**MSc in Quantum Fields and Fundamental Forces**

# Quantum Field Theory Test

**Thursday, 12th January 2012, 10:00 to 12:00**

*Please answer all three questions.*

Each question is worth 20 marks.

Use a separate booklet for each question. Make sure that each booklet carries your name, the course title, and the number of the question attempted.

**1.** Consider the Lagrangian density for a classical complex scalar field $\phi(x)$ given by

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* .$$

(i) Treating $\phi$ and $\phi^*$ as independent fields, show that the Euler–Lagrange equations imply that the equations of motion are given by

$$(\partial^2 + m^2)\phi = 0$$

[2 marks]

(ii) Define the momentum density fields $\pi$ and $\pi^*$ that are conjugate to $\phi$ and $\phi^*$, respectively, and derive the Hamiltonian density $\mathcal{H}$ and the Hamiltonian $H$.

[4 marks]

(iii) The energy-momentum density is defined by

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi^* + \partial^\mu \phi^* \partial^\nu \phi - \eta^{\mu\nu} \left( \partial_\lambda \phi \partial^\lambda \phi^* - m^2 \phi \phi^* \right) ,$$

Using the equations of motion show that $\partial_\mu T^{\mu\nu} = 0$. With what symmetry is this tensor related?

[4 marks]

(iv) Define four corresponding Noether charges $P^\mu$ and show that they are time independent. Show that $P^0$ is the Hamiltonian $H$. What is the significance of $P^i$?

[4 marks]

Consider now the quantum theory of the scalar field $\phi$.

(v) Write down the equal time commutation relations satisfied by the Heisenberg operators $\phi$ and $\pi$. Use them to show that

$$[P^0, \phi(x)] = \ -i\partial^0 \phi(x)$$

(Do *not* use the expansion of $\phi$ in terms of creation and annihilation operators.)

[6 marks]

[Total 20 marks]

**2.** The free Dirac field in the Heisenberg picture can be expanded as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{2} \left( a^s_{\boldsymbol{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\boldsymbol{p}} v^s(p) e^{ip\cdot x} \right)$$

$$\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{2} \left( b^s_{\boldsymbol{p}} v^{s\dagger}(p) e^{-ip\cdot x} + a^{s\dagger}_{\boldsymbol{p}} u^{s\dagger}(p) e^{ip\cdot x} \right)$$

where $p^0 = E_{\boldsymbol{p}}$ and the non-vanishing anti-commutation relations are given by

$$\{a^r_{\boldsymbol{p}}, a^{\dagger s}_{\boldsymbol{q}}\} = \{b^r_{\boldsymbol{p}}, b^{\dagger s}_{\boldsymbol{q}}\} = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \delta^{rs}$$

In addition

$$u^{r\dagger}(p) u^s(p) = 2E_{\boldsymbol{p}} \delta^{rs}, \qquad v^{r\dagger}(p) v^s(p) = 2E_{\boldsymbol{p}} \delta^{rs}$$

and

$$u^{r\dagger}(p^0, \boldsymbol{p}) v^s(p^0, -\boldsymbol{p}) = 0, \qquad v^{r\dagger}(p^0, \boldsymbol{p}) u^s(p^0, -\boldsymbol{p}) = 0$$

(i) Define the vacuum state, a single fermion state and a single anti-fermion state. [4 marks]

(ii) Show that the time-independent Noether charge defined by

$$Q = \int d^3x \psi^\dagger(x) \psi(x)$$

can be written in the form

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left( a^{s\dagger}_{\boldsymbol{p}} a^s_{\boldsymbol{p}} + b^s_{\boldsymbol{p}} b^{s\dagger}_{\boldsymbol{p}} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left( a^{s\dagger}_{\boldsymbol{p}} a^s_{\boldsymbol{p}} - b^{s\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}} \right)$$

where an infinite c-number is discarded to get to the last line. [10 marks]

(iii) Show that vacuum state, the single fermion state and the single anti-fermion state that you constructed in (i) are eigenstates of $Q$. What is the physical interpretation of $Q$? [6 marks]

[Total 20 marks]

**3.** Consider the free quantised real scalar field in the Heisenberg picture

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left( a_{\boldsymbol{p}} \mathrm{e}^{-\mathrm{i}p\cdot x} + a^\dagger_{\boldsymbol{p}} \mathrm{e}^{\mathrm{i}p\cdot x} \right),$$

where $x^\mu = (t, \boldsymbol{x})^\mu$ and $p^0 = E_{\boldsymbol{p}} \equiv \sqrt{|\boldsymbol{p}|^2 + m^2}$ and

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}),$$
$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}\right] = 0, \qquad \left[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = 0.$$

(i) Show that

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} e^{-ip\cdot(x-y)}$$

[3 marks]

(ii) The Feynman propagator is defined as $D_F(x-y) = \langle 0|T(\phi(x)\phi(y))|0\rangle$. Show that this can be written in the form

$$D_F(x-y) = \lim_{\epsilon\to 0} \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)},$$

explaining carefully the precise contour on which the integral is defined.

[8 marks]

(iii) Show that $D_F(x)$ satisfies

$$(\partial^2 + m^2) D_F(x) = -i\delta^4(x)$$

[2 marks]

Now assume that $\phi(x)$ is a quantised field in the interaction picture for $\phi$ to the fourth theory, with $S$-matrix given by

$$\mathcal{S} = T \exp\left( -\frac{1}{4!}\mathrm{i}\lambda \int \mathrm{d}^4 x \phi(x)^4 \right).$$

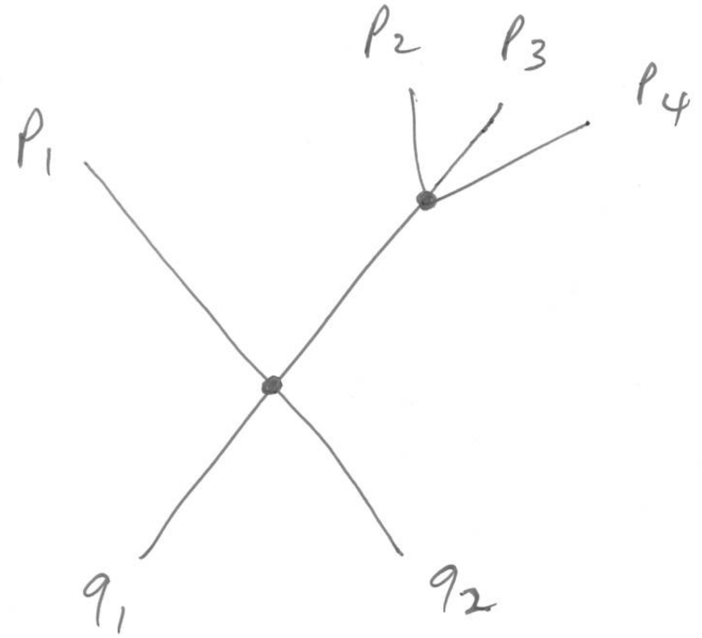
Consider the scattering amplitude $\mathrm{i}\mathcal{M}$ for two incoming particles with momentum $q_1$, $q_2$ scattering to four outgoing particles with momenta $p_1$, $p_2$, $p_3$ and $p_4$ defined by

$$\langle p_1, p_2, p_3, p_4|\mathrm{i}\mathcal{T}|q_1, q_2\rangle = (2\pi)^4 \delta^{(4)}\left(p_1 + p_2 + p_3 + p_4 - q_1 - q_2\right) \mathrm{i}\mathcal{M}$$

where $\mathcal{S} = \mathbb{1} + \mathrm{i}\mathcal{T}$.

**Please turn over**

(iv) Use the position-space Feynman rules to calculate the contribution of the following Feynman diagram to i$\mathcal{M}$. (You will need to use the expression for $D_F$ given in question 3. (ii) above and carry out some integrations) [4 marks]




(v) Draw the other Feynman diagrams that contribute to i$\mathcal{M}$ at the same order. Evaluate one of these diagrams using the momentum-space Feynman rules. [3 marks]

[Total 20 marks]



**End of examination paper**