{s\dagger}_{\boldsymbol{p}}|0\rangle.$$

The label $\boldsymbol{p}$ *denotes the three-momentum of the state,* $s = 1, 2$ *denotes the two spin states, and* $\pm$ *denote the charge of the state,* $+$ *for a positron and* $-$ *for an electron. The two electron state is*

$$|\boldsymbol{p}, r, -; \boldsymbol{q}, s, -\rangle = \sqrt{2E_{\boldsymbol{p}}}\sqrt{2E_{\boldsymbol{q}}} a^{r\dagger}_{\boldsymbol{p}} a^{s\dagger}_{\boldsymbol{q}}|0\rangle = -\sqrt{2E_{\boldsymbol{p}}}\sqrt{2E_{\boldsymbol{q}}} a^{s\dagger}_{\boldsymbol{q}} a^{r\dagger}_{\boldsymbol{p}}|0\rangle$$
$$= -|\boldsymbol{q}, s, -; \boldsymbol{p}, r, -\rangle,$$

demonstrating that the particles obey fermi statistics.

(ii) Show that

$$a^{r\dagger}_{\boldsymbol{q}} a^r_{\boldsymbol{q}}|\boldsymbol{p}, s, -\rangle = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \delta^{rs}\, |\boldsymbol{q}, r, -\rangle,$$
$$b^{r\dagger}_{\boldsymbol{q}} b^r_{\boldsymbol{q}}|\boldsymbol{p}, s, +\rangle = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \delta^{rs}\, |\boldsymbol{q}, r, +\rangle.$$

Hence show that $|0\rangle$, $|\boldsymbol{p}, s, -\rangle$ and $|\boldsymbol{p}, s, +\rangle$ are eigenstates of H and Q and give the eigenvalues. [6 marks]

ANSWER: *Using the properties of the ground state we have*

$$\begin{aligned}
a^{r\dagger}_{\boldsymbol{q}} a^r_{\boldsymbol{q}}|\boldsymbol{p}, s, -\rangle &= \sqrt{2E_{\boldsymbol{p}}} a^{r\dagger}_{\boldsymbol{q}} a^r_{\boldsymbol{q}} a^{s\dagger}_{\boldsymbol{q}}|0\rangle \\
&= \sqrt{2E_{\boldsymbol{p}}} a^{r\dagger}_{\boldsymbol{q}} \{a^r_{\boldsymbol{q}}, a^{s\dagger}_{\boldsymbol{q}}\}|0\rangle \\
&= (2\pi)^3 \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta^{rs}\sqrt{2E_{\boldsymbol{p}}} a^{r\dagger}_{\boldsymbol{q}}|0\rangle, \\
&= (2\pi)^3 \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta^{rs}\,|\boldsymbol{q}, r, -\rangle.
\end{aligned}$$

Similarly

$$\begin{aligned}
b^{r\dagger}_{\boldsymbol{q}} b^r_{\boldsymbol{q}}|\boldsymbol{p}, s, +\rangle &= \sqrt{2E_{\boldsymbol{p}}} b^{r\dagger}_{\boldsymbol{q}} b^r_{\boldsymbol{q}} b^{s\dagger}_{\boldsymbol{q}}|0\rangle \\
&= \sqrt{2E_{\boldsymbol{p}}} b^{r\dagger}_{\boldsymbol{q}} \{b^r_{\boldsymbol{q}}, b^{s\dagger}_{\boldsymbol{q}}\}|0\rangle \\
&= (2\pi)^3 \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta^{rs}\sqrt{2E_{\boldsymbol{p}}} b^{r\dagger}_{\boldsymbol{q}}|0\rangle, \\
&= (2\pi)^3 \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta^{rs}\,|\boldsymbol{q}, r, +\rangle.
\end{aligned}$$

Since $a^r_{\boldsymbol{q}}|0\rangle = b^r_{\boldsymbol{q}}|0\rangle = 0$ *we have*

$$H|0\rangle = Q|0\rangle = 0,$$

while using the above results we have

$$H|\boldsymbol{p}, s, \pm\rangle = \int \frac{d^3q}{(2\pi)^3} \sum_{r=1,2} E_{\boldsymbol{q}} (2\pi)^3 \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta^{rs}\,|\boldsymbol{q}, r, \pm\rangle = E_{\boldsymbol{p}}|\boldsymbol{p}, s, \pm\rangle,$$

$$Q|\boldsymbol{p}, s, \pm\rangle = \pm \int \frac{d^3q}{(2\pi)^3} \sum_{r=1,2} (2\pi)^3 \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta^{rs}\,|\boldsymbol{q}, r, \pm\rangle = \pm|\boldsymbol{p}, s, \pm\rangle,$$

where we have used the factor that the commutators between $a^s_{\boldsymbol{p}}$ *(or* $a^{s\dagger}_{\boldsymbol{p}}$*) and* $b^s_{\boldsymbol{p}}$ *(or* $b^{s\dagger}_{\boldsymbol{p}}$*) vanish.*

(iii) Using the results of part ii, write down an expression, similar to those for H and Q, for an operator S_z with eigenvalues

$$S_z|0\rangle = 0, \quad S_z|\boldsymbol{p}, 1, \pm\rangle = \pm\tfrac{1}{2}|\boldsymbol{p}, 1, \pm\rangle, \quad S_z|\boldsymbol{p}, 2, \pm\rangle = \mp\tfrac{1}{2}|\boldsymbol{p}, 2, \pm\rangle.$$

What does this operator measure? [3 marks]

ANSWER: *Consider the operator*

$$S_z = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \left(a^{1\dagger}_{\boldsymbol{p}} a^1_{\boldsymbol{p}} - a^{2\dagger}_{\boldsymbol{p}} a^2_{\boldsymbol{p}} - b^{1\dagger}_{\boldsymbol{p}} b^1_{\boldsymbol{p}} + b^{2\dagger}_{\boldsymbol{p}} b^2_{\boldsymbol{p}} \right).$$

As before $S_z|0\rangle = 0$ *and*

$$S_z|\boldsymbol{p}, 1, \pm\rangle = \pm\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \sum_{r=1}^{2} E_{\boldsymbol{q}} (2\pi)^3 \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta^{r1}\,|\boldsymbol{q}, r, \pm\rangle = \pm\tfrac{1}{2}|\boldsymbol{p}, 1, \pm\rangle,$$

$$S_z|\boldsymbol{p}, 2, \pm\rangle = \mp\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \sum_{r=1}^{2} E_{\boldsymbol{q}} (2\pi)^3 \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta^{r2}\,|\boldsymbol{q}, r, \pm\rangle = \mp\tfrac{1}{2}|\boldsymbol{p}, 2, \pm\rangle,$$

as required.

(iv) One could also try to quantize the Dirac field using commutation relations. Specifically the only non-vanishing commutators are then

$$\left[a^r_{\boldsymbol{p}}, a^{s\dagger}_{\boldsymbol{q}}\right] = -\left[b^r_{\boldsymbol{p}}, b^{s\dagger}_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\delta^{rs}.$$

and the sign of the $b^{s\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}}$ term in the Hamiltonian H changes.

If we define states in the Hilbert space in the conventional way, what are the two problems with this quantization prescription?

How can one problem be alleviated by changing the definition of the Hilbert space? [5 marks]

ANSWER: *Consider the normalization of the putative electron states*

$$\begin{aligned}\langle \boldsymbol{p}, r, -|\boldsymbol{q}, s, -\rangle &= \sqrt{2E_{\boldsymbol{p}}}\sqrt{2E_{\boldsymbol{q}}}\langle 0|a^r_{\boldsymbol{p}} a^{s\dagger}_{\boldsymbol{q}}|0\rangle = \sqrt{2E_{\boldsymbol{p}}}\sqrt{2E_{\boldsymbol{q}}}\langle 0|\left[a^r_{\boldsymbol{p}}, a^{s\dagger}_{\boldsymbol{q}}\right]|0\rangle \\ &= -(2\pi)^3(2E_{\boldsymbol{p}})\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\langle 0|0\rangle \\ &= -(2\pi)^3(2E_{\boldsymbol{p}})\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\end{aligned}$$

Thus for $\boldsymbol{p}=\boldsymbol{q}$ we have a negative norm state. This corresponds to negative probabilities and hence violates the standard interpretation of quantum mechanics. We also have

$$H = \int \frac{d^3p}{(2\pi)^3}\sum_{s=1}^{2} E_{\boldsymbol{p}}\left(a^{s\dagger}_{\boldsymbol{p}} a^s_{\boldsymbol{p}} - b^{s\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}}\right).$$

implying that the positron states have negative energy. This is unphysical, and gives a theory unstable to the production of the more and more positrons. One can correct the norm problem be defining the positron states and vacuum using

$$\tilde{b}^s_{\boldsymbol{p}} = b^{s\dagger}_{\boldsymbol{p}},$$

so that $\left[\tilde{b}^r_{\boldsymbol{p}}, \tilde{b}^{s\dagger}_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\delta^{rs}$. However the negative energy problem still remains since the form of H is unchanged (after normal ordering).

[Total 20 marks]

5. In both the interaction picture and the Heisenberg picture operators evolve with time. One has

$$\partial_t \mathcal{O}_I = \mathrm{i}\big[H_{0,I}, \mathcal{O}_I\big], \qquad \partial_t \mathcal{O}_H = \mathrm{i}\big[H_H, \mathcal{O}_H\big],$$

where in the interaction picture, $\mathcal{O}_I$ is the operator and $H_{0,I}$ is the free Hamiltonian, while in the Heisenberg picture $\mathcal{O}_H$ is the operator and H_H is the total Hamiltonian.

(i) What are the corresponding expressions for the evolution of states $|\psi(t)\rangle_H$ and $|\psi(t)\rangle_I$ in the Heisenberg and interaction pictures?

Comment briefly on why the interaction picture is useful in perturbation theory.

Write down the full Lagrangian density for "phi-fourth" theory and identify $\mathcal{L}_0$ and $\mathcal{L}_{\text{int}}$, the free and interaction Lagrangian densities, that contribute to the free and interaction Hamiltonians respectively. [5 marks]

ANSWER: *We have*

$$\partial_t |\psi(t)\rangle_I = -\mathrm{i}H_{int,I}|\psi(t)\rangle_I, \qquad \partial_t|\psi(t)\rangle_H = 0.$$

In the perturbation expansion, one expands as a power series in H_{int}. In the interaction picture, the fields still evolve with $H_{0,I}$ so take the same form as in the free theory. All the dependence on $H_{int,I}$ is then incorporated in the evolution of the states. In "phi-fourth" theory we have $\mathcal{L} + \mathcal{L}_0 + \mathcal{L}_{int}$ where

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2, \qquad \mathcal{L}_{int} = -\tfrac{1}{4!}\lambda\phi^4.$$

(ii) Show that in the two pictures $\partial_t H_H = 0$ and $\partial_t H_{0,I} = 0$. Hence show that

$$\mathcal{O}_H(t) = \mathrm{e}^{\mathrm{i}H_H(t-t_0)}\mathcal{O}_H(t_0)\,\mathrm{e}^{-\mathrm{i}H_H(t-t_0)},$$
$$\mathcal{O}_I(t) = \mathrm{e}^{\mathrm{i}H_{0,I}(t-t_0)}\mathcal{O}_I(t_0)\,\mathrm{e}^{-\mathrm{i}H_{0,I}(t-t_0)},$$

are solutions of the evolution equations for $\mathcal{O}_H(t)$ and $\mathcal{O}_I(t)$. [4 marks]

ANSWER: *By definition*

$$\partial_t H_H = \mathrm{i}\big[H_H, H_H\big] = 0, \qquad \partial_t H_{0,I} = \mathrm{i}\big[H_{0,I}, H_{0,I}\big] = 0.$$

We then have

$$\begin{aligned}
\partial_t \mathcal{O}_H(t) &= \partial_t\left(\mathrm{e}^{\mathrm{i}H_H(t-t_0)}\mathcal{O}_H(t_0)\,\mathrm{e}^{-\mathrm{i}H_H(t-t_0)}\right)\\
&= \partial_t\left(\mathrm{e}^{\mathrm{i}H_H(t-t_0)}\right)\mathcal{O}_H(t_0)\,\mathrm{e}^{-\mathrm{i}H_H(t-t_0)} + \mathrm{e}^{\mathrm{i}H_H(t-t_0)}\mathcal{O}_H(t_0)\,\partial_t\left(\mathrm{e}^{-\mathrm{i}H_H(t-t_0)}\right)\\
&= \mathrm{i}H_H\mathrm{e}^{\mathrm{i}H_H(t-t_0)}\mathcal{O}_H(t_0)\,\mathrm{e}^{-\mathrm{i}H_H(t-t_0)} + \mathrm{e}^{\mathrm{i}H_H(t-t_0)}\mathcal{O}_H(t_0)\,(-\mathrm{i}H_H)\mathrm{e}^{-\mathrm{i}H_H(t-t_0)}\\
&= \mathrm{i}H_H\mathcal{O}_H(t) - \mathrm{i}\mathcal{O}_H(t)H_H\\
&= \mathrm{i}\big[H_H, \mathcal{O}_H(t)\big].
\end{aligned}$$

Similarly

$$\begin{aligned}
\partial_t \mathcal{O}_I(t) &= \partial_t \left(\mathrm{e}^{\mathrm{i}H_{0,I}(t-t_0)} \mathcal{O}_I(t_0)\, \mathrm{e}^{-\mathrm{i}H_{0,I}(t-t_0)} \right) \\
&= \partial_t \left(\mathrm{e}^{\mathrm{i}H_{0,I}(t-t_0)} \right) \mathcal{O}_I(t_0)\, \mathrm{e}^{-\mathrm{i}H_{0,I}(t-t_0)} + \mathrm{e}^{\mathrm{i}H_{0,I}(t-t_0)} \mathcal{O}_I(t_0)\, \partial_t \left(\mathrm{e}^{-\mathrm{i}H_{0,I}(t-t_0)} \right) \\
&= \mathrm{i}H_{0,I} \mathrm{e}^{\mathrm{i}H_{0,I}(t-t_0)} \mathcal{O}_I(t_0)\, \mathrm{e}^{-\mathrm{i}H_{0,I}(t-t_0)} + \mathrm{e}^{\mathrm{i}H_{0,I}(t-t_0)} \mathcal{O}_I(t_0)\, (-\mathrm{i}H_{0,I}) \mathrm{e}^{-\mathrm{i}H_{0,I}(t-t_0)} \\
&= \mathrm{i}H_{0,I} \mathcal{O}_I(t) - \mathrm{i}\mathcal{O}_I(t) H_{0,I} \\
&= \mathrm{i}\big[H_{0,I}, \mathcal{O}_I(t)\big],
\end{aligned}$$

as required.

(iii) Let operators in the two pictures be related by

$$\mathcal{O}_I(t) = U(t, t_0)\mathcal{O}_H(t)U(t, t_0)^{-1},$$

where $U(t_0, t_0) = \mathbb{1}$.
Using the results of part ii write an expression for $U(t, t_0)$. Show that it satisfies

$$\partial_t U = -\mathrm{i}H_{\mathrm{int},I} U,$$

where $H_{\mathrm{int},I}$ is the interaction Hamiltonian in the interaction picture.

[4 marks]

ANSWER: *We have $\mathcal{O}_H(t_0) = \mathcal{O}_I(t_0)$ and hence*

$$U(t, t_0) = \mathrm{e}^{\mathrm{i}H_{0,I}(t-t_0)} \mathrm{e}^{-\mathrm{i}H_H(t-t_0)}.$$

Thus

$$\begin{aligned}
\partial_t U &= \partial_t \left(\mathrm{e}^{\mathrm{i}H_{0,I}(t-t_0)} \mathrm{e}^{-\mathrm{i}H_H(t-t_0)} \right) \\
&= (\mathrm{i}H_{0,I}) \mathrm{e}^{\mathrm{i}H_{0,I}(t-t_0)} \mathrm{e}^{-\mathrm{i}H_H(t-t_0)} + \mathrm{e}^{\mathrm{i}H_{0,I}(t-t_0)} (-\mathrm{i}H_H) \mathrm{e}^{-\mathrm{i}H_H(t-t_0)} \\
&= \mathrm{i}H_{0,I} \left(\mathrm{e}^{\mathrm{i}H_{0,I}(t-t_0)} \mathrm{e}^{-\mathrm{i}H_H(t-t_0)} \right) - \mathrm{i} \left(\mathrm{e}^{\mathrm{i}H_{0,I}(t-t_0)} \mathrm{e}^{-\mathrm{i}H_H(t-t_0)} \right) H_H \\
&= \mathrm{i}H_{0,I} U - \mathrm{i}U H_H \\
&= \mathrm{i}H_{0,I} U - \mathrm{i}(U H_H U^{-1}) U \\
&= \mathrm{i}\,(H_{0,I} - H_I)\, U \\
&= -\mathrm{i}H_{int,I} U.
\end{aligned}$$

as required.

(iv) Define the time-ordered product

$$T H_{\mathrm{int},I}(t_1) H_{\mathrm{int},I}(t_2).$$

As an expansion in $H_{\text{int},I}$, show explicitly that

$$U(t,t_0) = \mathbb{1} - \mathrm{i}\int_{t_0}^{t} \mathrm{d}t_1 H_{\text{int},I}(t_1) + \tfrac{1}{2}(-\mathrm{i})^2 \int_{t_0}^{t}\int_{t_0}^{t} \mathrm{d}t_1 \mathrm{d}t_2 T H_{\text{int},I}(t_1) H_{\text{int},I}(t_2) + \dots$$

is a solution for $U(t,t_0)$ to second order in $H_{\text{int},I}$. [7 marks]

ANSWER: *By definition*

$$T H_{int,I}(t_1) H_{int,I}(t_2) = \theta(t_1 - t_2) H_{int,I}(t_1) H_{int,I}(t_2) + \theta(t_2 - t_1) H_{int,I}(t_2) H_{int,I}(t_1),$$

where $\theta(t)$ is the Heaviside step function. Now solve the equation for $U = U_{(0)} + U_{(1)} + \dots$ perturbatively in $H_{int,I}$. At each order we have

$$\partial_t U_{(n+1)} = \mathrm{i} H_{int,I} U_{(n)}.$$

We can satisfy $U(t_0,t_0) = \mathbb{1}$ by taking $U_{(0)}(t_0,t_0) = \mathbb{1}$ and $U_{(n)}(t_0.t_0) = 0$ for $n \geq 1$. To zeroth-order

$$\partial_t U_{(0)} = 0 \quad \Rightarrow \quad U_{(0)}(t,t_0) = \mathbb{1},$$

since we have the boundary condition $U(t_0,t_0) = 1$. At first order we have

$$\partial_t U_{(1)}(t,t_0) = -\mathrm{i} H_{int,I}(t) U_{(0)}(t,t_0) = -\mathrm{i} H_{int,I}(t)$$
$$\Rightarrow \qquad U_{(1)}(t,t_0) = -\mathrm{i}\int_{t_0}^{t} \mathrm{d}t_1 H_{int,I}(t_1).$$

At second order we have

$$\partial_t U_{(2)} = -\mathrm{i} H_{int,I} U_{(1)} = (-\mathrm{i})^2 H_{int,I}(t) \int_{t_0}^{t} \mathrm{d}t_1 H_{int,I}(t_1)$$
$$\Rightarrow \qquad U_{(2)}(t,t_0) = (-\mathrm{i})^2 \int_{t_0}^{t} \mathrm{d}t_1 H_{int,I}(t_1) \int_{t_0}^{t_1} \mathrm{d}t_2 H_{int,I}(t_2).$$

By definition $T H_{int,I}(t_1) H_{int,I}(t_2) = T H_{int,I}(t_2) H_{int,I}(t_1)$. Thus

$$\begin{aligned}
\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2 T H_{int,I}(t_1) H_{int,I}(t_2) &= \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 T H_{int,I}(t_1) H_{int,I}(t_2) + \int_{t_0}^{t} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_1 T H_{int,I}(t_1) H_{int,I}(t_2) \\
&= 2\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 T H_{int,I}(t_1) H_{int,I}(t_2) \qquad (t_1 \leftrightarrow t_2 \text{ in second term}) \\
&= 2\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 H_{int,I}(t_1) H_{int,I}(t_2).
\end{aligned}$$

Hence

$$U(t,t_0) = \mathbb{1} - \mathrm{i}\int_{t_0}^{t} \mathrm{d}t_1 H_{int,I}(t_1) + \tfrac{1}{2}(-\mathrm{i})^2 \int_{t_0}^{t}\int_{t_0}^{t} \mathrm{d}t_1 \mathrm{d}t_2 T H_{int,I}(t_1) H_{int,I}(t_2) + \dots$$

as required.

[Total 20 marks]

6. This question is about Feynman diagrams in "phi-fourth" theory. Recall that in the interaction picture the $\mathcal{S}$-matrix is given by

$$\mathcal{S} = T \exp\left(\mathrm{i} \int \mathrm{d}^4x : \mathcal{L}_{\mathrm{int}}(x) : \right).$$

(i) Explain how $\mathcal{S}$ can be described as a perturbation expansion and write down the first three terms in the expansion.

The scattering amplitude $\mathrm{i}\mathcal{M}$ is usually defined to be

$$\langle \mathrm{out}|\mathrm{i}\mathcal{T}|\mathrm{in}\rangle = (2\pi)^4 \delta^{(4)}\left(p_{\mathrm{out}} - p_{\mathrm{in}}\right)\ \mathrm{i}\mathcal{M}$$

where $\mathcal{S} = \mathbb{1} + \mathrm{i}\mathcal{T}$. Why is the $\mathbb{1}$ contribution not included? What are p_{out} and p_{in} and what is the physical meaning of the δ-function? [4 marks]

ANSWER: *In perturbation theory we treat $\mathcal{L}_{int}$ as a small quantity. Thus we can expand S as*

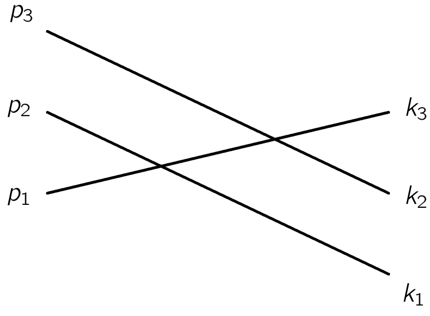
$$S = \mathbb{1} + \mathrm{i} \int \mathrm{d}^4x : \mathcal{L}_{int}(x) : \\ + \tfrac{1}{2}(\mathrm{i})^2 \int\int \mathrm{d}^4x_1 \mathrm{d}^4x_2 T : \mathcal{L}_{int}(x_1) : : \mathcal{L}_{int}(x_2) : + \ldots$$

and calculate the scattering amplitude term by term (using for instance Feynman diagrams). The $\mathbb{1}$ contribution is not included in the scattering amplitude since it corresponds to no scattering. The quantities p_{in} an p_{out} are the total momentum of the incoming and outgoing states. The δ-function implies momentum conservation.

(ii) Consider the scattering of three incoming ϕ particles with momenta k_1, k_2 and k_3 to three outgoing ϕ particles with momenta p_1, p_2 and p_3.

Define the $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$ states for this process.

Use the position-space Feynman rules to calculate the contribution of the following Feynman diagram to $\langle \mathrm{out}|\mathrm{i}\mathcal{T}|\mathrm{in}\rangle$ in terms of the propagator $D_F(x-y)$.



Given

$$D_F(x-y) = \mathrm{i}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2 - m^2 + \mathrm{i}\epsilon}$$

write this contribution as a function only of momenta.

Show that this agrees with the contribution to $\mathrm{i}\mathcal{M}$ calculated using the momentum-space rules. [7 marks]

ANSWER: *We have*

$$|in\rangle = \sqrt{2E_{\boldsymbol{k}_1}}\sqrt{2E_{\boldsymbol{k}_2}}\sqrt{2E_{\boldsymbol{k}_3}}a^\dagger_{\boldsymbol{k}_1}a^\dagger_{\boldsymbol{k}_2}a^\dagger_{\boldsymbol{k}_3}|0\rangle.$$
$$|out\rangle = \sqrt{2E_{\boldsymbol{p}_1}}\sqrt{2E_{\boldsymbol{p}_2}}\sqrt{2E_{\boldsymbol{p}_3}}a^\dagger_{\boldsymbol{p}_1}a^\dagger_{\boldsymbol{p}_2}a^\dagger_{\boldsymbol{p}_3}|0\rangle.$$

Using the Feynman rules,

contri. to $\langle out|\mathrm{i}T|in\rangle$

$$\begin{aligned}
&= (-\mathrm{i}\lambda)^2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{e}^{\mathrm{i}(p_3-k_2-k_3)\cdot x}\mathrm{e}^{\mathrm{i}(p_1+p_2-k_1)\cdot y}D_F(x-y)\\
&= (-\mathrm{i}\lambda)^2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}(p_3-k_2-k_3-p)\cdot x}\mathrm{e}^{\mathrm{i}(p_1+p_2-k_1+p)\cdot y}}{p^2-m^2+\mathrm{i}\epsilon}\\
&= -\mathrm{i}\lambda^2\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{(2\pi)^4\delta^{(4)}(p_3-k_2-k_3-p)(2\pi)^4\delta^{(4)}(p_1+p_2-k_1+p)}{p^2-m^2+\mathrm{i}\epsilon}\\
&= -\frac{\mathrm{i}\lambda^2}{p^2-m^2+\mathrm{i}\epsilon}(2\pi)^4\delta^{(4)}(p_1+p_2+p_3-k_1-k_2-k_3).
\end{aligned}$$

For the momentum space rules, the momentum flowing in the propagator is $k_2+k_3-p_1$ *so*

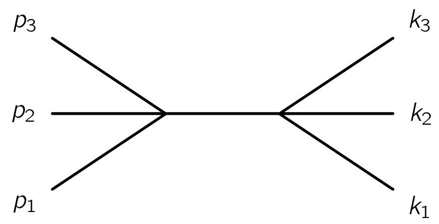
$$\begin{aligned}
\textit{contri. to } \mathrm{i}\mathcal{M} &= (-\mathrm{i}\lambda)^2\frac{\mathrm{i}}{(k_1+k_2-p_3)^3-m^2+\mathrm{i}\epsilon}\\
&= -\frac{\mathrm{i}\lambda^2}{(k_1+k_2-p_3)^3-m^2+\mathrm{i}\epsilon}.
\end{aligned}$$

This is in agreement with the position-space calculation.

(iii) Draw a second Feynman diagram that has no loops, is not related to the diagram in part ii by a permutation of the p_i momenta or of the k_i momenta, and contributes to $\mathrm{i}\mathcal{M}$ at the same order in λ.

Evaluate this diagram using the momentum-space Feynman rules. [3 marks]

ANSWER: *The diagram is*



The momentum flowing in the propagator is $k_1 + k_2 + k_3$, so using the Feynman rules we find

$$\begin{aligned} \textit{contri. to } \mathrm{i}\mathcal{M} &= (-\mathrm{i}\lambda)^2 \frac{\mathrm{i}}{(k_1+k_2+k_3)^3 - m^2 + \mathrm{i}\epsilon} \\ &= -\frac{\mathrm{i}\lambda^2}{(k_1+k_2+k_3)^3 - m^2 + \mathrm{i}\epsilon}. \end{aligned}$$

Now consider the scattering of two incoming ϕ particles with momenta q_1 and q_2 to two outgoing ϕ particles with momenta p_1 and p_2.

Taking the non-relativistic limit and comparing with the Born approximation gives

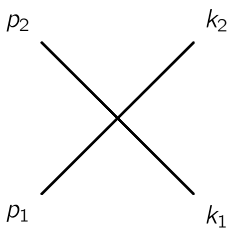
$$\mathrm{i}\mathcal{M}(q_1, q_2, p_1, p_2) = -\mathrm{i}\tfrac{1}{2}\left[\tilde{V}(\boldsymbol{p}_1 - \boldsymbol{q}_1) + \tilde{V}(\boldsymbol{p}_1 - \boldsymbol{q}_2)\right],$$

where $\tilde{V}(\boldsymbol{q})$ is the Fourier transform of the classical potential $V(\boldsymbol{x})$ between the two particles.

(iv) Use the momentum-space Feynman rules to identify $\tilde{V}(\boldsymbol{k})$ and hence calculate the form of $V(\boldsymbol{x})$ at order λ.

Draw a Feynman diagram that gives corrections to $\mathrm{i}\mathcal{M}$ (and hence potentially to $V(\boldsymbol{x})$) at order λ^2. [6 marks]

ANSWER: *At order λ, the only diagram is*



This contributes

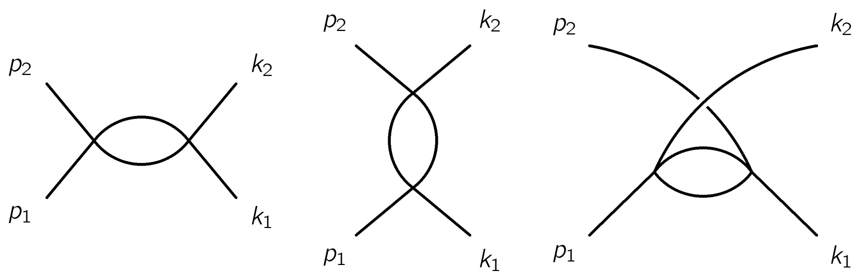
$$\textit{contri. to } \mathrm{i}\mathcal{M} = -\mathrm{i}\lambda.$$

Hence we have $\tilde{V}(\boldsymbol{q}) = \lambda$. Fourier transforming gives

$$V(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \tilde{V}(\boldsymbol{q}) \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \lambda \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} = \lambda\delta^{(3)}(\boldsymbol{x}).$$

(This is what is known as a contact interaction.) The relevant loop Feynman diagrams at order λ^2 are

[This question continues overleaf . . .]



(which lead to a renormalization of the vertex).

[Total 20 marks]