Quantum Electrodynamics Test

Wednesday, 9th January 2008

Section A contains one question worth 10 marks.
Section B contains two questions worth 10 marks each.

Please answer all of Section A and one question from Section B.

Use a separate booklet for each question. Make sure that each booklet carries your name, the course title, and the number of the question attempted.

Part A

Please answer all parts.

1. Consider a massive, non-interacting real scalar field $\phi(x)$ with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2.$$

The quantum field can be expanded as

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}p\cdot x} + a_{\mathbf{p}}^\dagger \mathrm{e}^{\mathrm{i}p\cdot x}\right),$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ and $p^\mu = (E_{\mathbf{p}}, \mathbf{p})$.

(i) Show that the classical Euler–Lagrange equation for ϕ is the Klein–Gordon equation

$$\left(\partial^2 + m^2\right)\phi = 0,$$

and that this equation admits classical plane wave solutions of the form

$$\phi(x) = \alpha\mathrm{e}^{-\mathrm{i}k\cdot x} + \alpha^*\mathrm{e}^{\mathrm{i}k\cdot x}.$$

What is the condition on $k^\mu = (\omega_{\mathbf{k}}, \mathbf{k})$? [3 marks]

(ii) In the quantum theory the operators $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ satisfy

$$\left[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger\right] = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}),$$

and $[a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0$.
To what commutation relations for $\phi(x)$ and $\pi(x) = \dot{\phi}(x)$ are these equivalent? In particular, use them to derive an expression for $\left[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})\right]$. [3 marks]

(iii) Describe how the states in the free-field Hilbert space are built using $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$. In particular define the vacuum state $|0\rangle$ and one-particle states $|\mathbf{p}\rangle$. (You may ignore the normalization of the states.) [1 marks]

(iv) The total energy (Hamiltonian) is given by

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_{\mathbf{p}}\, a_{\mathbf{p}}^\dagger a_{\mathbf{p}},$$

Show that the one particle states $|\mathbf{p}\rangle$ are eigenstates of H and give the eigenvalue.
Given the classical plane wave solutions discussed in part 1(i), comment on the relationship between the energy of the quantum state $|\mathbf{p}\rangle$ and the frequency of the corresponding classical excitation. In what classic experiment is the same relation seen to hold for photons? [3 marks]

Part B

Please answer *one* question.

1. Consider the classical Dirac field $\psi(x)$ satisfying the Dirac equation

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\,\psi(x) = 0,$$

where the gamma matrices γ^μ satisfy $\left\{\gamma^\mu, \gamma^\nu\right\} = 2\eta^{\mu\nu}\mathbb{1}$. Under a Lorentz transformation $x^\mu \mapsto x'^\mu = \left(\Lambda^{-1}\right)^\mu{}_\nu x^\nu$ the Dirac field transforms as

$$\psi(x) \mapsto \psi'(x) = \Lambda_{\frac{1}{2}}\,\psi(\Lambda^{-1}x),$$

where $\Lambda_{\frac{1}{2}}$ is a matrix acting on the spinor ψ.

(i) Show that

$$\partial_\mu = \left(\Lambda^{-1}\right)^\nu{}_\mu \partial'_\nu,$$

where $\partial'_\nu = \partial/\partial x'^\nu$. [1 marks]

(ii) You are given that $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$. Hence show that if $\psi(x)$ satisfies the Dirac equation then

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\,\psi'(x) = \Lambda_{\frac{1}{2}}\left(\mathrm{i}\gamma^\mu\partial'_\mu - m\right)\psi(x') = 0.$$

Comment on the significance of this result. [4 marks]

(iii) Using the definition $\bar{\psi}(x) = \psi^\dagger\gamma^0$ and $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ show that if $\psi(x)$ satisfies the Dirac equation then

$$\mathrm{i}\partial_\mu\bar{\psi}(x)\gamma^\mu + m\bar{\psi}(x) = 0.$$

[3 marks]

(iv) Hence show that the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved, that is $\partial_\mu j^\mu = 0$. [2 marks]

Question 2 is on the next page.

2. (i) In the Heisenberg picture operators $\mathcal{O}_H(t)$ evolve with time, while states $|\psi\rangle_H$ are fixed, that is

$$\partial_t \mathcal{O}_H = \mathrm{i}\big[H_H, \mathcal{O}_H\big], \qquad \partial_t |\psi\rangle_H = 0,$$

where H_H is the total Hamiltonian in the Heisenberg picture.

What are the corresponding expressions for the evolution of operators $\mathcal{O}_I(t)$ and states $|\psi, t\rangle_I$ in the interaction picture, in terms of $H_{0,I}$ and $H_{\mathrm{int},I}$ (the free and interaction Hamiltonians in the interaction picture)? [2 marks]

(ii) Let operators in the two pictures be related by

$$\mathcal{O}_I(t) = U(t,t_0)\mathcal{O}_H(t)U(t,t_0)^{-1},$$

where $U(t_0,t_0) = 1$. Show that, for any operator $\mathcal{O}_I(t)$,

$$\partial_t \mathcal{O}_I = \big[(\partial_t U)\, U^{-1} + \mathrm{i}H_I, \mathcal{O}_I\big],$$

where H_I is the total Hamiltonian in interaction picture. [4 marks]

(iii) Hence show that

$$\partial_t U = -\mathrm{i}H_{\mathrm{int},I}U.$$

Show that this equation is consistent with U being a unitary operator. [2 marks]

(iv) As an expansion in $H_{\mathrm{int},I}$, show explicitly that

$$U(t,t_0) = 1 - \mathrm{i}\int_{t_0}^{t} \mathrm{d}t_1 H_{\mathrm{int},I}(t_1) + \tfrac{1}{2}(-\mathrm{i})^2 \int_{t_0}^{t}\int_{t_0}^{t} \mathrm{d}t_1 \mathrm{d}t_2\, T H_{\mathrm{int},I}(t_1)H_{\mathrm{int},I}(t_2) + \ldots$$

is a solution for $U(t,t_0)$ up to second order in $H_{\mathrm{int},I}$, where "T" denotes time-ordering. [2 marks]