MSc in Quantum Fields and Fundamental Forces

Quantum Electrodynamics Test

Monday, 11th January 2010

Please answer all three questions.

All questions are worth 20 marks.

Use a separate booklet for each question. Make sure that each booklet carries your name, the course title, and the number of the question attempted.

1. Consider a free real scalar field $\phi(x)$ with conjugate momentum density $\pi(x) = \dot{\phi}(x)$. Define the operator

$$a_{\mathrm{p}} = \int \mathrm{d}^3x \frac{\mathrm{e}^{-\mathrm{i}p\cdot x}}{\sqrt{2E_{\mathrm{p}}}} \left[E_{\mathrm{p}}\phi(0,\mathbf{x}) + \mathrm{i}\pi(0,\mathbf{x})\right],$$

where $E_{\mathrm{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$.

(i) The equal-time commutation relations (ETCRs) state that

$$\left[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})\right] = \mathrm{i}\delta^{(3)}(\mathbf{x}-\mathbf{y}).$$

Are the fields $\phi(x)$ and $\pi(x)$ in the Schrödinger or Heisenberg picture? Why is this picture more natural in a relativistic theory?
What are the ETCRs for $\left[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})\right]$ and $\left[\pi(t,\mathbf{x}), \pi(t,\mathbf{y})\right]$? [4 marks]

(ii) Write down the expression for $a^\dagger_{\mathrm{p}}$ in terms of $\phi(0,\mathbf{x})$ and $\pi(0,\mathbf{x})$. Using the ETCRs show that

$$\left[a_{\mathrm{p}}, a^\dagger_{\mathrm{q}}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}),$$
$$\left[a_{\mathrm{p}}, a_{\mathrm{q}}\right] = 0, \qquad \left[a^\dagger_{\mathrm{p}}, a^\dagger_{\mathrm{q}}\right] = 0.$$

[6 marks]

(iii) Given

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathrm{p}}}}\left(a_{\mathrm{p}}\mathrm{e}^{-\mathrm{i}p\cdot x} + a^\dagger_{\mathrm{p}}\mathrm{e}^{\mathrm{i}p\cdot x}\right),$$

where $p^0 = E_{\mathrm{p}}$, use the results from part ii to show that at unequal times

$$\left[\phi(x), \phi(y)\right] = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathrm{p}}}\left(\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)}\right).$$

What properties must this commutator have if the field theory is to respect microcausality? [5 marks]

(iv) You are given that $\delta(x^2 - a^2) = (1/2a)\left[\delta(x-a) - \delta(x+a)\right]$. Show that the expression for $\left[\phi(x), \phi(y)\right]$ given in part iii can be rewritten as

$$\left[\phi(x), \phi(y)\right] = \int \frac{\mathrm{d}^4p}{(2\pi)^3}\delta(p^2 - m^2)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)},$$

where $\mathrm{d}^4p = \mathrm{d}^3p\,\mathrm{d}p^0$. Comment on the Lorentz transformation properties of this expression, and give a brief argument that the ETCRs take the same form in all inertial frames.
What is the significance of this result? [5 marks]

[Total 20 marks]

2. This question is about the free classical Dirac field $\psi(x)$. The Lagrangian is given by

$$\mathcal{L} = \mathrm{i}\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi,$$

where the gamma matrices γ^μ satisfy $\left\{\gamma^\mu, \gamma^\nu\right\} = 2\eta^{\mu\nu}\mathbb{1}$ and $\bar{\psi} = \psi^\dagger \gamma^0$.

(i) The field $\psi(x)$ is a spinor. How many components does it have? After quantization what is the spin of the corresponding single-particle states?

Treating $\psi(x)$ and $\bar{\psi}(x)$ as independent fields, show that the Euler–Lagrange equation for $\bar{\psi}(x)$ is the Dirac equation

$$\mathrm{i}\gamma^\mu \partial_\mu \psi - m\psi = 0.$$

[4 marks]

(ii) Using $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ show that if $\psi(x)$ satisfies the Dirac equation then

$$\mathrm{i}\partial_\mu \bar{\psi}(x)\gamma^\mu + m\bar{\psi}(x) = 0.$$

Hence show that the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved if ψ satisfies the Dirac equation. [4 marks]

(iii) Show that the complex conjugate of $\mathcal{L}$ is given by

$$\mathcal{L}^* = \mathcal{L} - \mathrm{i}\partial_\mu \left(\bar{\psi}\gamma^\mu\psi\right).$$

Consider the real Lagrangian

$$\mathcal{L}' = \tfrac{1}{2}\mathrm{i}\left[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi\right] - m\bar{\psi}\psi.$$

Treating $\psi(x)$ and $\bar{\psi}(x)$ as independent fields, show that the corresponding Euler–Lagrange equation for $\bar{\psi}(x)$ is again the Dirac equation. [5 marks]

(iv) Defining the matrix $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, show that $\left\{\gamma_5, \gamma_\mu\right\} = 0$. Show that the "axial vector" current $j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ satisfies

$$\partial_\mu j_5^\mu = 2\mathrm{i}m\bar{\psi}\gamma_5\psi.$$

What does this result imply about the symmetries of the Dirac equation when $m = 0$?

Identify the corresponding infinitesimal transformation of ψ and demonstrate directly whether or not $\mathcal{L}$ is invariant under this transformation. [7 marks]

[Total 20 marks]

3. This question is about Feynman diagrams in "phi-fourth" theory. Recall that in the interaction picture the $\mathcal{S}$-matrix is given by

$$\mathcal{S} = T \exp\left(\mathrm{i} \int \mathrm{d}^4x : \mathcal{L}_{\text{int}}(x) :\right).$$

(i) Explain how $\mathcal{S}$ can be described as a perturbation expansion and write down the first three terms in the expansion.

The scattering amplitude $\mathrm{i}\mathcal{M}$ is usually defined to be

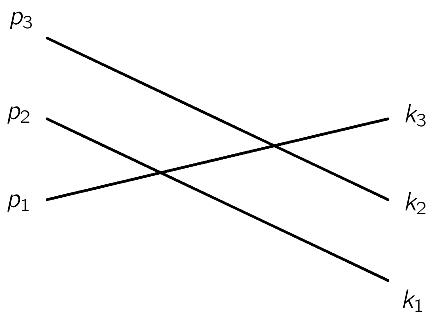
$$\langle \text{out} | \, \mathrm{i}\mathcal{T} \, | \text{in} \rangle = (2\pi)^4 \delta^{(4)} \left(p_{\text{out}} - p_{\text{in}}\right) \, \mathrm{i}\mathcal{M}$$

where $\mathcal{S} = \mathbb{1} + \mathrm{i}\mathcal{T}$. Why is the $\mathbb{1}$ contribution not included? What are p_{out} and p_{in} and what is the physical meaning of the δ-function? [4 marks]

(ii) Consider the scattering of three incoming ϕ particles with momenta k_1, k_2 and k_3 to three outgoing ϕ particles with momenta p_1, p_2 and p_3.

Define the $|\text{in}\rangle$ and $|\text{out}\rangle$ states for this process.

Use the position-space Feynman rules to calculate the contribution of the following Feynman diagram to $\langle \text{out} | \, \mathrm{i}\mathcal{T} \, | \text{in} \rangle$ in terms of the propagator $D_F(x - y)$.



Given

$$D_F(x - y) = \mathrm{i} \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2 - m^2 + \mathrm{i}\epsilon}$$

write this contribution as a function only of momenta.

Show that this agrees with the contribution to $\mathrm{i}\mathcal{M}$ calculated using the momentum-space rules. [7 marks]

(iii) Draw a second Feynman diagram that has no loops, is not related to the diagram in part ii by a permutation of the p_i momenta or of the k_i momenta, and contributes to $\mathrm{i}\mathcal{M}$ at the same order in λ.

Evaluate this diagram using the momentum-space Feynman rules. [3 marks]

Now consider the scattering of two incoming ϕ particles with momenta q_1 and q_2 to two outgoing ϕ particles with momenta p_1 and p_2.

Taking the non-relativistic limit and comparing with the Born approximation gives

$$\mathrm{i}\mathcal{M}(q_1, q_2, p_1, p_2) = -\mathrm{i}\tfrac{1}{2}\left[\tilde{V}(\mathbf{p}_1 - \mathbf{q}_1) + \tilde{V}(\mathbf{p}_1 - \mathbf{q}_2)\right],$$

where $\tilde{V}(\mathbf{q})$ is the Fourier transform of the classical potential $V(\mathbf{x})$ between the two particles.

(iv) Use the momentum-space Feynman rules to identify $\tilde{V}(\mathbf{k})$ and hence calculate the form of $V(\mathbf{x})$ at order λ.

Draw a Feynman diagram that gives corrections to $\mathrm{i}\mathcal{M}$ (and hence potentially to $V(\mathbf{x})$) at order λ^2. [6 marks]

[Total 20 marks]

End of examination paper