Imperial College London

MSci EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY

For 4th-Year Physics Students

23rd May 2012: 14:00 to 16:00

Answer THREE questions
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. Consider a complex scalar field $\phi(x)$ and a real scalar field $\sigma(x)$ described by the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma - m^2 \phi\phi^* - \frac{1}{2}M^2\sigma^2 - \frac{\lambda}{2}\phi^*\phi\sigma^2$$

(i) Using the Euler-Lagrange equations, write down the equations of motion for $\sigma(x)$, $\phi(x)$ and $\phi^*(x)$ (treating the latter two as independent fields). [3 marks]

(ii) Derive the momentum conjugate to ϕ, ϕ^* and σ and construct the Hamiltonian for this field theory. [4 marks]

(iii) Show that the Lagrangian is invariant under the symmetry transformation

$$\phi(x) \mapsto \phi'(x) = e^{i\alpha}\phi(x),$$

where α is a constant. Using the equations of motion, show that the current

$$j^\mu = i\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right)$$

is conserved. [3 marks]

(iv) Consider the integral

$$Q = \int d^3x\, j^0(t, \boldsymbol{x}).$$

Using the equations of motion show that Q is independent of t. [2 marks]

Now consider the quantum theory corresponding to $\mathcal{L}$.

(v) In the free theory with $\lambda = 0$, briefly describe, without derivation, the single particle states that exist corresponding to the complex field ϕ and the real field σ, including the mass and properties with respect to the operator corresponding to Q. [4 marks]

(vi) Consider now the impact of including interactions in the quantum theory by allowing $\lambda \neq 0$. Write down the basic vertex for the momentum space Feynman rules (numerical coefficients need not be given). [4 marks]

[Total 20 marks]

2. Consider a free real scalar field with, in the Heisenberg picture,

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}p\cdot x} + a_{\boldsymbol{p}}^\dagger \mathrm{e}^{\mathrm{i}p\cdot x} \right)$$

$$\pi(x) = -\mathrm{i} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sqrt{\frac{E_{\boldsymbol{p}}}{2}} \left(a_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}p\cdot x} - a_{\boldsymbol{p}}^\dagger \mathrm{e}^{\mathrm{i}p\cdot x} \right),$$

where $p^0 = E_{\boldsymbol{p}}$ and $\pi(x)$ is the momentum density conjugate to $\phi(x)$.

(i) The equal time commutation relations (ETCRs) state that

$$\left[\phi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})\right] = \mathrm{i}\delta^3(\boldsymbol{x} - \boldsymbol{y}).$$

What are the ETCRs for $\left[\phi(t, \boldsymbol{x}), \phi(t, \boldsymbol{y})\right]$ and $\left[\pi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})\right]$? [2 marks]

(ii) Show that

$$a_{\boldsymbol{p}} = \int \mathrm{d}^3x \frac{\mathrm{e}^{\mathrm{i}p\cdot x}}{\sqrt{2E_{\boldsymbol{p}}}} \left[E_{\boldsymbol{p}} \phi(t, \boldsymbol{x}) + \mathrm{i}\pi(t, \boldsymbol{x})\right].$$

[4 marks]

(iii) Hence, using the ETCRs, show that

$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}^\dagger\right] = (2\pi)^2 \delta^3(\boldsymbol{p} - \boldsymbol{q}),$$
$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}\right] = 0, \qquad \left[a_{\boldsymbol{p}}^\dagger, a_{\boldsymbol{q}}^\dagger\right] = 0.$$

[6 marks]

(iv) Using the results from part (iii), show that at unequal times

$$\left[\phi(x), \phi(y)\right] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)} \right).$$

[4 marks]

(v) Show that when $x - y$ is spacelike, $\left[\phi(x), \phi(y)\right] = 0$. What is the physical implication of this result? [4 marks]

[Total 20 marks]

3. Let $\psi(x)$ be a classical Dirac spinor field. The gamma matrices γ^μ satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$$

Work in a basis in which $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. The Lagrangian density for the Dirac field is defined to be

$$\mathcal{L}(x) = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)$$

(i) Show that $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$. Derive the Euler-Lagrange equations for $\mathcal{L}$ treating $\psi(x)$ and $\bar{\psi}(x)$ separately, showing that for both cases one obtains the Dirac equation. [6 marks]

Under a Lorentz transformation the Dirac field transforms as

$$\psi(x) \mapsto \psi'(x) = \Lambda_{\frac{1}{2}}\,\psi(\Lambda^{-1}x),$$

where $\Lambda_{\frac{1}{2}} = exp[-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}]$ with $S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$.

(ii) Given $\Lambda^\dagger_{\frac{1}{2}}\gamma^0 = \gamma^0(\Lambda_{\frac{1}{2}})^{-1}$ and also that $(\Lambda_{\frac{1}{2}})^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$ show that under a Lorentz transformation the Lagrangian density transforms via $\mathcal{L}(x) \to \mathcal{L}(\Lambda^{-1}x)$. What is the significance of this result? [7 marks]

Now define a conjugate spinor by $\psi^{(c)}(x) = C\psi^*(x)$ where C is a matrix satisfying $C^\dagger C = 1$ and $C^\dagger\gamma^\mu C = -(\gamma^\mu)^*$.

(iii) Show that $\psi^{(c)}(x)$ transforms under an *infinitesimal* Lorentz transformation in the same way that $\psi(x)$ does . [5 marks]

(iv) A Majorana spinor has the property $\psi(x) = \psi^{(c)}(x)$. Briefly comment on the implications of this property for the particle content of the quantum theory. [2 marks]

[Total 20 marks]

4. Consider a free Dirac field $\psi(x)$ given by

$$\psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{2} \left[a_{\boldsymbol{p}}^s u^s(p) \mathrm{e}^{-\mathrm{i}p\cdot x} + b_{\boldsymbol{p}}^{s\dagger} v^s(p) \mathrm{e}^{\mathrm{i}p\cdot x} \right],$$

where $p^0 = E_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2}$. The only non-vanishing anti-commutation relations are

$$\left\{ a_{\boldsymbol{p}}^r, a_{\boldsymbol{q}}^{s\dagger} \right\} = \left\{ b_{\boldsymbol{p}}^r, b_{\boldsymbol{q}}^{s\dagger} \right\} = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \delta^{rs}.$$

The Hamiltonian H and the operator Q are given by

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{s=1,2} E_{\boldsymbol{p}} \left(a_{\boldsymbol{p}}^{s\dagger} a_{\boldsymbol{p}}^s + b_{\boldsymbol{p}}^{s\dagger} b_{\boldsymbol{p}}^s \right),$$

$$Q = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{s=1,2} \left(a_{\boldsymbol{p}}^{s\dagger} a_{\boldsymbol{p}}^s - b_{\boldsymbol{p}}^{s\dagger} b_{\boldsymbol{p}}^s \right).$$

(i) Define the vacuum state $|0\rangle$ and the single particle states $|\boldsymbol{p}, s, e^-\rangle$ and $|\boldsymbol{p}, s, e^+\rangle$. What do the labels $\boldsymbol{p}$, s and $e^\pm$ denote?

Define a two-particle state, and show that the particles are fermions.

[6 marks]

(ii) Show that

$$a_{\boldsymbol{q}}^{r\dagger} a_{\boldsymbol{q}}^r |\boldsymbol{p}, s, e^-\rangle = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \delta^{rs} \, |\boldsymbol{q}, r, e^-\rangle,$$
$$b_{\boldsymbol{q}}^{r\dagger} b_{\boldsymbol{q}}^r |\boldsymbol{p}, s, e^+\rangle = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \delta^{rs} \, |\boldsymbol{q}, r, e^+\rangle.$$

Hence show that $|0\rangle$, $|\boldsymbol{p}, s, e^-\rangle$ and $|\boldsymbol{p}, s, e^+\rangle$ are eigenstates of H and Q and give the eigenvalues. [6 marks]

(iii) Next consider the interacting Yukawa theory with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2} \partial_\mu \phi \partial^\mu \phi - \tfrac{1}{2} M^2 \phi^2 + \bar{\psi} \left(\mathrm{i}\gamma^\mu \partial_\mu - m \right) \psi - g \phi \bar{\psi} \psi.$$

Consider the process of an e^+ state with momentum p and spin r scattering with a ϕ particle with momentum k to produce an e^+ state with momentum p' and spin r' and a ϕ particle with momentum k'. Draw two Feynman diagrams that contribute to this process at order g^2. Using the Feynman rules in momentum space (which need not be stated or proved), write down the contribution to the matrix element $i\mathcal{M}$ of *both* of the diagrams. (One may ignore the overall sign of the diagrams). [8 marks]

[Total 20 marks]

5. This question concerns ϕ^4-theory. The scattering matrix $\mathcal{S}$ can be written

$$\mathcal{S} = T[exp(i \int d^4x \mathcal{L}'(x))]$$

with interaction Lagrangian density

$$\mathcal{L}' = -\frac{\lambda}{4!} \int d^4x \phi^4(x)$$

where $\phi(x)$, in the interaction picture, can be written $\phi(x) = \phi^+(x) + \phi^-(x)$ where

$$\phi^+(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} a_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}p\cdot x},$$

$$\phi^-(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} a^\dagger_{\boldsymbol{p}} \mathrm{e}^{\mathrm{i}p\cdot x},$$

where $E_{\boldsymbol{p}} = p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$.

(i) Show that

$$\left[\phi^+(x), \phi^-(y)\right] = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

[2 marks]

(ii) Show that

$$T(\phi(x)\phi(y)) = N(\phi(x)\phi(y)) + D_F(x-y)$$

where the Feynman propagator is given by

$$D_F(z) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(\theta(z^0)\mathrm{e}^{-\mathrm{i}p\cdot z} + \theta(-z^0)\mathrm{e}^{\mathrm{i}p\cdot z}\right).$$

Also, state, without proof, Wicks theorem. [4 marks]

(iii) Consider the scattering of 2 particles with momenta k_1, k_2 into n particles with momenta $p_1, \ldots, p_n$. The scattering amplitude $\mathrm{i}\mathcal{M}$ can be written as

$$\langle p_1, \ldots, p_n | \mathrm{i}\mathcal{T} | k_1, k_2 \rangle = (2\pi)^4 \delta^{(4)}(X)\ \mathrm{i}\mathcal{M}$$

where $\mathcal{T}$ is defined via $\mathcal{S} = \mathbb{1} + \mathrm{i}\mathcal{T}$. Why is the $\mathbb{1}$ contribution not included? What is the argument, X, of the delta function. What is the physical meaning of the δ-function? [3 marks]

(iv) Write down the Feynman rules in momentum space for the calculation of the connected amputated diagrams contributing to $i\mathcal{M}$. (You do not need to discuss possible symmetry factors). [5 marks]

(v) Consider the scattering of 2 particles into 6 particles at order λ^3. Write down two Feynman diagrams with distinct topology (i.e. they cannot be obtained by relabelling momenta), contributing to $i\mathcal{M}$ at this order and use the Feynman rules in momentum space to calculate the contribution for both diagrams. [6 marks]

[Total 20 marks]