Imperial College London

MSci EXAMINATION May 2018

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY

For 4th-Year Physics Students

Wednesday 23rd May 2018: 10:00 to 12:00

Answer THREE questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. *Note: in this question we do* **not** *use natural units so c and $\hbar$ are not equal to one.*

Consider a ring of N identical balls, all of mass m. In equilibrium, balls lie equally spread around the ring with a distance a between nearest neighbours. The Hamiltonian operator for the normal modes may be written as

$$\widehat{H} = \sum_k \left[\frac{1}{2m} \widehat{P}_{-k} \widehat{P}_k + \frac{m\omega_k^2}{2} \widehat{U}_{-k} \widehat{U}_k \right] . \qquad (1)$$

Wavenumbers are labelled k, p, q etc. Sums over wavenumbers, such as $\sum_k$, are taken over all allowed wavenumbers and are symmetric (that is if a value k is present in the sum then the value $-k$ is also in the sum). You need not consider the $k = 0$ mode explicitly. For a normal mode of wave number k, $\widehat{U}_k$ is the hermitian position operator for that mode (with units of length), and $\widehat{P}_k$ is the hermitian momentum operator (with units of momentum). The dispersion relation is $\omega_k = |\sqrt{4\omega^2 \sin^2(ka/2) + \Omega^2}|$ where ω and Ω are both fixed characteristic frequencies.

The operators $\widehat{U}_k$ and $\widehat{P}_k$ satisfy

$$\widehat{U}_k^\dagger = \widehat{U}_{-k} , \quad \widehat{P}_k^\dagger = \widehat{P}_{-k} , \quad [\widehat{U}_p, \widehat{P}_q] = i\hbar\delta_{p,-q} , \quad [\widehat{U}_p, \widehat{U}_q] = 0 , \quad [\widehat{P}_p, \widehat{P}_q] = 0 . \quad (2)$$

(i) Show that $\ell_k = (\hbar/m\omega_k)^{1/2}$ has units of length.

Annihilation operators may be defined as

$$\hat{a}_k = \sqrt{\frac{m\omega_k}{2\hbar}} \left(\widehat{U}_k + \frac{i}{m\omega_k} \widehat{P}_k \right) . \qquad (3)$$

What are the units of the annihilation operator?

Prove that this annihilation operator and its hermitian conjugate satisfy the commutation relations $[\hat{a}_p, \hat{a}_q^\dagger] = \delta_{p,q}$, $[\hat{a}_p, \hat{a}_q] = 0$ and $[\hat{a}_p^\dagger, \hat{a}_q^\dagger] = 0$.

[10 marks]

(ii) Show that the Hamiltonian $\widehat{H}$ of (1) may be rewritten as

$$\widehat{H} = \frac{1}{2} \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger \right) . \qquad (4)$$

Hint: you may start from (4) and work towards (1). [10 marks]

(iii) The Hamiltonian for a single scalar relativistic field $\hat{\phi}$ in one spatial dimension with conjugate momentum operator $\hat{\Pi}$ is

$$\widehat{H} = \frac{1}{2} \int dx \left(\left(\hat{\Pi}(t,x)\right)^2 + c^2 \left(\frac{\partial\hat{\phi}(t,x)}{\partial x} \right)^2 + \frac{M^2c^4}{\hbar^2} (\hat{\phi}(t,x))^2 \right) . \qquad (5)$$

By writing $\hat{\phi}(t,x) = (2\pi)^{-1} \int dk\, e^{ikx} \hat{\phi}(t,k)$, show how to match the $\hat{\phi}$ terms with the $\widehat{U}$ terms of (1) in the limit $a \to 0$. You may ignore the $\hat{\Pi}$ and $\widehat{P}$ terms. As part of your answer you must express c and M from (5) in terms of a, ω and Ω of (1) such that the dimensions are compatible. [10 marks]

[Total 30 marks]

2. Consider a single real scalar field operator $\hat{\phi}(x)$ in a non-interacting theory. We will work in the Heisenberg picture (which is equal here to the Interaction picture) so $\hat{\phi}(x)$ is given in (17) of the "Useful Definitions" section at the end of this exam paper.

(i) Show that the commutation relation $[\hat{\phi}(x), \hat{\phi}(y)]$ at arbitrary spacetime points x and y may be written as

$$[\hat{\phi}(x), \hat{\phi}(y)] \quad = \quad \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\boldsymbol{p})} \left(e^{-ip(x-y)} - e^{+ip(x-y)} \right) . \tag{6}$$

The conjugate momentum field $\hat{\pi}(x)$ may be defined as $\hat{\pi}(x) = \partial\hat{\phi}(x)/\partial t$ where t is the time coordinate of the four-vector x^μ. By applying time derivatives directly to the expression (6) for $[\hat{\phi}(x), \hat{\phi}(y)]$, or otherwise, find similar expressions for

(a) $[\hat{\phi}(x), \hat{\pi}(y)]$,

(b) $[\hat{\pi}(x), \hat{\pi}(y)]$,

in terms of integrals over a three-momentum $\boldsymbol{p}$ involving x, y and $\omega(\boldsymbol{p})$.

[12 marks]

(ii) State the Equal Time Commutation Relations for the field $\hat{\phi}$ and its conjugate momentum $\hat{\pi}$.

Show that the real field $\hat{\phi}$ and its conjugate momentum $\hat{\pi}$ satisfy the Equal Time Commutation Relations using the expressions derived in part (i).

[6 marks]

(iii) The retarded propagator, $\Delta_R(x)$, for a free real scalar field of mass m may be written as

$$\Delta_R(x) \quad = \quad \int \frac{\mathrm{d}^4p}{(2\pi)^4} \mathrm{e}^{-ip_0t+i\boldsymbol{p}\cdot\boldsymbol{x}} \frac{i}{(p_0 + i\epsilon)^2 - \boldsymbol{p}^2 - m^2} , \tag{7}$$

where ϵ is a positive infinitesimal real number. By performing the p_0 integration, express this in terms of Heaviside functions of time, such as $\theta(x^0)$, and an integral over a three-momentum $\boldsymbol{p}$ involving x and $\omega(\boldsymbol{p})$.

Hence find a relationship between the retarded propagator $\Delta_R(x)$ and one of the three commutators found in part (i) of this question. [12 marks]

[Total 30 marks]

3. A real scalar field operator $\hat{\phi}(x)$ is split into two *arbitrary* parts,

$$\hat{\phi}(x) = \hat{\phi}^{+}(x) + \hat{\phi}^{-}(x)\,. \tag{8}$$

(i) Define normal ordering in terms of the arbitrary split of this field.

Define time ordering.

Define a contraction $\Delta(x-y) = \overbrace{\hat{\phi}(x)\hat{\phi}(y)}$.

State Wick's theorem. [10 marks]

(ii) In the following $\hat{\phi}(i) \equiv \hat{\phi}(x_i)$.

Assume that $\left[\hat{\phi}(j)^{+}, \hat{\phi}(i)^{+}\right] = \left[\hat{\phi}(j)^{-}, \hat{\phi}(i)^{-}\right] = 0$. This ensures that the contraction, all normal-ordered products and all time-ordered products are invariant under interchange of fields, i.e.

$$\begin{aligned} \overbrace{\hat{\phi}(1)\hat{\phi}(2)} &= \overbrace{\hat{\phi}(2)\hat{\phi}(1)} & (9)\\ \mathrm{T}\left(\hat{\phi}(1)\hat{\phi}(2)\ldots\hat{\phi}(n)\right) &= \mathrm{T}\left(\hat{\phi}(a_1)\hat{\phi}(a_2)\ldots\hat{\phi}(a_n)\right) & (10)\\ \mathrm{N}\left(\hat{\phi}(1)\hat{\phi}(2)\ldots\hat{\phi}(n)\right) &= \mathrm{N}\left(\hat{\phi}(a_1)\hat{\phi}(a_2)\ldots\hat{\phi}(a_n)\right) & (11)\end{aligned}$$

where $(a_1, a_2, \ldots, a_n)$ is any permutation of $(1, 2, \ldots, n)$.

You may also assume that equations (10) and (11) are also true if some (or all) of the fields are replaced by just one part of the split field, for instance if we replace $\phi(1)$ by $\hat{\phi}^{+}(1)$ and $\phi(2)$ by $\hat{\phi}^{-}(2)$.

You may assume that the contractions are always a c-number, i.e. they will commute with any operators.

A useful identity is that for a set of n-operators, $\widehat{A}_i$ $(i = 1, 2, \ldots, n)$, we have that

$$[\widehat{A}_1\,,\ \widehat{A}_2\ldots\widehat{A}_m] = \sum_{i=2}^{m} \widehat{A}_2\ldots\widehat{A}_{i-1}[\widehat{A}_1, \widehat{A}_i]\widehat{A}_{i+1}\ldots\widehat{A}_m\,. \tag{12}$$

Hence, prove Wick's theorem for the single real scalar field $\hat{\phi}$.

[20 marks]

[Total 30 marks]

4. The scalar Yukawa theory has a real scalar field ϕ of mass m and a complex scalar field ψ with mass M with a cubic interaction proportional to a real coupling constant g, giving a Lagrangian density of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu \psi^\dagger)(\partial^\mu \psi) - M^2\psi^\dagger\psi - g\psi^\dagger\psi\phi \,. \qquad (13)$$

The field operators in the interaction picture are given in equations (17) for $\hat{\phi}$ and (19) for $\hat{\psi}$ in the "Useful Definitions" section at the end of this exam paper.

(i) Define the Feynman rules for calculating the Green functions in coordinate space of the scalar Yukawa theory of (13). [10 marks]

(ii) Write down the Feynman diagrams which contribute to terms proportional to g and g^2 in the perturbation expansion of $Z = \langle 0|S|\,0\rangle$ where S is the S-matrix for this theory.

Hence show that $Z = 1 + g^2V_1 + g^2V_2 + O(g^3)$ where g^2V_1 and g^2V_2 correspond to two different diagrams.

Give explicit expressions for V_1 and V_2 in terms of appropriate propagators. You need not evaluate any integrations in your expression. [8 marks]

(iii) Write down all the Feynman diagrams which contribute terms up to and including g^2 in the perturbation expansion of $\Pi_0(z-y) = \langle 0|\mathrm{T}\left(\hat{\phi}(z)\hat{\phi}(y)S\right)|\,0\rangle$.

Hence show that

$$\begin{aligned}\Pi_0(z-y) \;=\; & \Delta(z-y) + g^2 D_1(z-y) + g^2 D_2(z-y) \\ & + g^2\Delta(z-y)V_1 + g^2\Delta(z-y)V_2 + O(g^3) \quad (14)\end{aligned}$$

where you should identify each of the five terms with a different diagram.

Identify $\Delta(z-y)$ in terms of one of the propagators of this theory.

Give explicit expressions for $D_1(z-y)$ and $D_2(z-y)$ in terms of appropriate propagators. You need not evaluate any integrations in your expression.

[8 marks]

(iv) Expand $\Pi_c(z-y)$ as a series in g up to and including g^2 where

$$\Pi_c(z-y) = \frac{1}{Z}\Pi_0(z-y)\,. \qquad (15)$$

Your answer should be given in terms of V_1, V_2, D_1, D_2, and Δ.

What diagrams contribute to $\Pi_c(z-y)$ in general? Illustrate your answer using your $O(g^2)$ result for $\Pi_c(z-y)$. [4 marks]

[Total 30 marks]

Useful Definitions

Units

Unless otherwise specified, natural units are used so $\hbar = c = 1$.

Metric

The metric is diagonal with $g^{00} = +1$ and $g^{ii} = -1$ for $i = 1, 2, 3$.

Annihilation and Creation Operators

The annihilation and creation operators obey the following commutation relations

$$\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \,, \qquad \left[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}\right] = \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = 0 \,. \tag{16}$$

The vacuum state $|\,0\rangle$ is destroyed by any annihilation operator, that is $\hat{a}_{\boldsymbol{p}}|\,0\rangle = 0$ for all $\boldsymbol{p}$.

Fields

In the following expressions $px \equiv p^\mu x_\mu = p_0 t - \boldsymbol{p}.\boldsymbol{x}$ where $p^\mu = (p_0, \boldsymbol{p})$, $x^\mu = (t, \boldsymbol{x})$ and $\boldsymbol{p}.\boldsymbol{x}$ is the usual three-vector scalar product.

In the interaction picture, a real field $\hat{\phi}(x)$ of mass m has the form

$$\hat{\phi}(x) \;=\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\boldsymbol{p})}} \left(\hat{a}_{\boldsymbol{p}} e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}} e^{+ipx}\right) , \tag{17}$$

$$\text{where} \quad p_0 = \omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2 + m^2}\right| \,. \tag{18}$$

A complex field $\hat{\psi}(x)$ of mass M in the interaction picture has the form

$$\hat{\psi}(x) \;=\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\Omega(\boldsymbol{p})}} \left(\hat{b}_{\boldsymbol{p}} e^{-ipx} + \hat{c}^\dagger_{\boldsymbol{p}} e^{ipx}\right) , \tag{19}$$

$$\text{where} \quad p_0 = \Omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2 + M^2}\right| \,. \tag{20}$$

Both the $\hat{b}_{\boldsymbol{p}}$, $\hat{b}^\dagger_{\boldsymbol{p}}$ pairs and the $\hat{c}_{\boldsymbol{p}}$, $\hat{c}^\dagger_{\boldsymbol{p}}$ pairs of annihilation and creation operators obey similar commutation relations to those of (16) for the $\hat{a}_{\boldsymbol{p}}$, $\hat{a}^\dagger_{\boldsymbol{p}}$ pairs. Different types of annihilation and creation operator always commute e.g. $\left[\hat{a}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{q}}\right] = 0$.