Imperial College London

MSci EXAMINATION May 2016

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY

For 4th-Year Physics Students

Tuesday, 17th May 2016: 14:00 to 16:00

Answer THREE questions.
Unless otherwise specified, natural units are used so $\hbar = c = 1$and the metric is diagonal with $g^{00} = +1$, and $g^{ii} = -1$ for $i = 1, 2, 3$.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) Consider a theory for a single complex scalar field Φ with Lagrangian density $\mathcal{L}$ given by

$$\mathcal{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - f(\Phi^* \Phi)\,. \tag{1}$$

where f can be any function. The equations of motion are given by

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0, \qquad \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} - \frac{\partial \mathcal{L}}{\partial \Phi^*} = 0\,. \tag{2}$$

This Lagrangian is invariant under $\Phi \to \Phi' = e^{i\theta}\Phi$ where θ is an arbitrary constant, i.e. $\partial_\mu \theta = 0$. Show that the conserved Noether current J^μ associated with this symmetry, i.e. where $\partial_\mu J^\mu = 0$, may be written as

$$J^\mu = -i(\partial^\mu \Phi^*)\Phi + i\Phi^*(\partial^\mu \Phi)\,. \tag{3}$$

[10 marks]

(ii) Consider the complex scalar field in the interaction picture given by

$$\hat{\Phi}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\boldsymbol{p})}} \left(\hat{b}(\boldsymbol{p}) e^{-ipx} + \hat{c}^\dagger(\boldsymbol{p}) e^{+ipx}\right), \tag{4}$$

$$p_0 = \omega(\boldsymbol{p}) = +\sqrt{\boldsymbol{p}^2 + m^2}\,. \tag{5}$$

Here

$$[\hat{b}(\boldsymbol{p}), \hat{b}^\dagger(\boldsymbol{q})] = [\hat{c}(\boldsymbol{p}), \hat{c}^\dagger(\boldsymbol{q})] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \tag{6}$$

but all other commutators between $\hat{b}(\boldsymbol{p})$, $\hat{b}^\dagger(\boldsymbol{p})$, $\hat{c}(\boldsymbol{p})$, and $\hat{c}^\dagger(\boldsymbol{p})$ are zero. By replacing the classical fields by the quantum field operator $\hat{\Phi}$ of (4) find an expression for the conserved charge operator $\widehat{Q} = \int d^3x\ \widehat{J}^0$ in terms of the annihilation and creation operators.

Interpret $\widehat{Q}$ in terms of the charges of the particles of the theory. [12 marks]

(iii) Show that $[\widehat{Q}, \hat{\Phi}]$ is proportional to $\hat{\Phi}$ and find the constant of proportionality. Find $[\widehat{Q}, \hat{\Phi}^\dagger]$.

If an operator $\hat{\eta}$ is transformed as $\hat{\eta} \to \hat{\eta}' = e^{i\theta\widehat{Q}} \hat{\eta} e^{-i\theta\widehat{Q}}$ where θ is a real parameter, then we find that $\hat{\eta}' = e^{i\theta q}\hat{\eta}$ if $\left[\widehat{Q}, \hat{\eta}\right] = q\hat{\eta}$. Hence give an interpretation of the $[\widehat{Q}, \hat{\Phi}]$ and $[\widehat{Q}, \hat{\Phi}^\dagger]$ commutators in terms of the symmetry and conserved charge of the fields $\hat{\Phi}$ and $\hat{\Phi}^\dagger$. [8 marks]

[Total 30 marks]

2. Consider a set of real bosonic field operators ϕ_i in the interaction picture and with corresponding masses m_i. The field operators are split into two parts $\phi_i = \phi_i^+ + \phi_i^-$ as follows

$$\phi_i^+(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_i}}\, \hat{a}_i(\boldsymbol{p}) e^{-ipx}\,, \qquad \phi_i^-(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_i}}\, \hat{a}_i^\dagger(\boldsymbol{p}) e^{+ipx}\,,$$
$$p_0 = \omega_i = +\sqrt{\boldsymbol{p}^2 + m_i^2}\,, \qquad (1)$$

where

$$\left[\hat{a}_i(\boldsymbol{p}), \hat{a}_j^\dagger(\boldsymbol{q})\right] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q})\, \delta_{ij}\,, \qquad [\hat{a}_i(\boldsymbol{p}), \hat{a}_j(\boldsymbol{q})] = \left[\hat{a}_i^\dagger(\boldsymbol{p}), \hat{a}_j^\dagger(\boldsymbol{q})\right] = 0\,. \quad (2)$$

(i) Show that

$$\left[\phi_i^+(x), \phi_j^+(y)\right] = \left[\phi_i^-(x), \phi_j^-(y)\right] = 0\,. \qquad (3)$$

Define the normal ordered product of fields, here denoted as $\mathrm{N}(\phi_1 \ldots \phi_n)$, in terms of $\phi_i^\pm$.

Give an expression for $\mathrm{N}(\phi_i\phi_j)$ in terms of $\phi_i^\pm$ and $\phi_j^\pm$.

Consider the vacuum state where $\hat{a}_i(\boldsymbol{p})|\,0\rangle = 0$ for all i and $\boldsymbol{p}$. Show that all vacuum expectation values of normal ordered products are zero, i.e.

$$\langle 0|\mathrm{N}(\phi_1 \ldots \phi_n)|\,0\rangle = 0\,. \qquad (4)$$

[7 marks]

(ii) Define the time-ordered product, $\mathrm{T}\,(\phi(x_1) \ldots \phi(x_n))$, for scalar fields.

Define the contraction $\contraction{}{\phi}{_i(x)}{\phi}\phi_i(x)\phi_j(y)$ between any two of these fields in terms of normal-ordered and time-ordered products.

Show that

$$\contraction{}{\phi}{_i(x)}{\phi}\phi_i(x)\phi_j(y) \;=\; \theta(x^0 - y^0)\left[\phi_i^+(x), \phi_j^-(y)\right] + \theta(y^0 - x^0)\left[\phi_j^+(y), \phi_i^-(x)\right] \quad (5)$$

where x^0 and y^0 are the time components of the coordinates. [7 marks]

(iii) Show that the contractions are equal to

$$\contraction{}{\phi}{_i(x)}{\phi}\phi_i(x)\phi_j(y) \;=\; \delta_{ij}\Delta_i(x - y)\,, \qquad \text{(no sum over repeated indices)} \quad (6)$$

where you should give an explicit form for $\Delta_i(x - y)$ in terms of the coordinate difference $x - y$, the mass of the i-th field, m_i, and an integral over three-momentum (a four-momentum representation is not required) . [8 marks]

(iv) State Wick's theorem for arbitrary numbers of several different scalar fields.

Write down an expression for the time-ordered product of a single scalar field ϕ evaluated at four different coordinates $T_{1234} = \mathrm{T}\,(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))$. This should be given in terms of the normal-ordered products of fields and in terms of the appropriate contractions $\contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y) = \Delta(x - y)$.

Hence give the vacuum expectation value of the time-ordered products of four fields in terms of $\Delta(x - y)$. [8 marks]

[Total 30 marks]

3. In this question all states and operators are in the Interaction picture.

States in the Interaction picture evolve as $|\,\psi, t\rangle = U(t, t_0)|\,\psi, t_0\rangle$ where the operator U satisfies

$$i\frac{d}{dt}U(t, t_0) \quad = \quad H_{\text{int}}(t)U(t, t_0) \tag{1}$$

and $H_{\text{int}}(t)$ is the interaction part of the full Hamitonian operator. Assuming $t > s$ then a solution of (1) is

$$U(t, s) = \mathrm{T}\left(\exp\{-i\int_s^t dt'\, H_{\text{int}}(t')\}\right) \tag{2}$$

where $\mathrm{T}(\ldots)$ indicates the operators are time-ordered.

(i) Show that, for an arbitrary state $|\,\psi\rangle$, we have

$$\lim_{s\to-\infty}\langle\psi|U(t, s)|\,0\rangle \quad = \quad \langle\psi|\,\Omega\rangle\langle\Omega|\,0\rangle \tag{3}$$

where $|\,0\rangle$ is the free vacuum and $|\,\Omega\rangle$ is the vacuum for the fully interacting theory. [8 marks]

(ii) The free propagator Π_0 and the full propagator Π_c are defined as

$$\Pi_0(z - y) \quad = \quad \langle 0|\mathrm{T}\left(\phi(z)\phi(y)S\right)|\,0\rangle\,, \tag{4}$$
$$\Pi_c(z - y) \quad = \quad \langle\Omega|\mathrm{T}\left(\phi(z)\phi(y)S\right)|\,\Omega\rangle\,, \tag{5}$$

where the S-matrix is given by $S = U(+\infty, -\infty)$.
Show that

$$\Pi_c(z - y) = \frac{1}{Z}\Pi_0(z - y) \text{ where } Z = \langle 0|S|\,0\rangle\,. \tag{6}$$

[6 marks]

(iii) Consider a theory of a single real scalar field ϕ with Feynman propagator $\Delta(x)$ and an interaction Hamiltonian given by

$$H_{\text{int}} = \frac{\lambda}{4!}\int d^3\boldsymbol{x}\,\phi^4(x)\,. \tag{7}$$

By using Wick's theorem (which you may quote without proof) or otherwise, find an expression for Z in terms of λ and the Feynman propagator $\Delta(x)$ to first order in λ. [6 marks]

(iv) Find an expression for the free propagator $\Pi_0(z - y)$ defined in (4) to first order in λ. This should be given in terms of λ and $\Delta(x)$. [6 marks]

(v) By using your expressions for Z and $\Pi_0(z-y)$, derive an explicit expression for the full propagator Π_c (5) to first order in λ, in terms of λ and $\Delta(x)$.
Interpret your result in terms of the different types of Feynman diagram which contribute to Z, Π_0 and Π_c. [4 marks]

[Total 30 marks]

4. In this question all quantities are given in the Interaction picture.

The scalar Yukawa theory for a real scalar field ϕ of mass m and a complex scalar field ψ of mass M has a cubic interaction with real coupling constant g and the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu \psi^\dagger)(\partial^\mu \psi) - M^2\psi^\dagger\psi - g\psi^\dagger(x)\psi(x)\phi(x) . \quad (1)$$

The field operators take the form

$$\hat{\phi}(x) \;=\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left(\hat{a}(\boldsymbol{p})e^{-ipx} + \hat{a}^\dagger(\boldsymbol{p})e^{+ipx}\right), \qquad p_0 = \omega(\boldsymbol{p}) , \quad (2)$$

$$\hat{\psi}(x) \;=\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega(\boldsymbol{p})}}\left(\hat{b}(\boldsymbol{p})e^{-ipx} + \hat{c}^\dagger(\boldsymbol{p})e^{+ipx}\right), \qquad p_0 = \Omega(\boldsymbol{p}) , \quad (3)$$

$$\omega(\boldsymbol{p}) = +\sqrt{\boldsymbol{p}^2 + m^2} , \qquad \Omega(\boldsymbol{p}) = +\sqrt{\boldsymbol{p}^2 + M^2} , \quad (4)$$

where the annihilation and creation operators obey their usual commutation relations

$$\left[\hat{a}(\boldsymbol{p}), \hat{a}^\dagger(\boldsymbol{q})\right] = (2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{q}) , \qquad [\hat{a}(\boldsymbol{p}), \hat{a}(\boldsymbol{q})] = \left[\hat{a}^\dagger(\boldsymbol{p}), \hat{a}^\dagger(\boldsymbol{q})\right] = 0 . \quad (5)$$

The $\hat{b}$, $\hat{b}^\dagger$ pair and the $\hat{c}$ and $\hat{c}^\dagger$ pair of annihilation and creation operators obey identical commutation relations to those of the $\hat{a}$ and $\hat{a}^\dagger$ pair. Different types of annihilation and creation operator always commute e.g. $\left[\hat{a}(\boldsymbol{p}), \hat{b}^\dagger(\boldsymbol{q})\right] = 0$.

You may assume that the vacuum expectation values of both fields are zero, $\langle 0|\phi|\, 0\rangle = \langle 0|\psi|\, 0\rangle = 0$, so no tadpole diagram contributions are needed.

Consider the decay of a ϕ particle into a ψ-$\bar{\psi}$ (particle – anti-particle) pair in the scalar Yukawa theory with an incoming ϕ particle of three-momentum $\boldsymbol{p}$ while the outgoing ψ-$\bar{\psi}$ pair have three-momenta $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ respectively. The relevant matrix element $\mathcal{M}$ is

$$\mathcal{M} \;=\; \langle f|\hat{S}|\, i\rangle = A\langle 0|\hat{b}(q_1)\hat{c}(q_2)\hat{S}\hat{a}^\dagger(p)|\, 0\rangle , \quad A = (8\Omega(q_1)\Omega(q_2)\omega(p))^{1/2} . (6)$$

The vacuum $|\, 0\rangle$ is the free vacuum state annihilated by $\hat{a}(\boldsymbol{p})$, $\hat{b}(\boldsymbol{p})$ and $\hat{c}(\boldsymbol{p})$.

(i) Derive the relationship between the matrix element $\mathcal{M}$ of (6) and the corresponding Green function G where

$$G(z_1, z_2, y) \;=\; \langle 0|\mathrm{T}\hat{\psi}(z_1)\hat{\psi}^\dagger(z_2)\hat{\phi}(y)\hat{S}|\, 0\rangle \quad (7)$$

[10 marks]

(ii) State the Feynman rules needed to calculate Green functions in coordinate space in this theory. No derivation is required. [5 marks]

(iii) Draw the only non-trivial Feynman diagram for G which is $O(g^1)$. Remember you may assume diagrams with tadpole contributions are zero.

Hence derive an expression for $G(z_1, z_2, y)$ to $O(g^1)$ in terms of g and the appropriate propagators. [5 marks]

[This question continues on the next page . . .]

(iv) Draw *all* the Feynman diagrams, except for those containing tadpole subdiagrams, which contribute g^3 terms to the Green function $G(z_1, z_2, y)$.

In addition:-

(a) For each of these diagrams specify (a) the symmetry factor and (b) the number of loop momenta.

(b) For each of these diagrams state if (a) they contain a vacuum diagram contribution, (b) contain self-energy corrections to external propagators.

[10 marks]

[Total 30 marks]